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ON THE SZEGED INDEX OF NON-COMMUTATIVE GRAPH OF GENERAL LINEAR GROUP

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ABSTRACT. Let G be a non-abelian group and let $Z(G)$ be the center of G. Associate with G there is a graph Γ_G as follows: Take $G\setminus Z(G)$ as vertices of Γ_G and joint two distinct vertices x and y whenever $yx \neq yx$. Γ_G is called the non-commuting graph of G. In recent years many interesting works have been done in non-commutative graph of groups. Computing the clique number, chromatic number, Szeged index and Wiener index play important role in graph theory. In particular, the clique number of non-commuting graph of some the general linear groups has been determined.
Recently, Wiener and Szeged indices have been computed for $\Gamma_{PSL(2,q)}$, where $q \equiv 0(mod\ 4)$. In this paper we will compute the Szeged index for $\Gamma_{PSL(2,q)}$, where $q \not\equiv 0(mod\ 4)$.

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1. **Introduction**

Let G be a non-abelian group and let $Z(G)$ be the center of G. Associate with G there is a graph Γ_G as follows: Take $G \setminus Z(G)$ as vertices of Γ_G and joint two distinct vertices x and y whenever $yx \neq yx$. Γ_G is called the non-commuting graph of G and many of the graph theoretical properties of Γ_G is studied in [1, 2, 3, 4, 15]. In particular, it is proved that if G is a non-abelian group, then Γ_G is a connected.

Let Γ be an undirected connected graph without loops or multiple edges. The sets of vertices and edges of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. For vertices x and y in $V(\Gamma)$, we denote by $d(x, y)$ the topological distance i.e., the number of edges on the shortest path, joining the two vertices of Γ. Since Γ is connected, $d(x, y)$ exists for all x, y in $V(\Gamma)$. The name Wiener number or Wiener index is nowadays in standard use in chemistry and is sometimes encountered also in the mathematical literature(see [5, 6, 7, 8, 15]). The Wiener index of the graph Γ is the half sum of distances over all its vertex pairs (u, v):

$$W(\Gamma) = \frac{1}{2} \sum_{u,v \in V(\Gamma)} d(u, v).$$

For an edge $e = uv$ of a graph Γ, let $n_u(e)$ denotes the set of vertices of Γ lying closer to u than to v and let $n_v(e)$ denoted the set of vertices of Γ lying closer to v than to u. The sets $n_u(e)$ and $n_v(e)$ play an important role in metric graph theory. For more information see [9, 13, 14]. Ivan Gutman [10] has defined the Szeged index, $Sz(\Gamma)$, for a graph Γ as follows:

$$Sz(\Gamma) = \sum_{uv=e \in E(\Gamma)} |n_u(e)| \cdot |n_v(e)|.$$

If Γ is a tree, then $Sz(\Gamma) = W(\Gamma)$. We recall that this is not true for any graph.

Recently, Wiener and Szeged indices have been computed for $\Gamma_{PSL(2,q)}$, where $q \equiv 0(mod\ 4)$, but the computing Szeged index of $\Gamma_{PSL(2,q)}$, $q \not\equiv 0(mod\ 4)$, is an open problem in [2]. In this paper we will answer the mentioned open question. Actually, we compute the Szeged index of $\Gamma_{PSL(2,q)}$, where $q \not\equiv 0(mod\ 4)$ i.e.

Theorem 1.1. *Let $G = PSL(2, q)$, where $q \not\equiv 0(mod4)$. Then*

$$Sz(\Gamma_G) = \begin{cases} \frac{q(q-1)(q+1)(q^5+2q^4-q^3-26q^2+132q-172)}{32}, & \text{if } q \equiv 1(mod4)\ ; \\ \frac{q(q-1)(q+1)(q^5+2q^4-q^3-26q^2+68q-28)}{32}, & \text{if } q \equiv 3(mod4). \end{cases}$$

2. **Some properties of** $PSL(2, q)$

In this section we give some basic results of the projective special linear groups.

