

Research Paper

**LOCAL AND 2-LOCAL  $\frac{1}{2}$ -DERIVATIONS ON QUASI-FILIFORM LIE ALGEBRAS OF MAXIMUM LENGTH**

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**ABSTRACT.** We determine all local  $\frac{1}{2}$ -derivations of quasi-filiform Lie algebras of maximum length and compute the dimensions of the corresponding spaces. In every case, the space of local  $\frac{1}{2}$ -derivations properly contains the space of  $\frac{1}{2}$ -derivations. We also construct explicit non-additive 2-local  $\frac{1}{2}$ -derivations on each algebra in the classification.

1. INTRODUCTION

The notion of  $\delta$ -derivations was introduced by V. Filippov for Lie algebras in [19, 20]. The space of  $\delta$ -derivations includes usual derivations ( $\delta = 1$ ), anti-derivations ( $\delta = -1$ ), and elements from the centroid. In [20] it was proved that prime Lie algebras, as a rule, do not have nonzero  $\delta$ -derivations (provided  $\delta \neq 1, -1, 0, \frac{1}{2}$ ), and all  $\frac{1}{2}$ -derivations of an arbitrary prime Lie algebra  $A$  over a field  $F$  of characteristic  $p \neq 2, 3$  with a non-degenerate symmetric invariant bilinear form were described. It was proved that if  $A$  is a central simple Lie algebra

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over a field of characteristic  $p \neq 2, 3$  with a non-degenerate symmetric invariant bilinear form, then any  $\frac{1}{2}$ -derivation  $\varphi$  has the form  $\varphi(x) = \lambda x$  for some  $\lambda \in F$ .

In [21],  $\delta$ -derivations were investigated for prime alternative and non-Lie Malcev algebras over the ring of operators  $F$ , and it was proved that alternative and non-Lie Malcev algebras with certain restrictions on  $F$  have no non-trivial  $\delta$ -derivations. The description of  $\delta$ -derivations of classical Lie superalgebras was given in [26]. The  $\delta$ -derivations of finite-dimensional semisimple Jordan algebras over a field of characteristic  $p \neq 2$  and  $\delta$ -superderivations of finite-dimensional simple Lie and Jordan superalgebras were investigated in [27]. Zusmanovich [41] described  $\delta$ -(super)derivations of prime Lie superalgebras. Specifically, he proved that a prime Lie superalgebra has no non-trivial  $\delta$ -(super)derivations for  $\delta \neq 1, -1, 0, \frac{1}{2}$ . Moreover, in [40] it is proved that any  $\delta$ -derivation of a finite-dimensional simple Jordan algebra with values in a finite-dimensional unital irreducible bimodule is, in a sense, trivial.

Nowadays, local and 2-local operators have become popular for non-associative algebras such as Lie, Jordan, and Leibniz algebras. The notions of local derivations were introduced in 1990 by Kadison [25] and Larson and Sourour [32]. Later, in 1997, Šemrl introduced the notions of 2-local derivations and 2-local automorphisms of algebras [34].

The investigation of local derivations on Lie algebras was initiated by Ayupov and Kuydaybergenov in [12]. They proved that every local derivation on a semisimple Lie algebra is a derivation and gave examples of finite-dimensional nilpotent Lie algebras with local derivations which are not derivations. In [9], local derivations of solvable Lie algebras are investigated, and it is shown that the class of solvable Lie algebras contains both algebras admitting local derivations which are not derivations and algebras for which every local derivation is a derivation. Several authors have investigated local derivations and related local-type operators on finite- and infinite-dimensional Lie, Leibniz, and associative algebras [5, 6, 8, 10, 13, 15, 17, 7, 2, 28, 36, 38]. Closely related investigations of almost inner derivations of nilpotent, solvable, and simple Leibniz algebras were carried out in [2, 30, 31]. It was proved that all local derivations of the following algebras are derivations: Borel subalgebras of finite-dimensional simple Lie algebras; Witt algebras; solvable Lie algebras of maximal rank; Cayley algebras; locally finite split simple Lie algebras; Schrödinger algebras; and conformal Galilei algebras.

Several papers have been devoted to similar notions and corresponding problems for 2-local derivations and automorphisms of Lie algebras [3, 11, 14, 15, 16, 18, 24, 35, 37]. Specifically, in [11] it is proved that every 2-local derivation on a semisimple Lie algebra is a derivation and that each finite-dimensional nilpotent Lie algebra of dimension greater than two admits a 2-local derivation which is not a derivation. The list of Lie algebras for which all 2-local derivations are derivations includes finite-dimensional semisimple Lie algebras, Witt algebras, locally

finite split simple Lie algebras, Virasoro algebras, the Virasoro-like algebra, the Schrödinger–Virasoro algebra, Jacobson–Witt algebras, and planar Galilean conformal algebras.

The investigation of local and 2-local  $\delta$ -derivations on Lie algebras was initiated by Khudoyberdiyev and Yusupov in [29]. In that paper the authors introduced local and 2-local  $\delta$ -derivations and described local and 2-local  $\frac{1}{2}$ -derivations of finite-dimensional solvable Lie algebras with filiform, Heisenberg, and abelian nilradicals. Moreover, they described local  $\frac{1}{2}$ -derivations of oscillator Lie algebras, conformal perfect Lie algebras, and Schrödinger algebras. Yusupov, Vaisova, and Madrakhimov proved similar results for local  $\frac{1}{2}$ -derivations of naturally graded quasi-filiform Leibniz algebras of type I in [39]. They proved that these algebras, as a rule, admit local  $\frac{1}{2}$ -derivations which are not  $\frac{1}{2}$ -derivations. Mamadaliyev, Sattarov, and Yusupov investigated local and 2-local  $\frac{1}{2}$ -derivations on solvable Leibniz algebras in [33].

In this work, we investigate local and 2-local  $\frac{1}{2}$ -derivations of quasi-filiform Lie algebras of maximum length. We describe all local  $\frac{1}{2}$ -derivations and show that these algebras admit local  $\frac{1}{2}$ -derivations which are not  $\frac{1}{2}$ -derivations. Moreover, we construct 2-local  $\frac{1}{2}$ -derivations which are not  $\frac{1}{2}$ -derivations.

## 2. PRELIMINARIES

Throughout the paper, all algebras are considered over the field  $\mathbb{C}$  unless otherwise stated.

**Definition 2.1.** Let  $(\mathcal{L}, [-, -])$  be an algebra. A linear map  $\varphi : \mathcal{L} \rightarrow \mathcal{L}$  is called a  $\delta$ -derivation if

$$\varphi([x, y]) = \delta([\varphi(x), y] + [x, \varphi(y)]),$$

for all  $x, y \in \mathcal{L}$ , where  $\delta$  belongs to the ground field  $F$ .

