

Research Paper

SOME PROPERTIES OF THE COMPLEMENT OF THE DOT PRODUCT GRAPH OF A COMMUTATIVE RING

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ABSTRACT. For a nonzero commutative ring A with identity and a positive integer n , let $R = A \times \cdots \times A$ (n times). The total dot product graph of R , denoted by $TD(R)$, is an undirected graph with vertex set equals $R \setminus \{(0, \dots, 0)\}$ and distinct vertices x and y are adjacent if and only if $x \cdot y = 0$, where $x \cdot y$ is the normal dot product of x and y . Let $Z(R)$ denote the set of all zero-divisors of R and let us denote $Z(R) \setminus \{(0, \dots, 0)\}$ by $Z(R)^*$. The zero-divisor dot product graph of R , denoted by $ZD(R)$, is the subgraph of $TD(R)$ induced by $Z(R)^*$. The graph $TD(R)$ (respectively, $ZD(R)$) was introduced and investigated by Badawi [*Comm. Algebra* **43**(1) (2015), 43-50]. We denote the complement of $TD(R)$ (respectively, $ZD(R)$) by $(TD(R))^c$ (respectively, $(ZD(R))^c$). In this paper, we determine the girth of $(TD(R))^c$ (respectively, $(ZD(R))^c$). Moreover, we characterize R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) is complemented. Furthermore, if $(TD(R))^c$ (respectively, $(ZD(R))^c$) is connected, then we characterize R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) admits a cut edge.

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1. INTRODUCTION

The rings considered in this paper are commutative with identity and are nonzero. For any ring T , we denote the set of all nonzero elements of T , the set of all zero-divisors of T , and the set of all nonzero zero-divisors of T by T^* , $Z(T)$, $Z(T)^*$, respectively. The graphs considered in this paper are undirected and simple. For a graph G , we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Let A be a ring. Let $n \in \mathbb{N}$ and let $R = A \times \cdots \times A$ (n times). Recall that the *total dot product graph* of R , denoted by $TD(R)$, is an undirected graph with $V(TD(R)) = R^*$ and distinct vertices x and y are adjacent if and only if $x \cdot y = 0$, where $x \cdot y$ denotes the normal dot product of x and y . Recall that the *zero-divisor dot product graph* of R , denoted by $ZD(R)$, is the subgraph of $TD(R)$ induced by $Z(R)^*$ [7]. This paper is motivated by the interesting results proved on $TD(R)$ and $ZD(R)$ in [7]. For $n = 1$, if R contains at least two nonzero elements (respectively, $Z(R)^* \neq \emptyset$), then it was observed in [7] that $TD(R)$ is disconnected (respectively, $ZD(R) = \Gamma(R)$ is connected), where $\Gamma(R)$ is the well-known zero-divisor graph of R . For any $n \geq 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$. For $n \geq 2$, it was shown in [7, Theorem 2.2] that $ZD(R) = \Gamma(R)$ if and only if either $n = 2$ and A is an integral domain or R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The diameter of a connected graph G is denoted by $diam(G)$. We denote the girth of a graph G by $gr(G)$. Let $n \geq 2$. In [7], Badawi characterized R such that $TD(R)$ (respectively, $ZD(R)$) is connected and moreover, he determined $diam(TD(R))$ (respectively, $diam(ZD(R))$) whenever $TD(R)$ (respectively, $ZD(R)$) is connected. Furthermore, many interesting results regarding $gr(TD(R))$ (respectively, $gr(ZD(R))$) were proved in [7]. For other interesting and inspiring results on $TD(R)$, $ZD(R)$, and $UD(R)$ (where $UD(R)$ is the subgraph of $TD(R)$ induced by the group of units of R), one can refer to [1, 13]. For a more detailed exposition of total dot product graphs, zero-divisor dot product graphs, and unit dot product graphs of commutative rings, the reader is referred to the book [4] and the authors of [4] have investigated in detail several graphs associated with commutative rings. For some recent research works on Algebraic graph theory by Moniri Hamzekolaee et al., Selvakumar et al., and Ali et al., one can refer to [2, 12], [14], and [3].

Let $G = (V, E)$ be a simple graph. We recall that the *complement* of G , denoted by G^c , is defined by taking $V(G^c) = V(G)$ and distinct vertices u and v are adjacent in G^c if and only if they are not adjacent in G [8, Definition 1.2.13]. This paper does not use much graph theory, except for some basic definitions and results. Suppose that G contains a cycle. Recall that the *girth* of G , denoted by $gr(G)$, is defined as the length of a shortest cycle in G . We set $gr(G) = \infty$ if G does not contain any cycle.

