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Research Paper

ON $CLESS$-MODULES AND MODULES OF BOUNDED ORDER

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Abstract. Let R be a commutative ring with identity. In this paper, we discuss about $CLESS$-modules and modules of bounded order. It is shown that if M is a $CLESS$-module then every $\mathscr{S}$-closed submodule K in M, with $Soc\, M \leq_e K$, is CLS and a direct summand of M; Furthermore, the converse is true if M is nonsingular. Also, we prove that every module of bounded order is a $CLESS$-module. Finally, we show that if M is a finitely generated module over nonsingular ring R and T is a complement to $Z(M)$ in M, then M/T is of bounded order.

1. Introduction

Throughout this paper, R is a commutative ring with identity and all modules are unitary. Let us give some basic notation and set the terminology. We shall denote the fact that a submodule N is essential in a module M by $N \leq_e M$. The socle and the singular submodule of a module M will be denoted by $Soc\, M$ and $Z(M) = \{m \in M \mid \text{ann}_R(m) \leq_e R\}$ respectively.

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In fact, $Z(M) = cl\,0$. Note that the closure of a submodule N of M, denoted by $cl\,N$ is defined by

$$cl\,N = \{m \in M | mI \subseteq N; I \text{ is an essential ideal of } R\}.$$

Also, the second singular submodule of M, denoted by $Z_2(M)$, is defined by $Z_2(M) = cl\,cl\,0$[9]. Also, by [14, 7.23] $Z(M/Z(M)) = Z_2(M)/Z(M)$. A module M is called singular (respectively nonsingular) if $Z(M) = M$ (respectively $Z(M) = 0$). If R is nonsingular, $Z(M) = Z_2(M)$ by [9, Proposition 2.3].

A submodule K of M is called closed if it has no proper essential extension in M. A submodule K of M is said to be a complement to a submodule S of M if $K \cap S = 0$ and K is maximal with respect to this property; in this case K is called a complement submodule of M. By Zorn's Lemma, any submodule S has a complement. It is well known (and easy to prove) that a submodule K of M is closed if and only if K is complement (see, for example, [7, 1.10]).

Recall from [10], [20] and [8] that a submodule N of a module M is said to be $\mathscr{S}$-closed if M/N is nonsingular (In [19], Tercan and Yücel call $\mathscr{S}$-closed by "z-closed"). Every $\mathscr{S}$-closed submodule is a complement, and every complement of a non-singular module is a $\mathscr{S}$-closed submodule [16, Lemma 2.3]. $CLESS$-modules was introduced by S. Crivei and S. Sahinkaya [6], which is a generalization of $CESS$-modules and CLS-modules, introduced by P. F. Smith [17] and A. Tercan [18], respectively. A module M is called a $CLESS$-module if every $\mathscr{S}$-closed submodule of M with essential socle is a direct summand of M, a CS-module (or extending module) if every complement submodule of M is a direct summand of M, a $CESS$-module if every complement submodule of M with essential socle is a direct summand of M, and a CLS-module if every $\mathscr{S}$-closed submodule of M is a direct summand of M. In section 2, we prove that if M is a $CLESS$-module then every $\mathscr{S}$-closed submodule K in M, with $Soc\,M \leq_e K$, is CLS and a direct summand of M. The converse is true if M is nonsingular. As a consequence, if M is a nonsingular $CLESS$-module, then $M = M_1 \oplus M_2$ for some CLS-module M_1 with essential socle and module M_2 with zero socle.

An R-module M is called of bounded order if there exists an essential ideal I of R such that MI=0 [3]. In section 3, we show that every module of bounded order is a $CLESS$-module and we construct an example where a $CLESS$-module is not of bounded order.

In section 4, we show that if M is a finitely generated module over nonsingular ring R and T is a complement to $Z(M)$ in M, then M/T is of bounded order.

An important question for a module over a commutative ring is: when does it split, in the sense that its singular submodule is a direct summand? Some classical results show that a commutative domain is a: (i) field if and only if every module splits (see J. Rotman [15]); (ii) Dedekind domain if and only if every module, whose singular submodule is of bounded order, splits. (see S.U. Chase [4] and I. Kaplansky [13]). Over a Dedekind domain, CLESS-modules

coincide with split modules (see [6, Corollary 4.2]). Recall from [3], R has BSP if every R-module, whose singular submodule is of bounded order, splits. In section 5, we show that if R is a regular ring and R has BSP then R/I is a hereditary ring for every ideal I of R. This generalizes [3, Proposition 1.13].

The dual notion of CS-modules are lifting modules. A module M is called lifting if every submodule N of M lies over a direct summand of M, that is, N contains a direct summand X of M such that N/X is small in M/X. The readers are encouraged to contextualize their work alongside this dual notions (see [5, 22.2] and [1]).