Note that a 1-derivation is a usual derivation and a  $(-1)$ -derivation is called an anti-derivation. If  $\varphi_1$  and  $\varphi_2$  are  $\delta_1$ - and  $\delta_2$ -derivations, respectively, then their commutator  $[\varphi_1, \varphi_2] = \varphi_1\varphi_2 - \varphi_2\varphi_1$  is a  $\delta_1\delta_2$ -derivation. The set of all  $\delta$ -derivations of  $\mathcal{L}$ , for fixed  $\delta$ , is denoted by  $\text{Der}_\delta(\mathcal{L})$ . For Lie algebras, anti-derivations coincide with reverse derivations, which were studied by Herstein in [23]. The main example of a  $\frac{1}{2}$ -derivation is multiplication by an element of the ground field, that is,  $\varphi(x) = \lambda x$  for all  $x \in \mathcal{L}$ . Such  $\frac{1}{2}$ -derivations are called trivial.

Since local operators may be defined for any type of operator, we use the following definitions.

**Definition 2.2.** A linear map  $\Delta : \mathcal{L} \rightarrow \mathcal{L}$  is called a local  $\delta$ -derivation if, for every  $x \in \mathcal{L}$ , there exists a  $\delta$ -derivation  $\varphi_x : \mathcal{L} \rightarrow \mathcal{L}$  (depending on  $x$ ) such that  $\Delta(x) = \varphi_x(x)$ . The set of all local  $\delta$ -derivations on  $\mathcal{L}$  is denoted by  $\text{LocDer}_\delta(\mathcal{L})$ .

**Definition 2.3.** A map  $\nabla : \mathcal{L} \rightarrow \mathcal{L}$ , not necessarily linear, is called a 2-local  $\delta$ -derivation if, for every  $x, y \in \mathcal{L}$ , there exists  $\varphi_{x,y} \in \text{Der}_\delta(\mathcal{L})$  such that

$$\nabla(x) = \varphi_{x,y}(x), \quad \nabla(y) = \varphi_{x,y}(y).$$

A 2-local  $\delta$ -derivation need not be linear, but it is homogeneous: for every  $x \in \mathcal{L}$  and every scalar  $\lambda$ ,

$$\nabla(\lambda x) = \varphi_{x,\lambda x}(\lambda x) = \lambda \varphi_{x,\lambda x}(x) = \lambda \nabla(x).$$

In this work, we focus on local and 2-local  $\frac{1}{2}$ -derivations. Since any  $\frac{1}{2}$ -derivation is both a local and a 2-local  $\frac{1}{2}$ -derivation, we are interested in local and 2-local  $\frac{1}{2}$ -derivations which are not  $\frac{1}{2}$ -derivations. We call them non-trivial local and non-trivial 2-local  $\frac{1}{2}$ -derivations, respectively.

The lower central series of a Lie algebra  $\mathcal{L}$  is defined by

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}], \quad k \geq 1.$$

**Definition 2.4.** A Lie algebra  $\mathcal{L}$  is called *nilpotent* if there exists  $k \in \mathbb{N}$  such that  $\mathcal{L}^k = \{0\}$ . The minimal such  $k$  is called the *index of nilpotency* of  $\mathcal{L}$ .

**Definition 2.5.** An  $n$ -dimensional Lie algebra is called *quasi-filiform* if its index of nilpotency is equal to  $n - 1$ .

We now define maximum length for nilpotent Lie algebras. A Lie algebra  $\mathcal{L}$  is  $\mathbb{Z}$ -graded if  $\mathcal{L} = \bigoplus_{i \in \mathbb{Z}} V_i$ , where  $[V_i, V_j] \subseteq V_{i+j}$  for all  $i, j \in \mathbb{Z}$  and only finitely many  $V_i$  are nonzero. A nilpotent Lie algebra  $\mathcal{L}$  admits a connected gradation  $\mathcal{L} = V_{k_1} \oplus \cdots \oplus V_{k_t}$  if  $V_{k_i} \neq \{0\}$  for every  $1 \leq i \leq t$ .

**Definition 2.6.** The number

$$l(\oplus \mathcal{L}) = l(V_{k_1} \oplus \cdots \oplus V_{k_t}) = k_t - k_1 + 1,$$

is called the *length of the gradation*. A gradation is of *maximum length* if  $l(\oplus \mathcal{L}) = \dim \mathcal{L}$ .

We denote by  $l(\mathcal{L})$  the maximum of  $l(\oplus \mathcal{L})$  over all connected gradations of  $\mathcal{L}$ .

**Definition 2.7.** A Lie algebra  $\mathcal{L}$  is called of *maximum length* if  $l(\mathcal{L}) = \dim \mathcal{L}$ .

The following classification of quasi-filiform Lie algebras of maximum length was obtained in [22].

**Theorem 2.8.** *Let  $L$  be an  $n$ -dimensional quasi-filiform Lie algebra of maximum length. Then  $L$  is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathfrak{g}_{(n,1)}^1 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_i, e_{n-i}] = (-1)^i e_n, & 2 \leq i \leq \frac{n-1}{2}, \end{cases} \quad n \geq 5 \text{ and } n \text{ odd};$$

$$\mathfrak{g}_{(n,1)}^2 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_i, e_n] = e_{i+2}, & 2 \leq i \leq n-3, \end{cases} \quad n \geq 5;$$

$$\mathfrak{g}_{(n,1)}^3 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_i, e_n] = e_{i+2}, & 2 \leq i \leq n-3, \\ [e_2, e_i] = e_{i+3}, & 3 \leq i \leq n-4, \end{cases} \quad n \geq 7;$$

$$\mathfrak{g}_7^1 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 5, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_i, e_{7-i}] = (-1)^i e_7, & 2 \leq i \leq 3; \end{cases}$$

$$\mathfrak{g}_9^2 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 7, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_2, e_5] = 3e_7, \quad [e_2, e_6] = 5e_8, \\ [e_3, e_i] = -2e_{i+3}, & 4 \leq i \leq 5, \\ [e_i, e_{9-i}] = (-1)^i e_9, & 2 \leq i \leq 4; \end{cases}$$

$$\mathfrak{g}_{11}^3 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 9, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_2, e_i] = -e_{i+2}, & 6 \leq i \leq 7, \\ [e_3, e_7] = -e_{10}, \\ [e_3, e_i] = e_{i+3}, & 4 \leq i \leq 5, \\ [e_4, e_i] = e_{i+4}, & 5 \leq i \leq 6, \\ [e_i, e_{11-i}] = (-1)^i e_{11}, & 2 \leq i \leq 5. \end{cases}$$

Here  $\{e_1, e_2, \dots, e_n\}$  is a basis of the algebra.