Before we give a brief account of the results that are proved in this paper, it is useful to recall the following. Let $G = (V, E)$ be a graph. Recall that two distinct vertices u, v of G are

said to be *orthogonal*, written $u \perp v$ if u and v are adjacent in G and there is no vertex of G which is adjacent to both u and v in G ; that is, the edge $u - v$ is not an edge of any triangle in G [5]. Let $u \in V$. A vertex v of G is said to be a *complement* of u if $u \perp v$ [5]. Moreover, recall that G is *complemented* if each vertex of G admits a complement in G [5]. Let T be a ring such that T is not an integral domain. Anderson et al. among other results, characterized rings T such that $\Gamma(T)$ is complemented, see [5, Section 3]. For a characterization of T such that $(\Gamma(T))^c$ is complemented, see [17].

It is useful to recall the following definitions from [8]. Let $G = (V, E)$ be a graph. Let S be a proper subset of V . Recall that $G[V \setminus S]$ (that is, the subgraph of G induced by $V \setminus S$) is said to be obtained from G by the *deletion* of S . This subgraph is denoted by $G - S$. If $S = \{v\}$, then this subgraph is simply denoted by $G - v$ [8, Definition 1.3.3].

We denote the cardinality of a set B by $|B|$. Let $G = (V, E)$ be a connected graph with $|V| \geq 2$. We recall that a subset S of V is said to be a *vertex cut* of G if $G - S$ is disconnected. Let $v \in V$. We recall that v is said to be a *cut vertex* of G if $\{v\}$ is a vertex cut of G [8, Definition 3.2.1(1)].

Let $G = (V, E)$ be a connected graph with $|V| \geq 2$. Let S be a nonempty subset of V . For $\bar{S} = V \setminus S \neq \emptyset$, let $[S, \bar{S}]$ denote the set of all edges of G that have one end vertex in S and the other in $\bar{S}$. A set of edges of the form $[S, \bar{S}]$ is called an *edge cut* of G . An edge e is a *cut edge* of G if $\{e\}$ is an edge cut of G [8, Definition 3.2.1(2)].

Let A be a ring. Let $n \in \mathbb{N}$ and let $R = A \times \cdots \times A$ (n times). If $n = 1$, whenever we want to characterize A in terms of a graph property of $(ZD(A))^c$, we assume that $Z(A)^* \neq \emptyset$. Motivated by the research work [7], in this paper, we study some graph properties of $(TD(R))^c$ (respectively, $(ZD(R))^c$) to identify the structure of R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) possesses a certain graph property. This paper has five sections. Section 1 is on introduction. In Section 2, we present some preliminaries. In Section 3, with proper assumptions, we prove $gr((TD(R))^c) \in \{3, \infty\}$ (respectively, $gr((ZD(R))^c) \in \{3, \infty\}$) and moreover, we characterize R such that $gr((TD(R))^c) = \infty$ (respectively, $gr((ZD(R))^c) = \infty$). In Section 4, we characterize R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) is complemented. In Section 5, we assume that $|R^*| \geq 2$ and $(TD(R))^c$ is connected (respectively, $|Z(R)^*| \geq 2$ and $(ZD(R))^c$ is connected) and we characterize R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) admits at least one cut edge.

2. PRELIMINARIES

Let A be a ring and let $n \in \mathbb{N}$. Let $R = A \times \cdots \times A$ (n times). In this section, we discuss some preliminary results that are needed for proving the main results of this paper. For any ring T , we denote the group of units of T by $U(T)$.

Let $G = (V, E)$ be a graph. If each edge of G is an edge of a triangle in G , then it is clear that no vertex of G admits a complement in G . Moreover, if G is connected, then G does not admit any cut edge by [8, Theorem 3.2.7]. As we are interested in characterizing R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) is complemented, we first determine R such that each edge of $(TD(R))^c$ (respectively, $(ZD(R))^c$) is an edge of a triangle in $(TD(R))^c$ (respectively, in $(ZD(R))^c$).

