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Research Paper

ISAACS-NAVARRO-WOLF CONJECTURE FOR CERTAIN GROUPS

SAJJAD MAHMOOD ROBATI*

Abstract. Let G be a finite group and let $\mathrm{F}_i(G)$ be the i-th term of the ascending Fitting series of G. In this paper, we show that if x is a non-vanishing element of non-nilpotent solvable group G and $G/\mathrm{F}_i(G)$ is supersolvable, then $x \in \mathrm{F}_i(G)$. Moreover, we prove that if G is an extension of a supersolvable group by a nilpotent group and x is a non-vanishing element, then x lies in the Fitting subgroup $\mathrm{F}(G)$, which is an affirmative answer to the open conjecture stated in [6] by Isaacs, Navarro, and Wolf.

1. Introduction

Let G be a finite group and $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. An element $x \in G$ is called a vanishing element if $\chi(x) = 0$ for some $\chi \in \mathrm{Irr}(G)$. Otherwise, x is called a non-vanishing element. We denote the set of vanishing elements of G by $\mathrm{Van}(G)$. The algebraic structure of a finite group is strongly influenced by its vanishing and non-vanishing elements (see [1, 2, 3, 4, 7, 8]).

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*Corresponding author

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A well-known open conjecture in 1999 by Isaacs, Navarro, and Wolf in [6] state that every non-vanishing element of a solvable group lies in the Fitting subgroup $\mathrm{F}(G)$. While this conjecture remains open in general, in Theorem B of [6], the authors proved that it holds true for supersolvable groups. In fact, they showed that all non-vanishing elements of a supersolvable group are contained in the center of the Fitting subgroup $\mathrm{Z}(\mathrm{F}(G))$.

To approach the general case for solvable groups, they investigated how non-vanishing elements interact with the ascending Fitting series of a group, leading to the following fundamental theorem:

Theorem 1.1 ([6], Theorem D)**.** *Let $\mathrm{F}_i(G)$ be the i-th term of the ascending Fitting series of a non-nilpotent solvable group G. If x is a non-vanishing element of G and $G/\mathrm{F}_i(G)$ is nilpotent, then $x \in \mathrm{F}_i(G)$, where $\mathrm{F}_i(G)$ denotes the i-th term of the ascending Fitting series of G.*

In other words, this theorem tells us that each non-vanishing element of a non-nilpotent solvable group lies in the penultimate term of the ascending Fitting series. Motivated by this result, we generalize Theorem 1.1 as the primary objective of this paper.

Investigating the relationship between vanishing properties of characters and the Fitting length of a group is a classical area of research. For instance, Dolfi et al. [3] studied how the vanishing prime graph bounds the Fitting length of solvable groups. Furthermore, the conjecture in [6] structurally asserts that the subgroup generated by all non-vanishing elements must have a Fitting length of exactly 1. Since this remains open, finding upper bounds for the Fitting length of this subgroup under weakened conditions is a significant step forward. In Section 2, we provide an upper bound for the Fitting length of the subgroup generated by non-vanishing elements.

2. Supersolvable Quotients

We begin by establishing some necessary tools. It is a straightforward observation from character theory that if the image of an element in a quotient group is vanishing, then the element itself is vanishing in the original group. Specifically, if $gN \in \mathrm{Van}(G/N)$, then $gN \subseteq \mathrm{Van}(G)$.

The following lemma by Dolfi et al. provides a crucial link between abelian quotients and vanishing elements.

Lemma 2.1. [3, Lemma 2.6] *Let G be a solvable group, and N be a normal subgroup of G. If $N/\mathrm{F}(N)$ is abelian, then $N \setminus \mathrm{F}(N) \subseteq \mathrm{Van}(G)$.*

We generalize this lemma to the higher terms of the ascending Fitting series. This proposition will be the main role for proving our results.