The following description of the  $\frac{1}{2}$ -derivations of quasi-filiform Lie algebras of maximum length was obtained in [1].

**Theorem 2.9.** *Any  $\frac{1}{2}$ -derivation  $\varphi$  of a quasi-filiform Lie algebra of maximum length has one of the following forms. All parameters occurring in each case are arbitrary complex numbers, while repeated symbols denote the same parameter.*

- For  $\mathfrak{g}_{(n,1)}^1$ ,  $n = 5$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, \\ \varphi(e_2) = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5, \\ \varphi(e_3) = \frac{1}{2}(\alpha_1 + \beta_2)e_3 + \frac{1}{2}\beta_3 e_4 - \frac{1}{2}\alpha_3 e_5, \\ \varphi(e_4) = \frac{1}{4}(3\alpha_1 + \beta_2)e_4 + \frac{1}{2}\alpha_2 e_5, \\ \varphi(e_5) = \frac{1}{2}\beta_1 e_4 + \frac{1}{4}(\alpha_1 + 3\beta_2)e_5. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^1$ ,  $n \geq 7$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \sum_{i=1}^n \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 e_2 + \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}\beta_{n-2} e_{n-1} - \frac{1}{2}\alpha_{n-2} e_n, \\ \varphi(e_i) = \alpha_1 e_i + \frac{(-1)^i}{2} \alpha_{n-i+1} e_n, \quad 4 \leq i \leq n-1, \\ \varphi(e_n) = \alpha_1 e_n. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^2$ ,  $n = 5$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, \\ \varphi(e_2) = \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5, \\ \varphi(e_3) = \frac{1}{2}(\alpha_1 + \beta_2)e_3 + \frac{1}{2}(\beta_3 - \alpha_5)e_4, \\ \varphi(e_4) = \frac{1}{4}(3\alpha_1 + \beta_2)e_4, \\ \varphi(e_5) = -\alpha_2 e_3 + \gamma_4 e_4 + \frac{1}{2}(3\alpha_1 - \beta_2)e_5. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^2$ ,  $n = 6$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6, \\ \varphi(e_2) = \alpha_1 e_2 - 3\alpha_6 e_3 + \beta_4 e_4 + \beta_5 e_5, \\ \varphi(e_3) = \alpha_1 e_3 - 2\alpha_6 e_4 + \frac{1}{2}\beta_4 e_5, \\ \varphi(e_4) = \alpha_1 e_4 - \frac{3}{2}\alpha_6 e_5, \\ \varphi(e_5) = \alpha_1 e_5, \\ \varphi(e_6) = -\alpha_2 e_3 - \alpha_3 e_4 + \gamma_5 e_5 + \alpha_1 e_6. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^2$ ,  $n \geq 7$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \sum_{i=1}^{n-1} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 e_2 + \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1}, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}\beta_{n-2} e_{n-1}, \\ \varphi(e_i) = \alpha_1 e_i, \quad 4 \leq i \leq n-1, \\ \varphi(e_n) = -\sum_{i=3}^{n-2} \alpha_{i-1} e_i + \gamma_{n-1} e_{n-1} + \alpha_1 e_n. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^3$ ,  $n = 7$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6, \\ \varphi(e_2) = \alpha_1 e_2 + 4\alpha_2 e_4 + \beta_5 e_5 + \beta_6 e_6, \\ \varphi(e_3) = \alpha_1 e_3 + 2\alpha_2 e_5 + \frac{1}{2}(\beta_5 - \alpha_3) e_6, \\ \varphi(e_4) = \alpha_1 e_4 + \frac{3}{2}\alpha_2 e_6, \\ \varphi(e_5) = \alpha_1 e_5, \quad \varphi(e_6) = \alpha_1 e_6, \\ \varphi(e_7) = -\alpha_2 e_3 - \alpha_3 e_4 - \alpha_4 e_5 + \gamma_6 e_6 + \alpha_1 e_7. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^3$ ,  $n \geq 8$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \alpha_1 e_1 + \sum_{i=3}^{n-1} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 e_2 + \sum_{i=5}^{n-3} \alpha_{i-2} e_i + \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1}, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}(\beta_{n-2} - \alpha_{n-4}) e_{n-1}, \\ \varphi(e_i) = \alpha_1 e_i, \quad 4 \leq i \leq n-1, \\ \varphi(e_n) = - \sum_{i=4}^{n-2} \alpha_{i-1} e_i + \gamma_{n-1} e_{n-1} + \alpha_1 e_n. \end{array} \right.$$

- For  $\mathfrak{g}_7^1$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \alpha_1 e_1 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7, \\ \varphi(e_2) = \alpha_1 e_2 - \frac{1}{3} \alpha_3 e_4 + \beta_5 e_5 + \beta_6 e_6 + \beta_7 e_7, \\ \varphi(e_3) = \alpha_1 e_3 - \frac{2}{3} \alpha_3 e_5 + \frac{1}{2}(\beta_5 - \alpha_4) e_6 - \frac{1}{2} \alpha_5 e_7, \\ \varphi(e_4) = \alpha_1 e_4 - \frac{1}{3} \alpha_3 e_6 + \frac{1}{2} \alpha_4 e_7, \\ \varphi(e_5) = \alpha_1 e_5 - \frac{1}{2} \alpha_3 e_7, \\ \varphi(e_6) = \alpha_1 e_6, \quad \varphi(e_7) = \alpha_1 e_7. \end{array} \right.$$

- For  $\mathfrak{g}_9^2$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \alpha_1 e_1 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7 + \alpha_8 e_8 + \alpha_9 e_9, \\ \varphi(e_2) = \alpha_1 e_2 + \frac{1}{3} \alpha_5 e_6 + \beta_7 e_7 + \beta_8 e_8 + \beta_9 e_9, \\ \varphi(e_3) = \alpha_1 e_3 - \frac{4}{3} \alpha_5 e_7 + \frac{1}{2}(\beta_7 - 5\alpha_6) e_8 - \frac{1}{2} \alpha_7 e_9, \\ \varphi(e_4) = \alpha_1 e_4 + \frac{1}{3} \alpha_5 e_8 + \frac{1}{2} \alpha_6 e_9, \\ \varphi(e_5) = \alpha_1 e_5 - \frac{1}{2} \alpha_5 e_9, \\ \varphi(e_i) = \alpha_1 e_i, \quad 6 \leq i \leq 9. \end{array} \right.$$