Proposition 2.1. *Let $G = PSL(2, q)$, where q is a power of a prime p and let $k = gcd(q - 1, 2)$. Then*

1: *a Sylow p-subgroup P of G is an elementary abelian group of order q and the number of Sylow p-subgroups of G is $q+1$.*

2: *G contains a cyclic subgroup A of order $t = \frac{q-1}{k}$ such that $N_G(\langle u\rangle)$ is a dihedral group of order $2t$ for every non-trivial element $u \in A$.*

3: *G contains a cyclic subgroup B of order $s = \frac{q+1}{k}$ such that $N_G(\langle u\rangle)$ is a dihedral group of order $2s$ for every non-trivial element $u \in B$.*

4: *The set $\{P^x, A^x, B^x | x \in G\}$ is a partition for G.*

Suppose that a is a non-trivial element of G.

5: *If $q > 5$ and $q \equiv 1(mod\ 4)$, then*

$$C_G(a) = \begin{cases} N_G(\langle a\rangle) & \text{if } a^2 = 1, a \in A^x \text{ for some } x \in G \\ A^x & \text{if } a^2 \neq 1, a \in A^x \text{ for some } x \in G \\ B^x & \text{if } a \in B^x \text{ for some } x \in G \\ P^x & \text{if } a \in P^x \text{ for some } x \in G \end{cases}$$

6: *If $q > 5$ and $q \equiv 3(mod\ 4)$, then*

$$C_G(a) = \begin{cases} N_G(\langle a\rangle) & \text{if } a^2 = 1, a \in B^x \text{ for some } x \in G \\ B^x & \text{if } a^2 \neq 1, a \in B^x \text{ for some } x \in G \\ A^x & \text{if } a \in A^x \text{ for some } x \in G \\ P^x & \text{if } a \in P^x \text{ for some } x \in G \end{cases}$$

Proof. For the proof we refer reader to Proposition 3.21 of [1]. □

Now suppose that $G = PSL(2, q)$, where $q \equiv 1(mod\ 4)$. Using the notation of Proposition 2.1, we have $k = 2$, $|PSL(2, q)| = \frac{1}{2}q(q-1)(q+1)$, $|A| = \frac{1}{2}(q-1)$ and $|B| = \frac{1}{2}(q+1)$. Also, it is clear, from the Proposition2.1, that the number of conjugates of P, A and B are $q+1, \frac{1}{2}q(q+1), \frac{1}{2}q(q-1)$, respectively. Let $\{P_i : i = 1, 2, \ldots, q+1\}, \{A_i : i = 1, 2, \ldots, \frac{1}{2}q(q+1)\}$ and $\{B_i : i = 1, 2, \ldots, \frac{1}{2}q(q-1)\}$ be the set of all conjugates of P, A and B respectively. Finally, since P_i, A_j and B_k are all abelian, so by using the definition of the non-commutative graph of a group, there exist no edge such that the two of its vertices belong to one of them.

Lemma 2.2. *Let $G = PSL(2, q)$, where $q \equiv 1(mod\ 4)$. If $a \in A_i$ and $b \in A_j$ are elements of order two such that $ab \neq ba$ and $i \neq j$, then $|C_G(a) \cap C_G(b)| \leq 2$.*

Proof. Since A_i and A_j are cyclic subgroups, $A_i = \langle y_i\rangle$ and $A_j = \langle y_j\rangle$. By Proposition2.1, $N_G(\langle a\rangle)$ and $N_G(\langle b\rangle)$ are dihedral groups of order $q-1$ and $C_G(a) = N_G(\langle a\rangle), C_G(b) = N_G(\langle b\rangle)$. Hence there exist

$x, z \in G$ of order two such that $x \in A_i, z \in A_j$ and $C_G(a) = \langle x, y_i \rangle, C_G(b) = \langle z, y_j \rangle$. Since $A_i \cap A_j = 1$, every non-trivial element of $C_G(a) \cap C_G(b)$ has order two. Suppose that $xy_i^r, xy_i^s \in C_G(a) \cap C_G(b)$. So $y_i^{s-r} \in C_G(a) \cap C_G(b)$. Hence $y_i^{s-r} \in A_i$. If $y_i^{s-r} = a$, then $ab = ba$ which is impossible. Thus $y_i^{s-r} \neq a$. Since the cyclic group A_i has only just one element of order two, y_i^{s-r} is not of order two and since $y_i^{s-r} \in C_G(b)$, $y_i^{s-r} \in A_j$. Therefore $y_i^{s-r} = 1$. Hence $r = s$. This completes the proof. □

Lemma 2.3. *If G is a non-abelian group, then Γ_G is a connected graph of diameter 2 and girth 3.*

Proof. See [1]. □

By the above Lemma it is clear that:

Remark 2.4. *Let G be a non-abelian group and $x \in G \setminus Z(G)$. Then*

$$d(x,y) = \begin{cases} 1, & \text{if } y \in G \backslash C_G(x)\ ; \\ 2, & \text{if } y \in C_G(x) \backslash Z(G). \end{cases}$$

The following definition and also the lemmas can be found in [2], but for the convenience we bring them here.