Proposition 2.1. *Let A be a ring. Then $(TD(A))^c$ admits at least one edge if and only if $|A^*| \geq 2$. If $|A^*| \geq 2$ and if A is not ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, then each edge of $(TD(A))^c$ is an edge of a triangle in $(TD(A))^c$.*

Proof. If $|A^*| = 1$, then $(TD(A))^c$ does not have any edge. Assume that $|A^*| \geq 2$. For any $a, b \in A$, $a \cdot b = ab$. Let $a \in A^* \setminus \{1\}$. Then $1 \cdot a = a \neq 0$, so $1 - a$ is an edge of $(TD(A))^c$. Assume that $|A^*| \geq 2$ and A is not ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $x - y$ be an edge of $(TD(A))^c$. If $x \neq 1$ and $y \neq 1$, then it is clear that $x - y - 1 - x$ is a cycle of length 3 in $(TD(A))^c$. If exactly one of x and y equals 1, then without loss of generality, we can assume that $y = 1$. By assumption, $|A^*| \geq 3$. Hence, there exists $a \in A^* \setminus \{x, 1\}$. If $xa \neq 0$, then $x - 1 - a - x$ is a cycle of length 3 in $(TD(A))^c$. Suppose that $xa = 0$. Then $x \notin U(A)$ and it is clear that $1 - x \in A^* \setminus \{1, x\}$. If $x - x^2 \neq 0$, then $x - 1 - (1 - x) - x$ is a cycle of length 3 in $(TD(A))^c$. Suppose that $x = x^2$. Then x is a nontrivial idempotent element of A . Let $f : A \rightarrow Ax \times A(1 - x)$ be defined by $f(c) = (cx, c(1 - x))$. Then f is an isomorphism of rings. Since $A \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings by assumption, it follows that either $|Ax| > 2$ or $|A(1 - x)| > 2$. If $|Ax| > 2$, then there exists $b \in A$ such that $bx \notin \{0, x\}$. In such a case, it is clear that $bx \in A^* \setminus \{1, x\}$ and $x - 1 - bx - x$ is a cycle of length 3 in $(TD(A))^c$. If $|A(1 - x)| > 2$, then there exists $c \in A$ such that $c(1 - x) \notin \{0, 1 - x\}$. Let $z = x + c(1 - x)$. Then it is not hard to verify that $z \in A^* \setminus \{1, x\}$ and $x - 1 - z - x$ is a cycle of length 3 in $(TD(A))^c$. Thus if $|A^*| \geq 2$ and A is not ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, then each edge of $(TD(A))^c$ is an edge of a triangle in $(TD(A))^c$. $\square$

Let A be a ring and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). If A is not ring-isomorphic to any of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, then we prove in Proposition 2.3 that any edge of $(TD(R))^c$ (respectively, $(ZD(R))^c$) is an edge of a triangle in $(TD(R))^c$ (respectively, in $(ZD(R))^c$). We use the following lemma in its proof.

Lemma 2.2. *Let A be a ring such that $|A^*| \geq 3$. Let $a, b \in A^*$ be such that $ab \neq 0$. If $A \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings, then there exists $c \in A^*$ with $c \neq a$, $c \neq b$ such that $ca \neq 0$ and $cb \neq 0$.*

Proof. By hypothesis, $|A^*| \geq 3$. Let $a, b \in A^*$ be such that $ab \neq 0$. Assume that $A \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. We consider the following cases.

Case(1) $a = b$.

Now, $a^2 \neq 0$. If $a = 1$, then $ca \neq 0$ for any $c \in A^* \setminus \{a\}$. Assume that $a \neq 1$. Then with $c = 1$, we get that $c \in A^* \setminus \{a\}$ and $ca = a \neq 0$.

Case(2) $a \neq b$.

From $ab \neq 0$, it follows that $a - b$ is an edge of $(TD(A))^c$. Since $|A^*| \geq 3$ and $A \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings by assumption, we obtain from Proposition 2.1 that there exists $c \in A^* \setminus \{a, b\}$ such that $ca \neq 0$ and $cb \neq 0$. $\square$

Proposition 2.3. *Let A be a ring and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). If A is not ring-isomorphic to any of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, then any edge of $(TD(R))^c$ (respectively, $(ZD(R))^c$) is an edge of a triangle in $(TD(R))^c$ (respectively, in $(ZD(R))^c$).*

