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# CONDITIONAL EXPECTATION IN THE KÔPKA'S $D$-POSETS

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Abstract. The notion of a $D$-poset was introduced in a connection with quantum mechanical models. In this paper, we introduce the conditional expectation of random variables on the Kôpka's $D$-Poset and prove the basic properties of conditional expectation on this structure.

## 1. Introduction

$D$-posets play an important role in quantum structures. They were described independently 25 years ago by Slovak authors as $D$-posets [5] and by American authors as effect algebras [3, 4]. Although the definitions of $D$-posets and effect algebras are formally different, actually their axioms and corresponding theories are equivalent. Very important examples of $D$-posets are MV-algebras with a

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corresponding sophisticated probability theory [9, 10]. The important tool in this theory is the notion of a product that was introduced independently by different authors with equivalent definitions, but from the distinct point of view. Therefore, it was of interest to follow $D$-posets with product [6, 7]. This notion is allowed to break new grounds in $D$-posets with promising probability applications. In [8], a new possibility for probability applications on a certain type of $D$-posets was shown. The main idea was to work only with a distribution function. Chovanec *et al.* [2] studied a new approach to a conditional probability in $D$-posets. They go into the inner structure of a conditional system which is a crucial notion for the existence of a conditional state. Samuel$\acute{\check{c}\acute{i}}$k and Holl$\acute{a}$ [11] studied the conditional probability on the K$\hat{o}$pka's $D$-posets. They only proved three basic properties for conditional probability on this structure. In this paper, we introduce the notion of the conditional expectation on the K$\hat{o}$pka's $D$-posets and prove the basic properties of conditional expectation for binary random variables. All results can be easily extended to $n$-ary random variables.

## 2. Preliminaries

**Definition 2.1.** A $D$-poset is an algebraic structure $D = (D, \leq, -, 0_D, 1_D)$ such that
1) $\leq$ is a partial ordering on $D$ with the least element $0_D$ and the greatest element $1_D$;
2) $- : D \times D \to D$ is a partial binary operation, where $b - a$ is defined if $a \leq b$ and there holds
i) $b - a \leq b$,
ii) $b - (b - a) = a$,
iii) $a \leq b \leq c \Longrightarrow c - b \leq c - a, (c - a) - (c - b) = b - a$.

**Definition 2.2.** Let an algebraic structure $D = (D, \leq, -, 0_D, 1_D)$ be a $D$-poset. The algebraic system $D = (D, \leq, -, *, 0_D, 1_D)$ is called the Kôpka's $D$-poset, if a binary operation $* : D \times D \to D$ is commutative, associative and there holds
i) $a * 1_D = a$,
ii) $a \leq b \Longrightarrow a * c \leq b * c,\ a, b, c \in D$,
iii) $a - (a * b) \leq 1_D - b,\ a, b, c \in D$.

**Definition 2.3.** A state on a Kôpka's $D$-poset $D$ is a mapping $m : D \to \langle 0, 1\rangle$ satisfying the following properties:
1) $m(0_D) = 0, m(1_D) = 1$,
2) if $a_n \uparrow a$, then $m(a_n) \uparrow m(a)$ for $a, a_n \in D$, $n = 1, 2, 3, ...$,
3) if $a_n \downarrow a$, then $m(a_n) \downarrow m(a)$ for $a, a_n \in D$, $n = 1, 2, 3, ....$

**Definition 2.4.** Suppose $J = \{(-\infty, t);\ t \in \mathbb{R}\}$. An observable on a Kôpka's $D$-poset $D$ is a mapping $x : J \to D$ satisfying the following conditions:
1) if $A_n \uparrow \mathbb{R}$, then $x(A_n) \uparrow 1_D$,
2) if $A_n \uparrow A$, then $x(A_n) \uparrow x(A)$ for $A_n \in J$, $n = 1, 2, 3, ...$,
3) if $A_n \downarrow \varphi$, then $x(A_n) \downarrow 0_D$ for $A_n \in J$, $n = 1, 2, 3, ....$