Definition 2.5. *Let Γ be a graph and $A, B \subseteq V(\Gamma)$. We define*

$$E_{A,B} = \{ab \in E(G) \mid a \in A,\ b \in B\},$$

and

$$S_{A,B} = \sum_{ab \in E_{A,B}} |n_{ab}(a)| \cdot |n_{ab}(b)|. \qquad (*)$$

If $E_{A,B} = \emptyset$, then we define $S_{A,B} = 0$. Also, for $a \in V(G)$ we put
$E_{\{a\},B} = E_{a,B}$.

Lemma 2.6. *Let G be a non-abelian group and $e = uv \in E(\Gamma_G)$. Then*

$$n_u(e) = ((C_G(v) \backslash C_G(u)) \backslash \{v\}) \cup \{u\}.$$

Proof. See [2]. □

3. **Proof of the main theorem**

In this section we present some lemmas which play important role in the proof of the Theorem 1.1.

Lemma 3.1. *Let $G = PSL(2, q)$, where $q \equiv 1 (mod\ 4)$. Then*

1: $S_{a,b} = S_{b,a}$ *for* $a, b \in G \setminus Z(G)$.

2: $S_{a,b} = |P_j - 1| \cdot |P_i - 1| = (q-1)^2$, *where* $a \in P_i$, $b \in P_j$ *and* $i \neq j$.

3: $S_{a,b} = |A_j - 1| \cdot |P_i - 1| = \frac{1}{2}(q-1) \cdot (q-3)$, *where* $a \in P_i$, $b \in A_j$ *and* $b^2 \neq 1$.

4: $S_{a,b} = |B_j - 1| \cdot |P_i - 1| = \frac{1}{2}(q-1)^2$, *where* $a \in P_i$ *and* $b \in B_j$.

5: $S_{a,b} = |A_j - 1| \cdot |A_i - 1| = \frac{1}{4}(q-3)^2$, *where* $a \in A_i$, $b \in A_j$, $i \neq j$, $a^2 \neq 1$ *and* $b^2 \neq 1$.

6: $S_{a,b} = |B_j - 1| \cdot |A_i - 1| = \frac{1}{4}(q-1) \cdot (q-3)$, *where* $a \in A_i$, $b \in B_j$ *and* $a^2 \neq 1$.

7: $S_{a,b} = |B_j - 1| \cdot |B_i - 1| = \frac{1}{4}(q-1)^2$, *where* $a \in B_i$, $b \in B_j$ *and* $i \neq j$.

8: $S_{a,b} = |N_G(\langle b \rangle) - 1| \cdot |P_i - 1| = (q-1)(q-2)$, *where* $a \in P_i$, $b \in A_j$ *and* $b^2 \neq 1$.

9: $S_{a,b} = |B_j - 1| \cdot |N_G(\langle a \rangle) - 1| = \frac{1}{2}(q-1)(q-2)$, *where* $a \in A_i$, $b \in B_j$ *and* $a^2 = 1$.

10: *If* $a \in A_i$, $b \in A_j$, $a^2 = 1$, $b^2 \neq 1$ *and* $i \neq j$ *then we have two cases:*

i- If $A_j \cap N_G(\langle a \rangle) = 1$, *then* $S_{a,b} = |A_j - 1| \cdot |N_G(\langle a \rangle) - 1| = \frac{1}{2}(q-2)(q-3)$.

ii-If $|A_j \cap N_G(\langle a \rangle)| = 2$, *then* $S_{a,b} = (|A_j| - 2) \cdot |N_G(\langle a \rangle)| - 2) = \frac{1}{2}(q-3)(q-5)$.