- For  $\mathfrak{g}_{11}^3$ ,

$$\left\{ \begin{array}{l} \varphi(e_1) = \alpha_1 e_1 + \alpha_6 e_6 + \alpha_7 e_7 + \alpha_8 e_8 + \alpha_9 e_9 + \alpha_{10} e_{10} + \alpha_{11} e_{11}, \\ \varphi(e_2) = \alpha_1 e_2 - \alpha_6 e_7 - \alpha_7 e_8 + \beta_9 e_9 + \beta_{10} e_{10} + \beta_{11} e_{11}, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2} \beta_9 e_{10} - \frac{1}{2} \alpha_9 e_{11}, \\ \varphi(e_4) = \alpha_1 e_4 + \frac{1}{2} \alpha_7 e_{10} + \frac{1}{2} \alpha_8 e_{11}, \\ \varphi(e_5) = \alpha_1 e_5 - \frac{1}{2} \alpha_6 e_{10} - \frac{1}{2} \alpha_7 e_{11}, \\ \varphi(e_6) = \alpha_1 e_6 + \frac{1}{2} \alpha_6 e_{11}, \\ \varphi(e_i) = \alpha_1 e_i, \quad 7 \leq i \leq 11. \end{array} \right.$$

### 3. LOCAL AND 2-LOCAL $\frac{1}{2}$ -DERIVATIONS ON QUASI-FILIFORM LIE ALGEBRAS OF MAXIMUM LENGTH

**3.1. Local  $\frac{1}{2}$ -derivations on quasi-filiform Lie algebras of maximum length.** In this subsection, we investigate local  $\frac{1}{2}$ -derivations on quasi-filiform Lie algebras of maximum length.

**Theorem 3.1.** *Any local  $\frac{1}{2}$ -derivation  $\Delta$  of a quasi-filiform Lie algebra of maximum length has one of the following forms. All parameters with different symbols in each case are arbitrary and independent complex numbers, while repeated symbols denote the same parameter.*

- For  $\mathfrak{g}_{(n,1)}^1$ ,  $n = 5$ ,

$$\left\{ \begin{array}{l} \Delta(e_1) = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \\ \Delta(e_2) = a_6 e_1 + a_7 e_2 + a_8 e_3 + a_9 e_4 + a_{10} e_5, \\ \Delta(e_3) = a_{11} e_3 + a_{12} e_4 + a_{13} e_5, \\ \Delta(e_4) = a_{14} e_4 + a_{15} e_5, \\ \Delta(e_5) = a_{16} e_4 + a_{17} e_5. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^1$ ,  $n \geq 7$ ,

$$(1) \quad \left\{ \begin{array}{l} \Delta(e_1) = \sum_{i=1}^n a_i e_i, \\ \Delta(e_2) = \lambda e_2 + b_2 e_{n-2} + b_3 e_{n-1} + b_4 e_n, \\ \Delta(e_3) = \lambda e_3 + c_2 e_{n-1} + c_3 e_n, \\ \Delta(e_i) = \lambda e_i + c_i e_n, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = d e_n. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^2$ ,  $n = 5$ ,

$$\left\{ \begin{array}{l} \Delta(e_1) = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5, \\ \Delta(e_2) = a_6e_2 + a_7e_3 + a_8e_4 + a_9e_5, \\ \Delta(e_3) = a_{10}e_3 + a_{11}e_4, \\ \Delta(e_4) = a_{12}e_4, \\ \Delta(e_5) = a_{13}e_3 + a_{14}e_4 + a_{15}e_5. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^2$ ,  $n = 6$ ,

$$\left\{ \begin{array}{l} \Delta(e_1) = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6, \\ \Delta(e_2) = a_7e_2 + a_8e_3 + a_9e_4 + a_{10}e_5, \\ \Delta(e_3) = a_{11}e_3 + a_{12}e_4 + a_{13}e_5, \\ \Delta(e_4) = a_{14}e_4 + a_{15}e_5, \\ \Delta(e_5) = a_{16}e_5, \\ \Delta(e_6) = a_{17}e_3 + a_{18}e_4 + a_{19}e_5 + a_{20}e_6. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^2$ ,  $n \geq 7$ ,

$$\left\{ \begin{array}{l} \Delta(e_1) = \sum_{i=1}^{n-1} a_i e_i, \\ \Delta(e_2) = a_1 e_2 + b_{n-2} e_{n-2} + b_{n-1} e_{n-1}, \\ \Delta(e_3) = a_1 e_3 + c e_{n-1}, \\ \Delta(e_i) = a_1 e_i, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = \sum_{i=3}^{n-1} d_i e_i + a_1 e_n. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^3$ ,  $n = 7$ ,

$$\left\{ \begin{array}{l} \Delta(e_1) = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6, \\ \Delta(e_2) = a_1e_2 + a_7e_4 + a_8e_5 + a_9e_6, \\ \Delta(e_3) = a_1e_3 + a_{10}e_5 + a_{11}e_6, \\ \Delta(e_4) = a_1e_4 + a_{12}e_6, \\ \Delta(e_5) = a_1e_5, \quad \Delta(e_6) = a_1e_6, \\ \Delta(e_7) = a_{13}e_3 + a_{14}e_4 + a_{15}e_5 + a_{16}e_6 + a_{17}e_7. \end{array} \right.$$

- For  $\mathfrak{g}_{(n,1)}^3$ ,  $n \geq 8$ ,

$$\left\{ \begin{array}{l} \Delta(e_1) = a_1 e_1 + \sum_{i=3}^{n-1} a_i e_i, \\ \Delta(e_2) = a_1 e_2 + \sum_{i=5}^{n-1} b_i e_i, \\ \Delta(e_3) = a_1 e_3 + a_2 e_{n-1}, \\ \Delta(e_i) = a_1 e_i, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = \sum_{i=4}^{n-1} c_i e_i + a_1 e_n. \end{array} \right.$$

- For  $\mathfrak{g}_7^1$ ,

$$\left\{ \begin{array}{l} \Delta(e_1) = a_1 e_1 + a_2 e_3 + a_3 e_4 + a_4 e_5 + a_5 e_6 + a_6 e_7, \\ \Delta(e_2) = a_1 e_2 + a_7 e_4 + a_8 e_5 + a_9 e_6 + a_{10} e_7, \\ \Delta(e_3) = a_1 e_3 + a_{11} e_5 + a_{12} e_6 + a_{13} e_7, \\ \Delta(e_4) = a_1 e_4 + a_{14} e_6 + a_{15} e_7, \\ \Delta(e_5) = a_1 e_5 + a_{16} e_7, \\ \Delta(e_6) = a_1 e_6, \quad \Delta(e_7) = a_1 e_7. \end{array} \right.$$

- For  $\mathfrak{g}_9^2$ ,

$$\left\{ \begin{array}{l} \Delta(e_1) = a_1 e_1 + a_2 e_5 + a_3 e_6 + a_4 e_7 + a_5 e_8 + a_6 e_9, \\ \Delta(e_2) = a_1 e_2 + a_7 e_6 + a_8 e_7 + a_9 e_8 + a_{10} e_9, \\ \Delta(e_3) = a_1 e_3 + a_{11} e_7 + a_{12} e_8 + a_{13} e_9, \\ \Delta(e_4) = a_1 e_4 + a_{14} e_8 + a_{15} e_9, \\ \Delta(e_5) = a_1 e_5 + a_{16} e_9, \\ \Delta(e_i) = a_1 e_i, \quad 6 \leq i \leq 9. \end{array} \right.$$