2. On $CLESS$-modules

Recall that a module M is called a $CLESS$-module if every $\mathscr{S}$-closed submodule of M with essential socle is a direct summand of M. Our next result gives some information about $CLESS$-modules parallel to [17, Proposition 1.5], which says that a module M is $CESS$ if and only if every closed submodule K in M, with $Soc\,M \leq_e K$, is CS and a direct summand of M.

Theorem 2.1. *Let M be a $CLESS$-module. Then every $\mathscr{S}$-closed submodule K in M, with $Soc\,M \leq_e K$, is CLS and a direct summand of M. The converse is true if M is nonsingular.*

Proof. Let K be a $\mathscr{S}$-closed submodule of M, with $Soc\,M \leq_e K$. We have $Soc\,M = Soc\,Soc\,M \leq Soc\,K \leq K$. Now $Soc\,M \leq_e K$ implies that $Soc\,K \leq_e K$. Hence K is a $\mathscr{S}$-closed submodule of M with essential socle, so by hypothesis K is a direct summand of M. On the other hand, since M is a $CLESS$-module and K is a $\mathscr{S}$-closed submodule of M, so by [6, Lemma 2.5 (i)] K is a $CLESS$-module. Furthermore, since K is a $CLESS$-module with essential socle, so by [6, Remark 2.2(1)] K is a CLS-module.

Now let M be nonsingular and suppose that every $\mathscr{S}$-closed submodule K in M, with $Soc\,M \leq_e K$, is CLS and a direct summand of M. Since M is nonsingular, every submodule of M is nonsingular. Then:

(a) By [16, Lemma 2.3], a submodule K of M is $\mathscr{S}$-closed if and only if it is a complement in M.

(b) By [18, Corollary 5], a nonsingular module K is CLS if and only if it is CS.

Hence the hypothesis implies that every complement K of M with $Soc\,M \leq_e K$ is CS and a direct summand of M. So M is $CESS$ by [17, Proposition 1.5], and therefore $CLESS$. It may also be worth remarking that $CESS \Rightarrow CLESS$ holds in general, since every $\mathscr{S}$-closed submodule is a complement. □

Let $N \leq M$. By Zorn's Lemma, the family of essential extensions of N incide in M, has a maximal member, say K. Such K is closed in M and it is called an essential closure of N (in M).

Corollary 2.2. *Let M be a nonsingular $CLESS$-module. Then $M = M_1 \oplus M_2$ for some CLS-module M_1 with essential socle and module M_2 with zero socle.*

Proof. Let M_1 be an essential closure of $Soc\, M$ in M. Hence $Soc\, M \leqq_e M_1$ and M_1 is closed in M. We have $Soc\, M = Soc\, Soc\, M \leq Soc\, M_1 \leq M_1$. Now $Soc\, M \leq_e M_1$ implies that $Soc\, M_1 \leq_e M_1$. On the other hand, since M is nonsingular and M_1 is closed in M, so M_1 is $\mathscr{S}$-closed in M. It follows from Theorem 2.1 that M_1 is CLS and a direct summand of M. This implies that $M = M_1 \oplus M_2$ for some submodule M_2 with zero socle. □

By [6, Example 3.1], the class of $CLESS$-modules is not closed under direct summands. Our next result gives some information about direct summand of a $CLESS$-module.

Proposition 2.3. *Let $M = M_1 \oplus M_2$ be a $CLESS$-module and M_2 be nonsingular. Then M_1 is $CLESS$. In particular, any direct summand of a nonsingular $CLESS$-module is $CLESS$.*

Proof. We have $Z(M/M_1) \cong Z(M_2) = 0$. Then M_1 is $\mathscr{S}$-closed in M. Hence by [6, Lemma 2.5 (i)] M_1 is a $CLESS$-module. The second part follows by previous part. □

3. Relationship between $CLESS$-modules and modules of bounded order

We say that an R-module M is of bounded order if there exists an essential ideal I of R such that MI=0 [3]. A module annihilated by an essential ideal is the most direct way to construct an example of a module that is of bounded order from the definition. The following result shows that every module of bounded order is $CLESS$.