11: *If* $a \in A_i$, $b \in A_j$, $a^2 = 1$, $b^2 = 1$ *and* $i \neq j$ *then we have two cases:*

i- If $N_G(\langle a \rangle) \cap N_G(\langle b \rangle) = 1$, *then* $S_{a,b} = |N_G(\langle a \rangle) - 1| \cdot |N_G(\langle a \rangle) - 1| = (q-1)^2$.

ii-If $|N_G(\langle a \rangle) \cap N_G(\langle b \rangle)| = 2$, *then* $S_{a,b} = (|N_G(\langle a \rangle)| - 2) \cdot |N_G(\langle b \rangle)| - 2) = (q-3)^2$.

Proof. Part (1) is trivial, the proofs of part (2) to (7) are similar, so we prove only part (3). The proofs of parts (8) and (9) also the proof of part (10) and (11) are similar. So we prove only (9) and (10).

The proof of part 3: Let $a \in P_i$ and $b \in A_j$. By Proposition 2.1, $C_G(a) = P_i$ and $C_G(b) = A_j$. Now by Lemma 2.6, $n_a(ab) = ((A_j \setminus P_i) \setminus \{b\}) \cup \{a\}$. By part 4 of Proposition 2.1, $A_j \cap P_i = 1$. Hence $n_a(ab) = (A_j \setminus \{1, b\}) \cup \{a\}$ and so $|n_a(ab)| = |A_j| - 1 = q - 2$. Similarly, $|n_b(ab)| = |P_i| - 1 = q - 1$. Thus $S_{a,b} = (q-2)(q-1)$.

9- Since $a \in A_i, a^2 = 1$ and $b \in B_j$, so by the proposition 2.1 we have $C_G(a) = N_G(\langle a \rangle)$ is a dihedral group of order $q-1$ and $C_G(b) = B_j$. Hence by Lemma 2.6, $|n_a(ab)| = |B_j \setminus N_G(\langle a \rangle)|$. It is easy to see that $B_j \cap N_G(\langle a \rangle) = 1$. So $|n_a(ab)| = \frac{1}{2}(q-1)$. Similarly, $|n_b(ab)| = |N_G(\langle a \rangle)| - 1 = q - 2$, so $S_{a,b} = \frac{1}{2}(q-1)(q-2)$.

10- By part 5 of Proposition 2.1, we have $C_G(a) = N_G(\langle a \rangle)$ is a dihedral group of order $q-1$ and $C_G(b) = A_j$. So $|n_a(ab)| = |A_j \setminus N_G(\langle a \rangle)| = |A_j| - |A_j \cap N_G(\langle a \rangle)|$. Suppose that $A_j = \langle y_i \rangle$. Since $N_G(\langle a \rangle)$ is a dihedral group, there exists an element x of order two such that $x \notin A_i$ and $N_G(\langle a \rangle) = \langle y_i, x \rangle$. For $0 \leq k \leq \frac{q-1}{2} - 1$ we know xy_i^k is a element of order two. So $xy_i^k \in \bigcup_{t=1}^{\frac{1}{2}q(q+1)} A_t$. Since A_t has just one element of order two and the dihedral group $N_G(\langle a \rangle)$ have $\frac{q-1}{2}$ elements of order two except a, so $|N_G(\langle a \rangle) \cap A_t| \leq 2$, where $1 \leq t \leq \frac{1}{2}q(q+1)$. Hence $|\{A_r\ :\ |A_r \cap N_G(\langle a \rangle)| = 2\}| = \frac{q-1}{2}$