**Definition 2.5.** A $D$-additive state on a Kôpka's $D$-poset $D$ is a state $m : D \to \langle 0, 1\rangle$ with the following property:

$$\text{if } a \leq b, \text{ then } m(b) = m(b - a) + m(a), \; a, b \in D.$$

## 3. Conditional Probability

**Proposition 3.1.** [11] *Let $D$ be a $\widehat{Kopka}$'s $D$-poset, $m : D \to \langle 0, 1\rangle$ be a state and $x : J \to D$ be an observable. If $F : \mathbb{R} \to \langle 0, 1\rangle$ is the function defined as $F(t) = m(x((-\infty, t)))$, then*
*1) $F$ is non-decreasing,*
*2) $F$ is left continuous in any point $t_0 \in \mathbb{R}$,*
*3) $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to+\infty} F(t) = 1$,*
*4) there exists exactly one measure $\lambda_F : \mathcal{B}(\mathbb{R}) \to \langle 0, 1\rangle$ such that*

$$\lambda_F(\langle u, v\rangle) = F(v) - F(u) = m(x((-\infty, v)) - x((-\infty, u))).$$

**Definition 3.2.** A Kôpka's $D$-poset $D$ with the following property:

$$\text{if } k \leq l, \text{ then } a * (l - k) = a * l - a * k, \; k, l, a \in D,$$

is called a strong Kôpka's $D$-poset.

**Proposition 3.3.** [11] *Let $D$ be a strong Kôpka's $D$-poset, $m : D \to \langle 0, 1\rangle$ be a state, $x : J \to D$ be an observable and $a \in D$. Define $G : \mathbb{R} \to \langle 0, 1\rangle$ by the formula*

$$G(t) = m(a * x((-\infty, t))), \; a \neq 0_D.$$

*Then*
*1) $G$ is non-decreasing,*
*2) $G$ is left continuous in any point $t_1 \in \mathbb{R}$,*
*3) $\lim_{t\to-\infty} G(t) = 0$ and $\lim_{t\to+\infty} G(t) = 1$,*
*4) there exists exactly one measure $\lambda_G : \mathcal{B}(\mathbb{R}) \to \langle 0, 1\rangle$ such that*

$$\lambda_G(\langle u, v\rangle) = G(v) - G(u).$$

**Theorem 3.4.** [11] *Let $D$ be a strong Kôpka's $D$-poset, $m : D \to \langle 0, 1\rangle$ be a $D$-additive state and $x : J \to D$ be an observable. Then there exists a function $p(a|x) : D \to \mathbb{R}$, such that*

$$\int_{(-\infty,t)} p(a|x) d\lambda_F = m(a * x((-\infty, t))).$$

*The function $p(a|x)$ is a version of conditional probability on the strong Kôpka's $D$-poset such that there almost everywhere holds:*
*1) $p(0_D|x) = 0, \; p(1_D|x) = 1$,*
*2) $0 \leq p(a|x) \leq 1$ for any $a \in D$,*
*3) if $a_n \uparrow a$, then $p(a_n|x) \uparrow p(a|x)$ for $a, a_n \in D$, $n = 1, 2, 3, ....$*

For every family $\mathcal{M}$, there exists a unique $\sigma$-algebra $\sigma(\mathcal{M})$, the $\sigma$-algebra generated by $\mathcal{M}$, such that

a) $\mathcal{M} \subseteq \sigma(\mathcal{M})$,

b) for any $\sigma$-algebra $\mathcal{H}$ with $\mathcal{M} \subseteq \mathcal{H}$, $\sigma(\mathcal{M}) \subseteq \mathcal{H}$ [1].

In this section, we first describe the following definitions:

**Definition 4.1.** Let $D$ be a Kôpka's $D$-poset, $m : D \to \langle 0, 1\rangle$ be a state, $x : J \to D$ be an observable. A Kôpka's $D$-poset $D$ with the following properties:

1) $m((a * b) * x(B)) = m(a * x(B)) = m(b * x(B)),\ a, b \in D, B \in \sigma(J)$,

2) $m((a \vee b) * x(B)) = m(a * x(B)) + m(b * x(B))$,

is called an $m$-strong K$\widehat{o}$pka's $D$-poset.

As mentioned earlier, the important tool in this theory is the notion of a product. In passing, we introduce a new version of product on $D$-poset $D$ similar to [6]. Let $\otimes : D \times \mathbb{R} \to \mathbb{R}$ be an operation with the following conditions:

1) $1_D \otimes r = r$, $r \in \mathbb{R}$,

2) $(a * b) \otimes r = a \otimes (b \otimes r),\ a, b \in D, r \in \mathbb{R}$.

3) $a$ and $r$ are commutative with respect to $\lambda_F$. i.e.,

$$a \otimes \int_B r\ d\lambda_F = \int_B (a \otimes r) d\lambda_F,\ a \in D, r \in \mathbb{R}, B \in \sigma(J).$$

**Definition 4.2.** A random variable $Y$ on a K$\widehat{o}$pka's $D$-poset $D$ is a function $Y : D \to \mathbb{R}$ such that $Y^{-1}(C) \in D$ for every $C \in \mathcal{B}(\mathbb{R})$.