• For  $\mathfrak{g}_{11}^3$ ,

$$\left\{ \begin{array}{l} \Delta(e_1) = a_1e_1 + a_2e_6 + a_3e_7 + a_4e_8 + a_5e_9 + a_6e_{10} + a_7e_{11}, \\ \Delta(e_2) = a_1e_2 + a_8e_7 + a_9e_8 + a_{10}e_9 + a_{11}e_{10} + a_{12}e_{11}, \\ \Delta(e_3) = a_1e_3 + a_{13}e_{10} + a_{14}e_{11}, \\ \Delta(e_4) = a_1e_4 + a_{15}e_{10} + a_{16}e_{11}, \\ \Delta(e_5) = a_1e_5 + a_{17}e_{10} + a_{18}e_{11}, \\ \Delta(e_6) = a_1e_6 + a_{19}e_{11}, \\ \Delta(e_i) = a_1e_i, \quad 7 \leq i \leq 11. \end{array} \right.$$

*Proof.* We give the details for  $\mathfrak{g}_{(n,1)}^1$ ,  $n \geq 7$ . The remaining cases follow from the same coefficient-comparison and evaluation-system argument described at the end of the proof. Let  $\Delta$  be an arbitrary local  $\frac{1}{2}$ -derivation on  $\mathfrak{g}_{(n,1)}^1$ . For every  $x = \sum_{i=1}^n x_i e_i$ , there exists a  $\frac{1}{2}$ -derivation  $\varphi_x$  such that  $\Delta(x) = \varphi_x(x)$ .

Choosing successively  $x = e_1, e_2, \dots, e_n$ , we obtain

$$\left\{ \begin{array}{l} \Delta(e_1) = \sum_{i=1}^n a_i e_i, \\ \Delta(e_2) = b_1 e_2 + b_2 e_{n-2} + b_3 e_{n-1} + b_4 e_n, \\ \Delta(e_3) = c_1 e_3 + c_2 e_{n-1} + c_3 e_n, \\ \Delta(e_i) = d_i e_i + c_i e_n, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = d e_n. \end{array} \right.$$

Consider

$$\Delta(e_2 + e_3) = b_1 e_2 + b_2 e_{n-2} + b_3 e_{n-1} + b_4 e_n + c_1 e_3 + c_2 e_{n-1} + c_3 e_n.$$

On the other hand,

$$\begin{aligned} \Delta(e_2 + e_3) &= \varphi_{e_2+e_3}(e_2) + \varphi_{e_2+e_3}(e_3) \\ &= \alpha_{1,e_2+e_3} e_2 + \beta_{n-2,e_2+e_3} e_{n-2} + \beta_{n-1,e_2+e_3} e_{n-1} + \beta_{n,e_2+e_3} e_n \\ &\quad + \alpha_{1,e_2+e_3} e_3 + \frac{1}{2} \beta_{n-2,e_2+e_3} e_{n-1} - \frac{1}{2} \alpha_{n-2,e_2+e_3} e_n. \end{aligned}$$

Comparing the coefficients of  $e_2$  and  $e_3$  gives  $\alpha_{1,e_2+e_3} = b_1 = c_1$ .

For  $4 \leq i \leq n-3$ , consider

$$\Delta(e_2 + e_i) = b_1 e_2 + b_2 e_{n-2} + b_3 e_{n-1} + b_4 e_n + d_i e_i + c_i e_n.$$

There exists a  $\frac{1}{2}$ -derivation  $\varphi_{e_2+e_i}$  such that

$$\begin{aligned}\Delta(e_2 + e_i) &= \varphi_{e_2+e_i}(e_2) + \varphi_{e_2+e_i}(e_i) \\ &= \alpha_{1,e_2+e_i}e_2 + \beta_{n-2,e_2+e_i}e_{n-2} + \beta_{n-1,e_2+e_i}e_{n-1} + \beta_{n,e_2+e_i}e_n \\ &\quad + \alpha_{1,e_2+e_i}e_i + \frac{(-1)^i}{2}\alpha_{n-i+1,e_2+e_i}e_n.\end{aligned}$$

Comparing the coefficients of  $e_2$  and  $e_i$  yields  $d_i = b_1$  for  $4 \leq i \leq n-3$ .

Similarly, for  $n-2 \leq i \leq n-1$ , consider

$$\Delta(e_4 + e_i) = b_1e_4 + c_4e_n + d_ie_i + c_ie_n.$$

On the other hand,

$$\begin{aligned}\Delta(e_4 + e_i) &= \varphi_{e_4+e_i}(e_4) + \varphi_{e_4+e_i}(e_i) \\ &= \alpha_{1,e_4+e_i}e_4 + \frac{1}{2}\alpha_{n-3,e_4+e_i}e_n + \alpha_{1,e_4+e_i}e_i + \frac{(-1)^i}{2}\alpha_{n-i+1,e_4+e_i}e_n.\end{aligned}$$

Comparing the coefficients of  $e_4$  and  $e_i$  gives  $d_i = b_1$  for  $n-2 \leq i \leq n-1$ .

It remains to show that every linear operator of the form

$$\left\{ \begin{array}{l} \Delta(e_1) = \sum_{i=1}^n a_ie_i, \\ \Delta(e_2) = b_1e_2 + b_2e_{n-2} + b_3e_{n-1} + b_4e_n, \\ \Delta(e_3) = b_1e_3 + c_2e_{n-1} + c_3e_n, \\ \Delta(e_i) = b_1e_i + c_ie_n, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = de_n \end{array} \right.$$

is a local  $\frac{1}{2}$ -derivation on  $\mathfrak{g}_{(n,1)}^1$ . For  $x = \sum_{i=1}^n x_ie_i$ , the identity  $\Delta(x) = \varphi_x(x)$  is equivalent to

$$a_1x_1 = \alpha_{1,x}x_1,$$

$$a_ix_1 + b_1x_i = \alpha_{i,x}x_1 + \alpha_{1,x}x_i, \quad 2 \leq i \leq n-3,$$

$$a_{n-2}x_1 + b_2x_2 + b_1x_{n-2} = \alpha_{n-2,x}x_1 + \beta_{n-2,x}x_2 + \alpha_{1,x}x_{n-2},$$

$$a_{n-1}x_1 + b_3x_2 + c_2x_3 + b_1x_{n-1} = \alpha_{n-1,x}x_1 + \beta_{n-1,x}x_2 + \frac{\beta_{n-2,x}}{2}x_3 + \alpha_{1,x}x_{n-1},$$

$$(2) \quad a_nx_1 + b_4x_2 + \sum_{i=3}^{n-1} c_ix_i + dx_n = \alpha_{n,x}x_1 + \beta_{n,x}x_2 + \sum_{i=3}^{n-1} \frac{(-1)^i \alpha_{n-i+1,x}}{2} x_i + \alpha_{1,x}x_n.$$

We distinguish the following cases.