Proposition 2.2. *Let* $\mathrm{F}_i(G)$ *be the* i*-th term of the ascending Fitting series of a non-nilpotent solvable group* G*. If* $N/\mathrm{F}_i(G)$ *is an abelian normal subgroup of* $G/\mathrm{F}_i(G)$*, then* $N \setminus \mathrm{F}_i(G) \subseteq \mathrm{Van}(G)$*.*

Proof. We proceed by induction on i. For $i = 1$, $\mathrm{F}(N)$ is a nilpotent characteristic subgroup of the normal subgroup N, implying $\mathrm{F}(N) = \mathrm{F}(G)$. Since $N/\mathrm{F}(N)$ is abelian, Lemma 2.1 yields $N \setminus \mathrm{F}(G) \subseteq \mathrm{Van}(G)$.

Assume the statement holds for $i = n$ and $N/\mathrm{F}_{n+1}(G)$ is an abelian normal subgroup of $G/\mathrm{F}_{n+1}(G)$. Using the third isomorphism theorem, we can write:

$$\frac{N/\mathrm{F}_n(G)}{\mathrm{F}_{n+1}(G)/\mathrm{F}_n(G)} \cong N/\mathrm{F}_{n+1}(G).$$

By the definition of the ascending Fitting series, $\mathrm{F}_{n+1}(G)/\mathrm{F}_n(G) = \mathrm{F}(G/\mathrm{F}_n(G))$. Furthermore, since N is a normal subgroup of G containing $\mathrm{F}_{n+1}(G)$, we have $\mathrm{F}(N/\mathrm{F}_n(G)) = \mathrm{F}(G/\mathrm{F}_n(G))$. Thus, the quotient

$$\frac{N/\mathrm{F}_n(G)}{\mathrm{F}(N/\mathrm{F}_n(G))},$$

is abelian. By applying the induction hypothesis and Lemma 2.1 to the group $G/\mathrm{F}_n(G)$ and its normal subgroup $N/\mathrm{F}_n(G)$, we obtain:

$$(N/\mathrm{F}_n(G)) \setminus (\mathrm{F}_{n+1}(G)/\mathrm{F}_n(G)) \subseteq \mathrm{Van}(G/\mathrm{F}_n(G)).$$

Since vanishing elements in the quotient pull back to vanishing elements in the whole group, this implies that $N \setminus \mathrm{F}_{n+1}(G) \subseteq \mathrm{Van}(G)$. This completes the induction. □

We are now in a position to state and prove our first main theorem, which weakens the nilpotency condition of Theorem 1.1.

Theorem 2.3. *Let* $\mathrm{F}_i(G)$ *be the* i*-th term of the ascending Fitting series of a non-nilpotent solvable group* G*. If* x *is a non-vanishing element of* G *and* $G/\mathrm{F}_i(G)$ *is supersolvable, then* $x \in \mathrm{F}_i(G)$*.*

Proof. Let $x \in G$ be a non-vanishing element. Since $G/\mathrm{F}_i(G)$ is supersolvable, Theorem B of [6] implies $x\mathrm{F}_i(G) \in \mathrm{Z}(G/\mathrm{F}_i(G))$. Let Z be the subgroup of G containing $\mathrm{F}_i(G)$ such that $Z/\mathrm{F}_i(G) = \mathrm{Z}(G/\mathrm{F}_i(G))$. Clearly, $Z/\mathrm{F}_i(G)$ is an abelian normal subgroup of $G/\mathrm{F}_i(G)$. By applying Proposition 2.2, we deduce that $Z \setminus \mathrm{F}_i(G) \subseteq \mathrm{Van}(G)$.

Since x belongs to Z and x is a non-vanishing element, it cannot be an element of the set $Z \setminus \mathrm{F}_i(G)$. Therefore, x must necessarily lie in $\mathrm{F}_i(G)$, which completes the proof. □

As an immediate consequence of Theorem 2.3, we can draw an answer to the conjecture mentioned.