**Theorem 4.3.** *Let $D$ be a strong K$\widehat{o}$pka's $D$-poset, $m : D \to \langle 0, 1\rangle$ be a state and $x : J \to D$ be an observable. Let $Y$ be a binary random variable with support $S_Y = \{a_1, a_2\}$, where $a_1, a_2 \in D$. Then there exists a function $\mathbb{E}(Y|x) : D \to \mathbb{R}$ such that*

$$\int_{(-\infty,t)} \mathbb{E}(Y|x) d\lambda_F = (a_1 \otimes m(a_1 * x((-\infty, t)))) + (a_2 \otimes m(a_2 * x((-\infty, t)))).$$

*The function $\mathbb{E}(Y|x)$ is a version of conditional expectation on the strong K$\widehat{o}$pka's $D$-poset such that $\mathbb{E}(1_D|x) = 1$ (a.e.) and $\mathbb{E}(0_D|x) = 0$ (a.e.).*

*Proof.* We have $\lambda_G(B) \leq \lambda_F(B)$ for every $B \in \sigma(J)$. Assume $\lambda_F(B) = 0$, then there holds

$$0 \leq m(a_1 * x(B)) = m(a_2 * x(B)) = \lambda_G(B) \leq \lambda_F(B) = 0.$$

Thus

$$0 \leq (a_1 \otimes m(a_1 * x(B))) + (a_2 \otimes m(a_2 * x(B))) \leq 0$$

and according to Radon-Nikodym theorem [1] there exists function $f$ such that

$$\lambda_G(B) = \int_B f d\lambda_F.$$

For every $B \in \sigma(J)$,

$$\begin{aligned}
\int_B \mathbb{E}(1_D|x) d\lambda_F &= 1_D \otimes m(1_D * x(B)) \\
&= m(x(B)) \\
&= \int_B p(1_D|x) d\lambda_F \\
&= \int_B 1 d\lambda_F.
\end{aligned}$$

Also

$$\int_B \mathbb{E}(0_D|x) d\lambda_F = 0_D \otimes m(0_D * x(B)) = 0.$$

□

**Theorem 4.4.** *Let $D$ be an $m$-strong Kôpka's $D$-poset and $x : J \to D$ be an observable. Let $Y_1$ be a binary random variable with support $S_{Y_1} = \{a_1, a_2\}$, where $a_1, a_2 \in D$ and $Y_2$ be a binary random variable with support $S_{Y_2} = \{b_1, b_2\}$, where $b_1, b_2 \in D$ and $a_i \leq b_i$ for $i = 1, 2$. Then*
*1) $\mathbb{E}(Y_1|x) \leq \mathbb{E}(Y_2|x)$ (a.e.)*
*2) $0 \leq \mathbb{E}(Y|x) \leq 1$ (a.e.).*

*Proof.* 1) For every $B \in \sigma(J)$,

$$\begin{aligned}
\int_B \mathbb{E}(Y_1|x) d\lambda_F &= (a_1 \otimes m(a_1 * x(B))) + (a_2 \otimes m(a_2 * x(B))) \\
&\leq (b_1 \otimes m(b_1 * x(B))) + (b_2 \otimes m(b_2 * x(B))) \\
&= \int_B \mathbb{E}(Y_2|x) d\lambda_F.
\end{aligned}$$

2) We have

$$\begin{aligned}
0 = 0_D \otimes m(0_D * x(\varphi)) &\leq (a_1 \otimes m(a_1 * x(B))) + (a_2 \otimes m(a_2 * x(B))) \\
&= \int_B \mathbb{E}(Y|x) d\lambda_F \\
&\leq \int_{\mathbb{R}} \mathbb{E}(1_D|x) d\lambda_F \\
&= 1.
\end{aligned}$$

□

**Theorem 4.5.** *Let $D$ be the $m$-strong Kôpka's D-poset and $d \in D$. If $Y$ is a random variable and $x : J \to D$ be an observable, then*

$$\mathbb{E}(Y * d|x) = d \otimes \mathbb{E}(Y|x) \ (a.e.).$$

*Proof.* By definition 4.1,

$$\begin{aligned} \int_B \mathbb{E}(Y * d|x) d\lambda_F &= (a_1 * d) \otimes m((a_1 * d) * x(B)) \\ &+ (a_2 * d) \otimes m((a_2 * d) * x(B)) \\ &= \int_B d \otimes \mathbb{E}(Y|x) d\lambda_F. \end{aligned}$$