and $|\{A_r\ :\ |A_r \cap N_G(\langle a\rangle)| = 1\}| = \frac{1}{2}q(q+1) - \frac{q-1}{2} - 1 = \frac{q^2-1}{2}$.Therefore
i) If $A_j \cap N_G(\langle a\rangle) = 1$, then $|n_a(ab)| = |N_G(\langle a\rangle)| - 1 = q-2$ and $|n_b(ab)| = |A_j| - 1 = \frac{q-3}{2}$. Hence $S_{a,b} = \frac{1}{2}(q-2)(q-3)$.
ii) If $A_j \cap N_G(\langle a\rangle) = 2$, then $|n_a(ab)| = |N_G(\langle a\rangle)| - 2 = q-3$ and $|n_b(ab)| = |A_j| - 2 = \frac{q-5}{2}$. Hence $S_{a,b} = \frac{1}{2}(q-3)(q-5)$.
11- Since $a \in A_i, a^2 = 1$ and $b \in A_j, b^2 = 1$, so $C_G(a) = N_G(\langle a\rangle)$ and $C_G(b) = N_G(\langle b\rangle)$ are dihedral groups of order $q-1$. Considering Lemma 2.2 we have:
i− If $C_G(a) \cap C_G(b) = 1$, then $|n_a(ab)| = |C_G(b) \setminus C_G(a)| = |N_G(\langle a\rangle)| - 1 = q-2$ and similarly $|n_b(ab)| = q-2$. Hence $S_{a,b} = (q-2)^2$.
ii− If $|C_G(a) \cap C_G(b)| = 2$, then $|n_a(ab)| = q-3$ and $|n_b(ab)| = q-3$. Thus $S_{a,b} = (q-2)^3$. □

Corollary 3.2. *Under the assumptions of Lemma 3.1 and letting $\beta = \frac{1}{2}q(q-1)$ we have:*

1: $S_{P_i,P_j} = (|P_i|-1)^2 \cdot (|P_j|-1)^2 = (q-1)^4$, *where* $1 \le i \ne j \le q+1$.
2: $S_{P_i,B_j} = (|P_i|-1)^2 \cdot (|B_j|-1)^2 = \frac{1}{4}(q-1)^4$, *where* $1 \le i \le q+1$ *and* $1 \le j \le \beta$.
3: $S_{B_i,B_j} = (|B_i|-1)^2 \cdot (|B_j|-1)^2 = \frac{1}{16}(q-1)^4$, *where* $1 \le i \ne j \le \beta$.
4: $S_{P_i,A_j} = \frac{1}{4}(q-1)^2(q^2-4q+7)$, *where* $1 \le i \le q+1$ *and* $1 \le j \le \alpha$.
5: $S_{A_i,B_j} = \frac{1}{16}(q-1)^2(q^2-4q+7)$, *where* $1 \le i \le \alpha$ *and* $1 \le j \le \beta$.
6: $S_{A_i,A_j} = \frac{1}{16}(q-3)^2(q-5)^2 + 2\sum_{a\in A_i\setminus\{1,x_i\}} S_{a,x_j} + S_{x_i,x_j}$, *where* $1 \le i \ne j \le \alpha$ *and* $x_i \in A_i, x_j \in A_j$ *and* $x_i^2 = 1 = x_j^2$.

Proof. The proofs of part (1), (2) and (3) are similar, so we prove only part (2). Also, the proofs of part (4) and (5) are similar, so we prove only part (4).
The proof of part 2: Let $a \in P_i$ and $b \in B_j$. By part 4 of Lemma 3.1, $S_{a,b} = \frac{1}{2}(q-1)^2$. Since $S_{P_i,B_j} = \sum_{a\in P_i\setminus\{1\}} \sum_{b\in B_j\setminus\{1\}} S_{a,b}$, so $S_{P_i,B_j} = \sum_{a\in P_i\setminus\{1\}} \sum_{b\in B_j\setminus\{1\}} \frac{1}{2}(q-1)^2$. Hence $S_{P_i,B_j} = \frac{1}{4}(q-1)^4$.
The proof of part 4: Since A_j is a cyclic group of order even, so A_j has a unique element of order 2, namely x_j. We know

$$S_{P_i,A_j} = \sum_{a\in P_i\setminus\{1\}} \sum_{b\in A_j\setminus\{1\}} S_{a,b} = \sum_{a\in P_i\setminus\{1\}} (\sum_{b\in A_j\setminus\{1,x_j\}} S_{a,b} + S_{a,x_j}).\ (*)$$

By Lemma 3.1, we have

$$\sum_{b\in A_j\setminus\{1,x_j\}} S_{a,b} + S_{a,x_j} = \frac{1}{2}(q-1)(q-3)(\frac{1}{2}(q-1)-2) + (q-1)(q-2) = (q-1)[\frac{1}{4}(q-3)(q-5) + (q-2)].$$

Now, by replacing the above value in $(*)$ we have

$$S_{P_i,A_j} = \sum_{a\in P_i\setminus\{1\}} (q-1)[\frac{1}{4}(q-3)(q-5) + (q-2)] = \frac{1}{4}(q-1)^2(q^2-4q+7).$$