Corollary 2.4. *Let G be an extension of a supersolvable group by a nilpotent group. If $x \in G$ is a non-vanishing element, then $x \in \mathrm{F}(G)$.*

Proof. Let G be an extension of a supersolvable group by a nilpotent group N. Since $N \subseteq \mathrm{F}(G)$, the quotient $G/\mathrm{F}(G)$ is a homomorphic image of the supersolvable group G/N, meaning $G/\mathrm{F}(G)$ is also supersolvable. The result follows from Theorem 2.3 for $i = 1$. □

Example 2.5. Let G be the symmetric group on four letters, $G = \mathrm{S}_4$. It is a well-known fact that G is solvable but not nilpotent. Its Fitting subgroup is the Klein four-group,

$$\mathrm{F}(G) = V_4 = \{1, (12)(34), (13)(24), (14)(23)\}.$$

The quotient group $G/\mathrm{F}(G)$ is isomorphic to S_3. Since S_3 is supersolvable but not nilpotent, we cannot use Theorem 1.1 to determine the location of the non-vanishing elements of S_4. However, since the quotient is supersolvable, Theorem 2.3 asserts that every non-vanishing element of S_4 must lie inside $\mathrm{F}(G)$. Indeed, by inspecting the character table of S_4, one can readily verify that the non-vanishing elements of G are exactly the four elements of V_4.

Let $h(G)$ denote the Fitting length of a solvable group G. Since the commutator subgroup of a supersolvable group H is nilpotent, $H/\mathrm{F}(H)$ is abelian, implying $h(H) \leq 2$.

Corollary 2.6. *Let G be a solvable group and let*

$$V(G) = \langle x \in G \mid x \text{ is non-vanishing in } G \rangle.$$

If $G/\mathrm{F}_i(G)$ is supersolvable, then $h(G) \leq i + 2$ and $h(V(G)) \leq i$.

Proof. Since $G/\mathrm{F}_i(G)$ is supersolvable, its Fitting length is at most 2. Thus, $\mathrm{F}_2(G/\mathrm{F}_i(G)) = G/\mathrm{F}_i(G)$, meaning $\mathrm{F}_{i+2}(G) = G$. Hence, $h(G) \leq i+2$. By Theorem 2.3, $V(G) \leq \mathrm{F}_i(G)$. Since the Fitting length is inherited by subgroups and $h(\mathrm{F}_i(G)) = i$, we obtain $h(V(G)) \leq i$. □

3. Formations

Recall that a class of groups $\mathfrak{F}$ is called a *formation* if it is closed under taking homomorphic images, and whenever $N, M \trianglelefteq G$ such that $G/N \in \mathfrak{F}$ and $G/M \in \mathfrak{F}$, we have $G/(N \cap M) \in \mathfrak{F}$ (see [5, Chapter II]). Both nilpotent and supersolvable groups are formations.

Motivated by Proposition 2.2, we define $\mathfrak{F}_{anv}$ as the class of all solvable groups H satisfying the *abelian non-vanishing property*: for every non-vanishing element $h \in H$, there exists an abelian normal subgroup $A \trianglelefteq H$ such that $h \in A$. We first show that this property itself is a formation.

Lemma 3.1. *The class $\mathfrak{F}_{anv}$ is a formation.*

Proof. To prove closure under homomorphic images, let $H \in \mathfrak{F}_{anv}$ and $N \trianglelefteq H$. Suppose $\bar{h} = hN$ is a non-vanishing element of H/N. Then h is non-vanishing in H. Since $H \in \mathfrak{F}_{anv}$, there exists an abelian normal subgroup $A \trianglelefteq H$ containing h. Consequently, $AN/N \cong A/(A \cap N)$ is an abelian normal subgroup of H/N containing $\bar{h}$, so $H/N \in \mathfrak{F}_{anv}$.