**Case 1.** If  $x_1 \neq 0$ , take  $\alpha_{1,x} = a_1$  and

$$\begin{aligned}\alpha_{i,x} &= \frac{a_i x_1 + b_1 x_i - \alpha_{1,x} x_i}{x_1}, \quad 2 \leq i \leq n-3, \\ \alpha_{n-2,x} &= \frac{a_{n-2} x_1 + b_2 x_2 + b_1 x_{n-2} - \beta_{n-2,x} x_2 - \alpha_{1,x} x_{n-2}}{x_1}, \\ \alpha_{n-1,x} &= \frac{a_{n-1} x_1 + b_3 x_2 + c_2 x_3 + b_1 x_{n-1} - \beta_{n-1,x} x_2 - \frac{1}{2} \beta_{n-2,x} x_3 - \alpha_{1,x} x_{n-1}}{x_1}, \\ \alpha_{n,x} &= \frac{a_n x_1 + b_4 x_2 + \sum_{i=3}^{n-1} c_i x_i + d x_n - \beta_{n,x} x_2 - \sum_{i=3}^{n-1} \frac{(-1)^i \alpha_{n-i+1,x}}{2} x_i - \alpha_{1,x} x_n}{x_1},\end{aligned}$$

where  $\beta_{n-2,x}, \beta_{n-1,x}, \beta_{n,x}$  are arbitrary.

**Case 2.** If  $x_1 = 0$  and  $x_2 \neq 0$ , take

$$\begin{aligned}\alpha_{1,x} &= b_1, \\ \beta_{n-2,x} &= \frac{b_2 x_2 + b_1 x_{n-2} - \alpha_{1,x} x_{n-2}}{x_2}, \\ \beta_{n-1,x} &= \frac{b_3 x_2 + c_2 x_3 + b_1 x_{n-1} - \frac{1}{2} \beta_{n-2,x} x_3 - \alpha_{1,x} x_{n-1}}{x_2}, \\ \beta_{n,x} &= \frac{b_4 x_2 + \sum_{i=3}^{n-1} c_i x_i + d x_n - \sum_{i=3}^{n-1} \frac{(-1)^i \alpha_{n-i+1,x}}{2} x_i - \alpha_{1,x} x_n}{x_2},\end{aligned}$$

where  $\alpha_{2,x}, \dots, \alpha_{n-2,x}$  are arbitrary.

**Case 3.** If  $x_1 = x_2 = 0$  and  $x_3 \neq 0$ , take

$$\begin{aligned}\alpha_{1,x} &= b_1, \\ \beta_{n-2,x} &= \frac{2(c_2 x_3 + b_1 x_{n-1} - \alpha_{1,x} x_{n-1})}{x_3}, \\ \alpha_{n-2,x} &= -2 \frac{\sum_{i=3}^{n-1} c_i x_i + d x_n - \sum_{i=4}^{n-1} \frac{(-1)^i \alpha_{n-i+1,x}}{2} x_i - \alpha_{1,x} x_n}{x_3},\end{aligned}$$

where  $\alpha_{2,x}, \dots, \alpha_{n-3,x}$  are arbitrary.

**Case 4.** If  $x_1 = x_2 = \dots = x_{i-1} = 0$  and  $x_i \neq 0$ , where  $4 \leq i \leq n-1$ , take  $\alpha_{1,x} = b_1$  and

$$\alpha_{n-i+1,x} = 2(-1)^i \frac{\sum_{j=i}^{n-1} c_j x_j + d x_n - \sum_{j=i}^{n-2} \frac{(-1)^{j+1} \alpha_{n-j,x} x_{j+1}}{2} - \alpha_{1,x} x_n}{x_i},$$

where  $\alpha_{2,x}, \dots, \alpha_{n-i,x}$  are arbitrary.

**Case 5.** If  $x_1 = x_2 = \dots = x_{n-1} = 0$  and  $x_n \neq 0$ , take  $\alpha_{1,x} = d$ . If  $x = 0$ , take  $\varphi_0 = 0$ .

Thus, for every choice of the coefficients in (1), there exist coefficients defining  $\varphi_x$  for which (2) holds. Hence every transformation of the form (1) is a local  $\frac{1}{2}$ -derivation on  $\mathfrak{g}_{(n,1)}^1$ ,  $n \geq 7$ .

For completeness, let  $D_1, \dots, D_r$  be a basis of  $\text{Der}_{\frac{1}{2}}(\mathcal{L})$ . By the definition of a local  $\frac{1}{2}$ -derivation, a linear operator  $T$  on  $\mathcal{L}$  is local if and only if

$$T(x) \in \text{span}\{D_1(x), \dots, D_r(x)\} \quad \text{for every } x \in \mathcal{L}.$$

Equivalently,

$$\text{rank}(D_1(x), \dots, D_r(x), T(x)) = \text{rank}(D_1(x), \dots, D_r(x)) \quad \text{for every } x \in \mathcal{L}.$$

Substituting the  $\frac{1}{2}$ -derivations listed in Theorem 2.9 into this criterion gives exactly the remaining forms displayed in the statement.

In particular, for  $\mathfrak{g}_{(7,1)}^3$ , applying locality to  $e_1 + e_7$  and comparing the coefficients of  $e_1$  and  $e_7$  gives  $a_{17} = a_1$ . Therefore

$$\Delta(e_7) = a_{13}e_3 + a_{14}e_4 + a_{15}e_5 + a_{16}e_6 + a_1e_7,$$

and hence

$$\dim \text{LocDer}_{\frac{1}{2}}(\mathfrak{g}_{(7,1)}^3) = 16.$$

The corresponding evaluation systems for the remaining algebras produce no additional relations and also prove the sufficiency of the displayed forms.  $\square$

The dimensions in the following table are obtained by counting the independent parameters in Theorems 2.9 and 3.1. Repeated parameters are counted only once.