Theorem 3.1. *Every module of bounded order is a $CLESS$-module.*

Proof. Let M be an R-module of bounded order. First,we claim that for every submodule $L \subseteq M$, $Z(M/L) = M/L$ that is the quotiant module M/L is singular. Since M is of bounded order, there exists an essential ideal I of R such that MI=0. Let $L \subseteq M$ be a submodule. Hence the quotient module M/L is also annihilated by I, i.e. $(M/L)I = 0$. Let $m + L \in M/L$ be arbitrary. Therefore, we have $I \subseteq \mathrm{Ann_R}\,(\mathrm{m} + \mathrm{L}) \subseteq \mathrm{R}$. Since I is essential in R, it follows that $\mathrm{Ann_R}\,(\mathrm{m}+\mathrm{L})$ is essential in R and thus $m+L \in Z(M/L)$. Hence we have proved that the quotient module M/L is singular. Now, let K be a $\mathscr{S}$-closed submodule of M, with essential socle. Inasmuch as, M is of bounded order, so the quotiant module M/K is singular by the

above. Hence, $M/K = Z(M/K) = 0$. This implies that $K = M$ and clearly K is a direct summand of M. Consequently, M is a $CLESS$-module. □

In the following, we construct an example where a $CLESS$-module is not of bounded order that is the converse of the above theorem is false.

Example 3.2. Let's construct a $CLESS$-module that is not of bounded order. We Choose the Ring $R = \mathbb{Z}$ and the Module $M = \mathbb{Q}$. Since M is torsion-free over $R = \mathbb{Z}$, $\operatorname{Soc} \mathrm{M} = 0$, so no nonzero submodule of M has essential socle. Thus the only $\mathscr{S}$-closed submodule of M with essential socle is 0, which is clearly a direct summand. Hence M is $CLESS$. For any two ideals $0 \neq \langle n \rangle = I$ and $0 \neq \langle m \rangle = J$ of $\mathbb{Z}$, we have $I \cap J = \langle [n, m] \rangle \neq 0$, where $[n, m]$ is the least common multiple of m, n. Therefore, the essential ideals of $\mathbb{Z}$ are the nonzero ideals. If M is of bounded order, there must exist a nonzero essential ideal I of $R = \mathbb{Z}$ such that $MI = 0$. That is, every rational number must be annihilated by some nonzero integer. However, in $\mathbb{Q}$, no nonzero integer annihilates all of $\mathbb{Q}$. In fact, for any $0 \neq n \in \mathbb{Z}$, there exists $q \in \mathbb{Q}$ such that $nq \neq 0$ (e.g., take $q = 1/n$). Hence M is not of bounded order. Thus, this provides an example where a $CLESS$ module is not of bounded order.

4. On modules of bounded order and singular submodule

Standard references for modules of bounded order are [3] and [11]. We begin with the following proposition:

Proposition 4.1. *Every module of bounded order is singular.*

Proof. Let M be an R-module of bounded order. Thus there exists an essential ideal I of R such that MI=0. Let $m \in M$ be arbitrary. Therefore, we have $I \subseteq \operatorname{Ann_R} \mathrm{m} \subseteq \mathrm{R}$. Since I is essential in R, it follows that $\operatorname{Ann_R} \mathrm{m}$ is essential in R and thus $m \in Z(M)$. Hence we have proved that M is singular. □

Now, we prove the converse of the previous proposition for finitely generated modules.

Proposition 4.2. *Every finitely generated singular module is of bounded order.*

Proof. Let M be a finitely generated singular R-module. Then $M = Z(M)$ and $M = Rm_1 + \cdots Rm_n$ for some $m_1, \cdots, m_n \in M$. Since $m_i \in Z(M)$, $\operatorname{ann} m_i \leq_e R$ for all i. Then $\cap_{i=1}^{n} \operatorname{ann} m_i \leq_e R$. Now by putting $I = \cap_{i=1}^{n} \operatorname{ann} m_i$, we have $MI = 0$ and $I \leq_e R$, as required. □

Corollary 4.3. *Let M be an R-module.*

(*i*) *If M is Noetherian then $Z(M)$ is of bounded order,*

(*ii*) *If M is finitely generated and $Z_2(M) = M$ then $M/Z(M)$ is of bounded order.*

Proof. (*i*) Since M is Noetherian, so $Z(M)$ is a finitely generated singular module, and so by Proposition 4.2, $Z(M)$ is of bounded order.

(*ii*) We have $Z(M/Z(M)) = Z_2(M)/Z(M) = M/Z(M)$. It follows that $M/Z(M)$ is a finitely generated singular module. Thus $M/Z(M)$ is of bounded order by Proposition 4.2. □

Theorem 4.4. *Let M be an R-module and T be a complement to $Z(M)$ in M.*

(*i*) *$Z_2(M/T) = M/T$.*

(*ii*) *If M is finitely generated, then $(M/T)/Z(M/T)$ is of bounded order.*

(*iii*) *If $Z(R) = 0$, then M/T is singular.*

(*iv*) *If M is finitely generated and $Z(R) = 0$, then M/T is of bounded order.*

Proof. (*i*) Since T is a complement to $Z(M)$, $T \cap Z(M) = 0$ and $T \oplus Z(M) \leq_e M$. Hence $T \oplus Z(M)/T \leq_e M/T$ and since $T \oplus Z(M)/T \subseteq Z(M/T) \subseteq M/T$, so $Z(M/T) \leq_e M/T$. It follows that $(M/T)/Z(M/T)$ is singular. Now we have that

$$(M/T)/Z(M/T) = Z((M/T)/Z(M/T)) = Z_2(M/T)/Z(M/T),$$

which implies that $Z_2(M/T) = M/T$.