Proof. Assume that $n \in \mathbb{N} \setminus \{1\}$ and A is not ring-isomorphic to any of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$. As $(1, 0, \dots, 0)$ and $(1, 1, \dots, 1) \in R^*$ are adjacent in $(TD(R))^c$, $(TD(R))^c$ admits at least one edge. Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in R^*$ be such that $x - y$ is an edge of $(TD(R))^c$. Then $\sum_{i=1}^n x_i y_i \neq 0$. Hence, $x_i y_i \neq 0$ for at least one $i \in \{1, 2, \dots, n\}$. Fix $i \in \{1, 2, \dots, n\}$ such that $x_i y_i \neq 0$. Since $|A^*| \geq 3$ and $A \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings by assumption, there exists $z_i \in A^*$, $z_i \neq x_i$, $z_i \neq y_i$ with $x_i z_i \neq 0$ and $y_i z_i \neq 0$ by Lemma 2.2. Let $w = (w_1, w_2, \dots, w_n) \in R^*$ be given by $w_i = z_i$ and $w_j = 0$ for all $j \in \{1, 2, \dots, n\} \setminus \{i\}$. Then it is clear that $w \in Z(R)^*$, $w \notin \{x, y\}$, and $x - y - w - x$ is a cycle of length 3 in $(TD(R))^c$. This shows that any edge of $(TD(R))^c$ is an edge of a triangle in $(TD(R))^c$.

As $|A^*| \geq 3$ by assumption, it is possible to find $a \in A^* \setminus \{1\}$. Since $(1, 0, \dots, 0), (a, 0, \dots, 0) \in Z(R)^*$, we obtain that $(1, 0, \dots, 0) - (a, 0, \dots, 0)$ is an edge of $(ZD(R))^c$. Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in Z(R)^*$ be such that $x - y$ is an edge of $(ZD(R))^c$. Then $x_i y_i \neq 0$ for at least one $i \in \{1, 2, \dots, n\}$. Fix $i \in \{1, 2, \dots, n\}$ such that $x_i y_i \neq 0$ and let $w \in Z(R)^*$ be as in the previous paragraph. Then $x - y - w - x$ is a cycle of length 3 in $(ZD(R))^c$. Therefore, any edge of $(ZD(R))^c$ is an edge of a triangle in $(ZD(R))^c$. $\square$

Let A, n, R be as in the beginning of this section. Let $n \geq 2$. For each $i \in \{1, 2, \dots, n\}$, let e_i denote the element of R whose i th coordinate equals 1 and j th coordinate equals 0 for all $j \in \{1, 2, 3, \dots, n\} \setminus \{i\}$.

Let $A \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$ and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). In the following propositions, we discuss some results about $(TD(R))^c$ (respectively, $(ZD(R))^c$) which are needed for our future use.

Proposition 2.4. *Let $A = \mathbb{Z}_2$ and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). Then the following statements hold:*

- (1) If $n = 2$, then $|E((TD(R))^c)| = 2$ but $(ZD(R))^c$ has no edges.
- (2) If $n \geq 3$, then any edge of $(ZD(R))^c$ is an edge of a triangle in $(ZD(R))^c$.
- (3) If $n \geq 4$, then any edge of $(TD(R))^c$ is an edge of a triangle in $(TD(R))^c$. If $n = 3$, then any edge of $(TD(R))^c$ is an edge of a cycle of length 3 or 4.

Proof. (1) Assume that $n = 2$. Then $R^* = \{(1, 0), (0, 1), (1, 1)\}$, $Z(R)^* = \{(1, 0), (0, 1)\}$, and $(TD(R))^c$ is the path of length 2 given by $(1, 0) - (1, 1) - (0, 1)$. Therefore, $|E((TD(R))^c)| = 2$. As $(1, 0)(0, 1) = (0, 0)$, it is clear that $(ZD(R))^c$ has no edges.

(2) Assume that $n \geq 3$. Then $(1, 0, 0, \dots, 0) - (1, 1, 0, \dots, 0)$ is an edge of $(ZD(R))^c$ and hence, it is an edge of $(TD(R))^c$.