The proof of part 6: Since x_i and x_j are of order 2 and A_i, A_j are cyclic groups, so

$$S_{A_i,A_j} = \sum_{a\in A_i\backslash\{1\}} \sum_{b\in A_j\backslash\{1\}} S_{a,b} = \sum_{a\in A_i\backslash\{1,x_i\}} \sum_{b\in A_j\backslash\{1,x_j\}} S_{a,b} + \sum_{b\in A_j\backslash\{1,x_j\}} S_{x_i,b} + \sum_{a\in A_i\backslash\{1,x_j\}} S_{a,x_j} + S_{x_i,x_j}. (**)$$

We know that

1) It is clear that $\sum_{b\in A_j\backslash\{1,x_j\}} S_{x_i,b} = \sum_{a\in A_i\backslash\{1,x_j\}} S_{a,x_j}$

2) By using the part 5 of Lemma 3.1, we have

$$\sum_{a\in A_i\backslash\{1,x_i\}} \sum_{b\in A_j\backslash\{1,x_j\}} S_{a,b} = (|A_i|-2)(|A_j|-2)\frac{1}{4}(q-3)^2 = \frac{1}{16}(q-3)^2(q-5)^2.$$

By replacing the above values in relative $(**)$ the proof will be completed. □

Lemma 3.3. *If $G = PSL(2, q)$, where $q \equiv 1(mod4)$, then in the non-comutativie graph Γ_G, the number of edges which their two vertices are elements of order 2 is equal $\frac{1}{8}q(q-1)(q+1)^2$.*

Proof. We know that the number of elements of order 2 in G is equal to the number of conjugates of A in G which is $\frac{1}{2}q(q+1)$ and we call it α and let $t = \frac{1}{2}(q-1)$. Suppose that $a_i \in A_i$ is of order two and let $A_i = \langle y_i \rangle$, where $1 \leq i \leq \alpha$. Since $C_G(a_i) = N_G(\langle a_i \rangle)$ is a dihedral groups of order $q-1$, so there exists $x \in G$ of order 2 such that $x \notin A_i$ and $C_G(a_i) = \langle y_i, x\rangle$. The dihedral group $C_G(a_i)$ have $t+1$ elements of order 2 which a_i does not form an edge with them. Hence the number of elements of order 2 in G which they form edges with a_i is equal to $\alpha - t - 1$. Therefore the number of edges in $E(\Gamma_G)$ which their two vertices are of order two is $\alpha(\alpha - t - 1)$ and so the number of their distinct edges is $\frac{1}{2}\alpha(\alpha - t - 1)$. Now, by replacing the values of α and t the proof will be completed. □

Corollary 3.4. *If $G = PSL(2, q)$, where $q \equiv 1(mod4)$, then the number of a and b of order 2 which do not make an edge is equal to $\frac{1}{8}q(q-1)(q+1)$.*

Proof. The number of the elements of order two in G is α, so by the countalility axiom the number of a, b such that $a^2 = b^2 = 1$ is $\alpha(\alpha-1)$. Now, by the Lemma 3.3, the number of edges which their vertices are of order 2 is $\frac{1}{8}q(q-1)(q+1)^2$. It follows that the number of a, b such that $a^2 = b^2 = 1$ and a, b do not form an edge is $\alpha(\alpha-1) - \frac{1}{8}q(q-1)(q+1)^2 = \frac{1}{4}q(q-1)(q+1)$. Since we count each vertex twice, so the actual number of a, b such that $a^2 = b^2 = 1$ and a and b do not form an edge is $\frac{1}{8}q(q-1)(q+1)$. □

Lemma 3.5. *Under the assumptions of Lemma 3.3, we have:*

1: *The number of $ab \in E(\Gamma_G)$ such that $a^2 = b^2 = 1$ and $|C_G(a) \cap C_G(b)| = 2$ is equal $\frac{1}{16}q(q-1)(q+1)(q-5)$.*

2: *The number of* $ab \in E(\Gamma_G)$ *such that* $a^2 = b^2 = 1$ *and* $|C_G(a) \cap C_G(b)| = 1$ *is equal* $\frac{1}{16}q(q-1)(q+1)(q+7)$.