| Algebra                            | $\dim \text{Der}_{\frac{1}{2}}$ | $\dim \text{LocDer}_{\frac{1}{2}}$ |
|------------------------------------|---------------------------------|------------------------------------|
| $\mathfrak{g}_{(5,1)}^1$           | 10                              | 17                                 |
| $\mathfrak{g}_{(n,1)}^1, n \geq 7$ | $n + 3$                         | $2n + 3$                           |
| $\mathfrak{g}_{(5,1)}^2$           | 10                              | 15                                 |
| $\mathfrak{g}_{(6,1)}^2$           | 9                               | 19                                 |
| $\mathfrak{g}_{(n,1)}^2, n \geq 7$ | $n + 2$                         | $2n - 1$                           |
| $\mathfrak{g}_{(7,1)}^3$           | 9                               | 16                                 |
| $\mathfrak{g}_{(n,1)}^3, n \geq 8$ | $n + 1$                         | $3n - 10$                          |
| $\mathfrak{g}_7^1$                 | 9                               | 16                                 |
| $\mathfrak{g}_9^2$                 | 9                               | 16                                 |
| $\mathfrak{g}_{11}^3$              | 10                              | 19                                 |

**Corollary 3.2.** *Quasi-filiform Lie algebras of maximum length admit local  $\frac{1}{2}$ -derivations which are not  $\frac{1}{2}$ -derivations.*

*Proof.* In every row of the preceding table,  $\dim \text{LocDer}_{\frac{1}{2}}(\mathcal{L}) > \dim \text{Der}_{\frac{1}{2}}(\mathcal{L})$ . Therefore  $\text{LocDer}_{\frac{1}{2}}(\mathcal{L})$  contains an operator which is not a  $\frac{1}{2}$ -derivation.  $\square$

**3.2. 2-local  $\frac{1}{2}$ -derivations on quasi-filiform Lie algebras of maximum length.** We now investigate 2-local  $\frac{1}{2}$ -derivations on quasi-filiform Lie algebras of maximum length.

**Theorem 3.3.** *Every quasi-filiform Lie algebra of maximum length admits a 2-local  $\frac{1}{2}$ -derivation which is not a  $\frac{1}{2}$ -derivation.*

*Proof.* For every algebra  $\mathcal{L}$  in Theorem 2.8, choose a nonzero vector  $z_{\mathcal{L}}$  as follows:

| $\mathcal{L}$                      | $z_{\mathcal{L}}$ | $\mathcal{L}$                      | $z_{\mathcal{L}}$ |
|------------------------------------|-------------------|------------------------------------|-------------------|
| $\mathfrak{g}_{(5,1)}^1$           | $e_5$             | $\mathfrak{g}_{(7,1)}^3$           | $e_6$             |
| $\mathfrak{g}_{(n,1)}^1, n \geq 7$ | $e_n$             | $\mathfrak{g}_{(n,1)}^3, n \geq 8$ | $e_{n-1}$         |
| $\mathfrak{g}_{(5,1)}^2$           | $e_4$             | $\mathfrak{g}_7^1$                 | $e_7$             |
| $\mathfrak{g}_{(6,1)}^2$           | $e_5$             | $\mathfrak{g}_9^2$                 | $e_9$             |
| $\mathfrak{g}_{(n,1)}^2, n \geq 7$ | $e_{n-1}$         | $\mathfrak{g}_{11}^3$              | $e_{11}$          |

It follows directly from Theorem 2.9 that, for arbitrary  $\mu, \nu \in \mathbb{C}$ , the linear map

$$(3) \quad D_{\mu, \nu}(x) = (\mu x_1 + \nu x_2)z_{\mathcal{L}}, \quad x = \sum_{i=1}^n x_i e_i,$$

is a  $\frac{1}{2}$ -derivation of  $\mathcal{L}$ .

Define a homogeneous function  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$  by

$$f(u, v) = \begin{cases} \frac{u^2}{v}, & v \neq 0, \\ 0, & v = 0. \end{cases}$$

For  $x = \sum_{i=1}^n x_i e_i \in \mathcal{L}$ , put

$$(4) \quad \nabla_{\mathcal{L}}(x) = f(x_1, x_2)z_{\mathcal{L}}.$$

It is easy to see that the function  $f$  is homogeneous but not additive. Consequently,  $\nabla_{\mathcal{L}}$  is not linear and hence is not a  $\frac{1}{2}$ -derivation.

Let  $x, y \in \mathcal{L}$ . To find a  $\frac{1}{2}$ -derivation of the form (3) which agrees with  $\nabla_{\mathcal{L}}$  at  $x$  and  $y$ , it is enough to solve

$$(5) \quad \begin{cases} x_1\mu + x_2\nu = f(x_1, x_2), \\ y_1\mu + y_2\nu = f(y_1, y_2). \end{cases}$$

If  $x_1y_2 - x_2y_1 \neq 0$ , this system has a unique solution.

Suppose that  $x_1y_2 - x_2y_1 = 0$ . If one of the rows  $(x_1, x_2)$  and  $(y_1, y_2)$  is zero, then its right-hand side is also zero. If both rows are nonzero, there exists  $t \in \mathbb{C}$  such that

$$(y_1, y_2) = t(x_1, x_2).$$

Since  $f$  is homogeneous,

$$f(y_1, y_2) = tf(x_1, x_2),$$

and hence the two equations in (5) are compatible. Thus, the system is solvable in every case.

Choose a solution  $(\mu, \nu)$  and set  $\varphi_{x,y} = D_{\mu,\nu}$ . Then

$$\varphi_{x,y}(x) = \nabla_{\mathcal{L}}(x), \quad \varphi_{x,y}(y) = \nabla_{\mathcal{L}}(y).$$

Therefore  $\nabla_{\mathcal{L}}$  is a 2-local  $\frac{1}{2}$ -derivation which is not a  $\frac{1}{2}$ -derivation. The construction applies to every algebra in the classification.  $\square$

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#### REFERENCES

- [1] K. Abdurasulov, F. Deraman, A. Saydaliyev and S. H. Sapar, *Transposed Poisson structures on low dimensional quasi-filiform Lie algebras of maximum length*, Lobachevskii J. Math., **45** (2024) 5735-5749.
- [2] J. K. Adashev and T. K. Kurbanbaev, *Almost inner derivations of some nilpotent Leibniz algebras*, J. Sib. Fed. Univ. Math. Phys., **13** No. 6 (2020) 733-745.
- [3] A. Alauadinov, Z. Normatov and B. Yusupov, *2-local derivations on the perfect Lie algebras  $\mathfrak{L}^m$* , Algebr. Struct. Appl., **12** No. 3 (2025) 263-271.
- [4] A. K. Alauadinov and B. B. Yusupov, *Local derivations on the Schrödinger Lie algebra*, Algebra Colloq., **32** No. 3 (2025) 467-480.
- [5] A. K. Alauadinov and B. B. Yusupov, *Local derivation on the Schrödinger Lie algebra in  $(n+1)$ -dimensional space-time*, Bull. Karaganda Univ. Math. Ser., No. 1(121) (2026) 55-66.
- [6] A. K. Alauadinov and B. B. Yusupov, *Local derivations of conformal Galilei algebra*, Comm. Algebra, **52** No. 6 (2024) 2489-2508.
- [7] Kh. Atajonov, T. Madrakhimov and B. B. Yusupov, *Local and 2-local anti-derivations on filiform associative algebras*, Bol. Soc. Paran. Mat., **43** (2025) 1-13.
- [8] Sh. Ayupov, A. Elduque and K. Kudaybergenov, *Local derivations and automorphisms of Cayley algebras*, J. Pure Appl. Algebra, **227** No. 5 (2023) 107277.
- [9] Sh. Ayupov and A. Khudoyberdiyev, *Local derivations on solvable Lie algebras*, Linear Multilinear Algebra, **69** No. 7 (2021) 1286-1301.
- [10] Sh. Ayupov, A. Khudoyberdiyev and B. Yusupov, *Local and 2-local derivations of solvable Leibniz algebras*, Internat. J. Algebra Comput., **30** No. 6 (2020) 1185-1197.