Now, let $N, M \trianglelefteq H$ such that $H/N, H/M \in \mathfrak{F}_{anv}$. Let $h(N \cap M) \in H/(N \cap M)$ be a non-vanishing element. Its images hN and hM are non-vanishing in H/N and H/M, respectively. Thus, there exist abelian normal subgroups $A_1/N \trianglelefteq H/N$ and $A_2/M \trianglelefteq H/M$ such that $hN \in A_1/N$ and $hM \in A_2/M$. This implies $h \in A_1 \cap A_2$. Notice that $A_1 \cap A_2$ is normal in H. Furthermore, since A_1/N is abelian, $A_1' \leq N$. Similarly, $A_2' \leq M$. Therefore, $(A_1 \cap A_2)' \leq A_1' \cap A_2' \leq N \cap M$. This means $A_1 \cap A_2/(N \cap M)$ is an abelian normal subgroup of $H/(N \cap M)$ containing $h(N \cap M)$. Hence, $H/(N \cap M) \in \mathfrak{F}_{anv}$. □

Since every non-vanishing element of a supersolvable group lies in $\mathrm{Z}(\mathrm{F}(H))$, the formation of supersolvable groups is a subclass of $\mathfrak{F}_{anv}$. We now generalize Theorem 2.3 to any formation contained in $\mathfrak{F}_{anv}$.

Theorem 3.2. *Let $\mathfrak{F}$ be a formation of solvable groups such that $\mathfrak{F} \subseteq \mathfrak{F}_{anv}$. Let $\mathrm{F}_i(G)$ be the i-th term of the ascending Fitting series of a solvable group G. If x is a non-vanishing element of G and $G/\mathrm{F}_i(G) \in \mathfrak{F}$, then $x \in \mathrm{F}_i(G)$.*

Proof. Let $x \in G$ be non-vanishing, and $\bar{G} = G/\mathrm{F}_i(G)$. The image $\bar{x} = x\mathrm{F}_i(G)$ is non-vanishing in $\bar{G}$. Since $\bar{G} \in \mathfrak{F} \subseteq \mathfrak{F}_{anv}$, there exists an abelian normal subgroup $\bar{A}$ of $\bar{G}$ such that $\bar{x} \in \bar{A}$.

Let A be the normal subgroup of G containing $\mathrm{F}_i(G)$ such that $A/\mathrm{F}_i(G) = \bar{A}$. By Proposition 2.2, $A \setminus \mathrm{F}_i(G) \subseteq \mathrm{Van}(G)$. Since $x \in A$ is non-vanishing, $x \notin A \setminus \mathrm{F}_i(G)$. Thus, $x \in \mathrm{F}_i(G)$.
□

4. Non-Solvable Groups

The results in this paper are restricted to solvable groups, because of the ascending Fitting series $\mathrm{F}_i(G)$. Instead, in non-solvable groups, the appropriate structural tool is the *generalized Fitting subgroup*, $\mathrm{F}^*(G) = \mathrm{F}(G)E(G)$, where $E(G)$ is the layer generated by all quasisimple components of G.

In arbitrary finite groups, the generalized Fitting subgroup is expected to restrict these elements exactly as the Fitting subgroup does in solvable ones. Defining the ascending generalized Fitting series by $\mathrm{F}_1^*(G) = \mathrm{F}^*(G)$ and $\mathrm{F}_{i+1}^*(G)/\mathrm{F}_i^*(G) = \mathrm{F}^*(G/\mathrm{F}_i^*(G))$, we propose the following structural extension as an open problem:

Conjecture. *Let G be a finite group and let $\mathrm{F}_i^*(G)$ be the i-th term of its ascending generalized Fitting series. If $x \in G$ is a non-vanishing element and the quotient $G/\mathrm{F}_i^*(G)$ belongs to $\mathfrak{F}_{anv}$, then $x \in \mathrm{F}_i^*(G)$.*

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Sajjad Mahmood Robati

Department of Pure Mathematics,

Faculty of Science,

Imam Khomeini International University,

Qazvin, Iran.

sajjad.robati@gmail.com, mahmoodrobati@sci.ikiu.ac.ir