Proof. 1) Let a be an element of order 2 in G. As we saw in Lemma 2.2, $C_G(a)$ contains $t = \frac{1}{2}(q-1)$ elements of order 2 except a. So a commutes with these t elements. Thus a is in the centralizer of t elements of order 2 and the intersection of all these t centralizers contains a. The number of 2 by 2 intersections of these centralizers is equal to $t(t-1)$, so there are $t(t-1)$ edges ab such that $|C_G(a) \cap C_G(b)| \neq 1$, and since the number of elements of order 2 is equal to α so, by the countability axiom, the total number of distinct edges ab such that $a^2 = b^2 = 1$ and $|C_G(a) \cap C_G(b)| > 1$ is equal $\frac{1}{2}\alpha t(t-1)$. If a and b form an edge, then by Lemma 2.2 we have $|C_G(a) \cap C_G(b)| \leq 2$. Thus, if a and b do not form an edge, we have $|C_G(a) \cap C_G(b)| > 2$. But by Corollary 3.4 the number of a and b which do not form an edge is equal to $\frac{1}{8}q(q-1)(q+1)$. So, the number of edges ab such that $a^2 = b^2 = 1$ and $|C_G(a) \cap C_G(b)| = 2$ is equal to $\frac{1}{2}\alpha t(t-1) - \frac{1}{8}q(q-1)(q+1) = \frac{1}{16}q(q-1)(q+1)(q-5)$.
2) By Lemma 3.3, the total number of edges which their vertices are of order 2 is equal to $\frac{1}{8}q(q-1)(q+1)^2$. On the other hand, if $a^2 = b^2 = 1$ and $ab \neq ba$, then by Lemma 2 we have $|C_G(a) \cap C_G(b)| \leq 2$. Therefore, by part (1), the number of edges ab which $a^2 = b^2 = 1$ and $|C_G(a) \cap C_G(b)| = 1$ is $\frac{1}{8}q(q-1)(q+1)^2 - \frac{1}{16}q(q-1)(q+1)(q-5) = \frac{1}{16}q(q-1)(q+1)(q+7)$. □

Now by using the above lemmas we are ready to prove the Theorem 1.1.

Proof of theorem 1.1

First of all we suppose $q \equiv 1(mod\ 4)$. By Definition 2.5 and the part 4 of Proposition 2.1, we have

$$\begin{aligned} 2Sz(\Gamma_G) &= S_{G,G} = S_{\cup_{i=1}^{q+1} P_i, G} + S_{\cup_{i=1}^{\alpha} A_i, G} + S_{\cup_{i=1}^{\beta} B_i, G} \\ &= \sum_{i=1}^{q+1} S_{P_i, G \setminus P_i} + \sum_{i=1}^{\alpha} S_{A_i, G \setminus A_i} + \sum_{i=1}^{\beta} S_{B_i, G \setminus B_i}. \end{aligned} \tag{3.1}$$

Again, by part 4 of Proposition 2.1 and Corollary 3.2, we have

$$\begin{aligned} S_{P_i, G \setminus P_i} &= S_{P_i, \cup_{j=1}^{q+1} P_j \setminus P_i} + S_{P_i, \cup_{j=1}^{\alpha} A_j} + S_{P_i, \cup_{j=1}^{\beta} B_j} \\ &= \sum_{j=1, j \neq i}^{q+1} S_{P_i, P_j} + \sum_{j=1}^{\alpha} S_{P_i, A_j} + \sum_{j=1}^{\beta} S_{P_i, B_j} \\ &= q(q-1)^4 + \frac{1}{4}\alpha(q-1)^2(q^2-4q+7) + \frac{1}{4}\beta(q-1)^4 \\ &= \frac{1}{4}q(q-1)^2(q^3+q^2-5q+7). \end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned}
S_{B_i,G\setminus B_i} &= S_{B_i,\cup_{j=1}^{q+1}P_j} + S_{B_i,\cup_{j=1}^{\alpha}A_j} + S_{B_i,\cup_{j=1}^{\beta}B_j\setminus B_i} \\
&= \sum_{j=1,}^{q+1} S_{B_i,P_j} + \sum_{j=1}^{\alpha} S_{B_i,A_j} + \sum_{j=1,j\neq i}^{\beta} S_{B_i,B_j} \\
&= \frac{1}{4}(q+1)(q-1)^4 + \frac{1}{16}\alpha(q-1)^2(q^2-4q+7) + \frac{1}{16}(\beta-1)(q-1)^4 \\
&= \frac{1}{16}(q+1)(q-1)^2(q^3+-2q+3). \qquad (3.3)
\end{aligned}$$