(*ii*) Since T is a complement to $Z(M)$, $Z_2(M/T) = M/T$ by part (*i*). Now since M/T is finitely generated and $Z_2(M/T) = M/T$, $(M/T)/Z(M/T)$ is of bounded order by Corollary 4.3(ii).

(*iii*) Put $N = M/T$. Since T is a complement to $Z(M)$, $Z_2(N) = N$ by (*i*). $Z(R) = 0$ means that R is nonsingular. Hence $Z_2(N) = Z(N)$, by [9, Proposition 2.3]. It follows that $Z(N) = N$ and thus N is singular.

(*iv*) Since T is a complement to $Z(M)$ and $Z(R) = 0$, M/T is singular by (*iii*). Now since M/T is a singular finitely generated module, so M/T is of bounded order by Proposition 4.2. □

An immediate consequence of Theorem 4.4, is the following corollary, which is the famous results [3, Proposition 1.6].

Corollary 4.5. *Let M be a finitely generated R-module and $Z(R) = 0$. Then $Z(M)$ is of bounded order*

Proof. Let T be a complement to $Z(M)$ in M. By Theorem 4.4(*iv*), M/T is of bounded order. Hence there exists an essential ideal J of R such that $(M/T)J = 0$. Hence $Z(M)J \subseteq MJ \subseteq T$.

It follows that $Z(M)J \subseteq Z(M) \cap T = 0$ and thus $Z(M)J = 0$. Therefore, $Z(M)$ is of bounded order. □

5. On splitting property

As in [3] and [20], we say that an R-module M splits if $Z(M)$ is a direct summand of M; R has splitting property (SP) if every R-module splits.

Remark 5.1. If R is a commutative domain, then $Z(M) = t(M)$, where $t(M)$ is the torsion submodule of M. Motivated by [3], If R is a Prufer domain then every finitely generated R-module splits. We prove this fact by [2, Chapt. VII, Proposition 3.2] and [2, Chapt. I, Proposition 6.2]. For, if R is a Prufer domain and M is a finitely generated R-module, then $M/t(M)$ is a finitely generated torsion-free R-module, so by [2, Chapt. VII, Proposition 3.2], $M/t(M)$ admits a monomorphism into a free R-module F with a finite base. Now since R is semi-hereditary, so by [2, Chapt. I, Proposition 6.2], $M/t(M)$ is projective and so the sequence $0 \to t(M) \to M \to M/t(M) \to 0$ splits, which implies that $Z(M) = t(M)$ is a direct summand of M. Conversely, Kaplansky [12] has shown that if every finitely generated R-module splits then R is a Prufer domain.

Proposition 5.2. *Let R be a ring. Then*

(*i*) *$Z(R)$ has no nonzero idempotent;*

(*ii*) *If R splits, then $Z(R) = 0$.*

Proof. (*i*) Let $e \in Z(R)$ be an idempotent. We have $\operatorname{ann} e \cap eR = 0$. For,let $t \in \operatorname{ann} e \cap eR$. It follows that $et = 0$ and $t = ez$ for some $z \in R$. Hence $t = ez = e^2z = et = 0$, as desired. Since $\operatorname{ann} e \leq_e R$, $eR = 0$ and so $e = 0$.

(*ii*) Since R splits, so $Z(R)$ is a direct summand of R. Then $R = Z(R) \oplus I$ for some ideal I of R. This implies that $1 = y + z$ sor some $y \in Z(R)$ and $z \in I$. Now $y = y^2 + yz$ and so $y - y^2 = yz \in Z(R) \cap I = 0$. It follows that y is idempotent and y=0 by (*i*). Therefore, $I = R$ and $Z(R) = 0$. □

Recall from [3], R has BSP if every R-module, whose singular submodule is of bounded order, splits. The following proposition generalizes [3, Proposition 1.13].

Proposition 5.3. *Let R be a regular ring. If R has BSP and I is an ideal of R, then R/I is a hereditary ring.*

Proof. Let R has BSP. Since R is regular, $BSP \Rightarrow SP$ over R by [3, Theorem 2.1]; SP passes to homomorphic images by [3, Remark, p.161]; hence by the end part of [3, Theorem 2.1], R/I

is a hereditary ring. Note that this argument avoids the pull-back of bounded-order modules from R/I to R. □

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