□

**Theorem 4.6.** *Let $D$ be an $m$-strong Kôpka's D-poset. Let $(Y_n)_{n \geq 1}$ be a sequence of random variables with support $S_{Y_n} = \{a_{1n}, a_{2n}\}$, where $a_{1n}, a_{2n} \in D$ $(n = 1, 2, ...)$ and $Y$ be a random variable with support $S_Y = \{a_1, a_2\}$, where $a_1, a_2 \in D$. If $a_{in} \uparrow a_i$ for each $i \in \{1, 2\}$, then $\mathbb{E}(Y_n|x) \uparrow \mathbb{E}(Y|x)$ (a.e.)*

*Proof.* This is an immediate consequence of the monotone convergence theorem (MCT) and the definition of a state $m$ [1]. □

**Theorem 4.7.** *Let $D$ be a $m$-strong Kôpka's D-poset. If $Y = d$ (a.e.) and there exists a real number $r$ such that*

$$d \otimes \int_B 1 d\lambda_F = \int_B r d\lambda_F \ (a.e.), \ for \ d \in D,$$

*then $\mathbb{E}(Y|x) = r$ (a.e.). Also, if*

$$d \otimes \int_B 1 d\lambda_F \leq \int_B r d\lambda_F \ (a.e.), \ for \ d \in D,$$

*then $\mathbb{E}(Y|x) \leq r$ (a.e.)*

*Proof.* For every $B \in \sigma(J)$,

$$\begin{aligned} \int_B \mathbb{E}(Y|x) d\lambda_F &= d \otimes m(d * x(B))) \\ &= d \otimes m(x(B))) \\ &= d \otimes \int_B 1 d\lambda_F \\ &= \int_B r d\lambda_F. \end{aligned}$$

Also

$$\begin{aligned}\int_B \mathbb{E}(Y|x)d\lambda_F &\leq d\otimes m(d*x(B))) + d\otimes m(d*x(B)))\\ &= d\otimes m(1_D * x(B)))\\ &\leq \int_B r d\lambda_F.\end{aligned}$$

□

**Example 4.8.** Consider the classical Kolmogorov probability space $(\Omega, S, P)$. Let

$$F = \{\mu_A : \Omega \to \langle 0,1\rangle,\ \mu_A \ is\ S - measurable\},$$

and define a partial operation on $F$as

$$\mu_A \leq \mu_B \Longleftrightarrow \mu_A(w) \leq \mu_B(w),\ \forall w \in \Omega,$$

where the least element is $0_\Omega$ and the greatest element is constant function $1_\Omega$. Also, define two partial binary operations "$-$" and "$*$" as

$$- : F \times F \to F,\ if\, \mu_A \leq \mu_B,\ then\ (\mu_B - \mu_A)(w) = \mu_B(w) - \mu_A(w)$$

and

$$* : F \times F \to F,\ \mu_A(w) * \mu_B(w) = \mu_B(w).\mu_A(w).$$

Now, using the scalar product

$$\otimes : F \times \mathbb{R} \to \mathbb{R},\ \mu_A(w) \otimes r = r\mu_A(w),$$

the algebraic structure $F = (F, \leq, -, *, 0_\Omega, 1_\Omega)$ becomes a Kôpka's $D$-poset. Note that, on this structure we have

$$\mathbb{E}(rY|x) = r\mathbb{E}(Y|x)\ (a.e.).$$

**Example 4.9.** MV-algebras are an examples of $D$-posets. We can obtain similar results by defining the appropriate operations on this structures (see [2]).

All of the mentioned results can be extended using a generalization of Theorem 4.3 as follows.

**Theorem 4.10.** *Let $D$ be a strong Kôpka's $D$-poset, $m : D \to \langle 0,1\rangle$ be a state and $x : J \to D$ be an observable. Let $Y$ be a $n$-ary random variable with support $S_Y = \{a_1, a_2, ..., a_n\}$, where $a_1, a_2, ..., a_n \in D$. Then there exists a function $\mathbb{E}(Y|x) : D \to \mathbb{R}$ such that*

$$\int_{(-\infty,t)} \mathbb{E}(Y|x)d\lambda_F = \sum_{i=1}^{n} a_i \otimes m(a_i * x((-\infty,t))).$$

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