$$\begin{aligned}
S_{A_i,G\setminus A_i} &= S_{A_i,\cup_{j=1}^{q+1}P_j} + S_{A_i,\cup_{j=1}^{\alpha}A_j\setminus A_i} + S_{A_i,\cup_{j=1}^{\beta}B_j} \\
&= \sum_{j=1,}^{q+1} S_{A_i,P_j} + \sum_{j=1,j\neq i}^{\alpha} S_{A_i,A_j} + \sum_{j=1}^{\beta} S_{A_i,B_j}. \qquad (3.4)
\end{aligned}$$

By part 4 and 5 Corollary 3.2, we have:

$$\sum_{j=1,}^{q+1} S_{A_i,P_j} = (q+1)[\frac{1}{4}(q-1)^2(q^2-4q+7)]$$

and

$$\sum_{j=1}^{\beta} S_{A_i,B_j} = \frac{1}{16}\beta(q-1)^2(q^2-4q+7).$$

But, by part 6 Corollary 3.2, we have

$$\begin{aligned}
\sum_{j=1,j\neq i}^{\alpha} S_{A_i,A_j} &= \sum_{j=1,j\neq i}^{\alpha} [\frac{1}{16}(q-3)^2(q-5)^2 + 2\sum_{a\in A_i\setminus\{1,x_i\}} S_{a,x_j} + S_{x_i,x_j}] \\
&= \frac{1}{16}(\alpha-1)(q-3)^2(q-5)^2 + 2\sum_{j=1,j\neq i}^{\alpha}\sum_{a\in A_i\setminus\{1,x_i\}} S_{a,x_j} + \sum_{j=1,j\neq i}^{\alpha} S_{x_i,x_j}, \qquad (3.5)
\end{aligned}$$

where $x_i \in A_i, x_j \in A_j$ and $x_i^2 = 1 = x_j^2$. Now by the proof of part 10 of Lemma 3.1,

$$\sum_{j=1,j\neq i}^{\alpha}\sum_{a\in A_i\setminus\{1,x_i\}} S_{a,x_j} = \frac{1}{4}t(q-3)(q-5)^2 + \frac{1}{4}(\alpha-t-1)(q-2)(q-3)(q-5),$$

where $t = \frac{1}{2}(q-1)$ and $\alpha = \frac{1}{2}q(q-1)$. It follows that

$$S_{A_i,G\setminus A_i} = \frac{1}{16}(q-1)(q^5-2q^4-9q^3+12q^2+116q-223) + \sum_{j=1,j\neq i}^{\alpha} S_{x_i,x_j}.$$

So

$$\sum_{i=1}^{\alpha} S_{A_i,G\setminus A_i} = \frac{1}{16}\alpha(q-1)(q^5-2q^4-9q^3+12q^2+116q-223) + \sum_{j=1,j\neq i}^{\alpha}\sum_{j=1,j\neq i}^{\alpha} S_{x_i,x_j}, \quad (\mathcal{I})$$

and finally that, by Lemma 3.3 and Corollary 3.4,

$$\sum_{j=1,j\neq i}^{\alpha}\sum_{j=1,j\neq i}^{\alpha} S_{x_i,x_j} = \frac{1}{8}q(q-1)(q+1)(q-5)(q-3)^2 + \frac{1}{8}q(q-1)(q+1)(q+7)(q-2)^2. \ (\mathcal{II})$$

Therefore, by replacing ($\mathcal{II}$) in ($\mathcal{I}$) and then replacing $\mathcal{I}$, (3.2)and (3.3) in (3.1), the proof for the case $q \equiv 1(mod\ 4)$ will be completed.
In the case $q \equiv 3(mod\ 4)$, considering the part 6 of the Proposition 2.1, if we change the role of the subgroup B by the subgroup A, then all the mentioned lemmas can be proved similarly. To avoid lengthy arguments, proofs are not presented here.

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