- [11] Sh. Ayupov, K. Kudaybergenov and I. Rakhimov, *2-local derivations on finite-dimensional Lie algebras*, Linear Algebra Appl., **474** (2015) 1-11.
- [12] Sh. Ayupov and K. Kudaybergenov, *Local derivations on finite-dimensional Lie algebras*, Linear Algebra Appl., **493** (2016) 381-398.
- [13] Sh. A. Ayupov, Kh. Atajonov and B. B. Yusupov, *Local and 2-local anti-derivations on solvable Lie algebras*, Eur. J. Math., **11** No. 3 (2025) 52.
- [14] Sh. A. Ayupov, K. K. Kudaybergenov and B. B. Yusupov, *Local and 2-local derivations of  $p$ -filiform Leibniz algebras*, J. Math. Sci., **245** No. 3 (2020) 359-367.
- [15] Sh. Ayupov, K. Kudaybergenov and B. Yusupov, *Local and 2-local derivations of locally simple Lie algebras*, J. Math. Sci., **278** No. 4 (2024) 613-622.
- [16] Sh. Ayupov, K. Kudaybergenov and B. Yusupov, *2-local derivations on generalized Witt algebras*, Linear Multilinear Algebra, **69** No. 16 (2021) 3130-3140.
- [17] Sh. Ayupov and B. Yusupov, *2-local derivations of infinite-dimensional Lie algebras*, J. Algebra Appl., **19** No. 5 (2020) 2050100.
- [18] Z. Chen and D. Wang, *2-local automorphisms of finite-dimensional simple Lie algebras*, Linear Algebra Appl., **486** (2015) 335-344.
- [19] V. Filippov, *On  $\delta$ -derivations of Lie algebras*, Siberian Math. J., **39** No. 6 (1998) 1218-1230.
- [20] V. Filippov,  *$\delta$ -derivations of prime Lie algebras*, Siberian Math. J., **40** No. 1 (1999) 174-184.
- [21] V. Filippov, *On  $\delta$ -derivations of prime alternative and Malcev algebras*, Algebra Logic, **39** No. 5 (2000) 354-358.
- [22] J. R. Gómez, A. Jiménez-Merchán and J. Reyes, *Quasi-filiform Lie algebras of maximum length*, Linear Algebra Appl., **335** (2001) 119-135.
- [23] I. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc., **8** (1957) 1104-1110.
- [24] Q. Jiang and X. Tang, *2-local derivations on the Schrödinger-Virasoro algebra*, Linear Multilinear Algebra, **72** No. 8 (2023) 1328-1345.
- [25] R. Kadison, *Local derivations*, J. Algebra, **130** (1990) 494-509.
- [26] I. Kaygorodov,  *$\delta$ -derivations of classical Lie superalgebras*, Siberian Math. J., **50** No. 3 (2009) 434-449.
- [27] I. Kaygorodov, *On  $\delta$ -derivations of simple finite-dimensional Jordan superalgebras*, Algebra Logic, **46** No. 5 (2007) 318-329.
- [28] K. Kudaybergenov, B. Omirov and T. Kurbanbaev, *Local derivations on solvable Lie algebras of maximal rank*, Comm. Algebra, **50** No. 9 (2022) 1-11.
- [29] A. Kh. Khudoyberdiyev and B. B. Yusupov, *Local and 2-local  $\frac{1}{2}$ -derivations on finite-dimensional Lie algebras*, Results Math., **79** (2024) 210.
- [30] T. K. Kurbanbaev, *Almost inner derivations of some Leibniz algebras*, J. Sib. Fed. Univ. Math. Phys., **16** No. 4 (2023) 457-474.
- [31] T. K. Kurbanbaev, *Almost inner derivations of finite-dimensional simple Leibniz algebras*, Ukrainian Math. J., **77** No. 11 (2025) 1524-1536.
- [32] D. Larson and A. Sourour, *Local derivations and local automorphisms of  $B(X)$* , Proc. Sympos. Pure Math., **51** (1990) 187-194.

- [33] U. Kh. Mamadaliyev, A. Sattarov and B. B. Yusupov, *Local and 2-local  $\frac{1}{2}$ -derivations of solvable Leibniz algebras*, Eurasian Math. J., **16** No. 2 (2025) 42-54.
- [34] P. Šemrl, *Local automorphisms and derivations on  $B(H)$* , Proc. Amer. Math. Soc., **125** (1997) 2677-2680.
- [35] X. Tang, M. Xiao and P. Wang, *Local properties of Virasoro-like algebra*, J. Geom. Phys., **186** (2023) 104772.
- [36] Y. F. Yao, *Local derivations on the Witt algebra in prime characteristic*, Linear Multilinear Algebra, **70** No. 15 (2022) 2919-2933.
- [37] Y. Yao and K. Zhao, *Local properties of Jacobson-Witt algebras*, J. Algebra, **586** (2021) 1110-1121.
- [38] Y. Yu and Zh. Chen, *Local derivations on Borel subalgebras of finite-dimensional simple Lie algebras*, Comm. Algebra, **48** No. 1 (2020) 1-10.
- [39] B. Yusupov, N. Vaisova and T. Madrakhimov, *Local  $\frac{1}{2}$ -derivations of  $n$ -dimensional naturally graded quasi-filiform Leibniz algebras of type I*, AIP Conf. Proc., **3147** (2024) 020008.
- [40] A. Zohrabi and P. Zusmanovich, *A  $\delta$ -first Whitehead lemma for Jordan algebras*, Commun. Math., **33** No. 1 (2025) 2.
- [41] P. Zusmanovich, *On  $\delta$ -derivations of Lie algebras and superalgebras*, J. Algebra, **324** No. 12 (2010) 3470-3486.

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