Observe that $Z(R)^* = R^* \setminus \{\sum_{i=1}^n e_i\}$ and for any $x \in Z(R)^*$, there exists a nonempty proper subset S of $\{1, 2, 3, \dots, n\}$ such that $x = \sum_{i \in S} e_i$. Let $x, y \in Z(R)^*$ be such that $x - y$ is an edge of $(ZD(R))^c$. Then there exist distinct nonempty proper subsets S_1, S_2 of $\{1, 2, 3, \dots, n\}$ such that $x = \sum_{i \in S_1} e_i$ and $y = \sum_{i \in S_2} e_i$. From $x \cdot y = 1$, it follows that $|S_1 \cap S_2|$ is odd. Either $|S_1 \cap S_2| > 1$ or $|S_1 \cap S_2| = 1$. Suppose that $|S_1 \cap S_2| = 1$. Let $S_1 \cap S_2 = \{i\}$. As $S_1 \neq S_2$, it follows that at least one of S_1 and S_2 must contain at least two elements. Without loss of generality, we can assume that $|S_1| \geq 2$. As S_1 is a proper subset of $\{1, 2, 3, \dots, n\}$, there exists $j \in \{1, 2, 3, \dots, n\}$ such that $j \notin S_1$. If $S_2 = \{i\}$, then $e_i + e_j \in Z(R)^* \setminus \{x, y\}$ and $x - y - (e_i + e_j) - x$ is a cycle of length 3 in $(ZD(R))^c$. If $|S_2| \geq 2$, then $e_i \in Z(R)^* \setminus \{x, y\}$ and $x - y - e_i - x$ is a cycle of length 3 in $(ZD(R))^c$. Suppose that $|S_1 \cap S_2| > 1$. Then $|S_i| > 1$ for each $i \in \{1, 2\}$. Let $i \in S_1 \cap S_2$. Then $e_i \in Z(R)^* \setminus \{x, y\}$ and $x - y - e_i - x$ is a cycle of length 3 in $(ZD(R))^c$. This shows that any edge of $(ZD(R))^c$ is an edge of a triangle in $(ZD(R))^c$.

(3) Assume that $n \geq 4$. Let $x, y \in R^*$ be such that $x - y$ is an edge of $(TD(R))^c$. Then $x \cdot y = 1$.

Case(i) Both x and y belong to $Z(R)^*$.

Then it follows from (2) that the edge $x - y$ is an edge of a triangle in $(ZD(R))^c$, so in $(TD(R))^c$. This is also true for $n = 3$ by (2).

Case(ii) Either x or y does not belong to $Z(R)^*$.

As $R^* = Z(R)^* \cup \{\sum_{i=1}^n e_i\}$, it follows that either $x = \sum_{i=1}^n e_i$ or $y = \sum_{i=1}^n e_i$ but not both. Without loss of generality, we can assume that $x = \sum_{i=1}^n e_i$. Then $y = \sum_{i \in S} e_i$ for some nonempty proper subset S of $\{1, 2, 3, 4, \dots, n\}$. From $x \cdot y = 1$, it follows that $|S|$ is odd. If $|S| \neq 1$, then for any $i \in S$, $e_i \in Z(R)^* \setminus \{x, y\}$ and $x - y - e_i - x$ is a cycle of length 3 in $(TD(R))^c$. Suppose that $|S| = 1$. Then $y = e_i$ for some $i \in \{1, 2, 3, 4, \dots, n\}$. Let $j, k \in \{1, 2, 3, 4, \dots, n\} \setminus \{i\}$ with $j \neq k$. Then $z = e_i + e_j + e_k \in Z(R)^* \setminus \{x, y\}$ and from $y \cdot z = x \cdot z = 1$, it follows that $x - y - z - x$ is a cycle of length 3 in $(TD(R))^c$. Thus if $n \geq 4$, then any edge of $(TD(R))^c$ is an edge of a triangle in $(TD(R))^c$.

Let $n = 3$. From $y \in Z(R)^*$ and $x \cdot y = 1$, it follows that $y = e_i$ for some $i \in \{1, 2, 3\}$. Let $z \in Z(R)^* \setminus \{x, y\}$ be such that $x - z$ is an edge of $(TD(R))^c$. Then $x \cdot z = 1$, so $z = e_j$ for some $j \in \{1, 2, 3\} \setminus \{i\}$. As $y \cdot z = 0$, y and z are not adjacent in $(TD(R))^c$. Thus the edges $(\sum_{i=1}^3 e_i) - e_1$, $(\sum_{i=1}^3 e_i) - e_2$, and $(\sum_{i=1}^3 e_i) - e_3$ are not edges of any triangle in $(TD(R))^c$. Let $i \in \{1, 2, 3\}$. Then for any $j \in \{1, 2, 3\} \setminus \{i\}$, $x = (\sum_{k=1}^3 e_k) - e_i - (e_i + e_j) - e_j - x$ is a cycle of length 4 in $(TD(R))^c$. This shows that any edge of $(TD(R))^c$ is an edge of a cycle of length 3 or 4 in $(TD(R))^c$. $\square$

Proposition 2.5. *Let $A = \mathbb{Z}_3$ and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). Then the following statements hold:*

- (1) *If $n = 2$, then any edge of $(TD(R))^c$ is an edge of a triangle in $(TD(R))^c$ and $(ZD(R))^c$ does not contain any cycle.*
- (2) *If $n \geq 3$, then any edge of $(TD(R))^c$ (respectively, $(ZD(R))^c$) is an edge of a triangle in $(TD(R))^c$ (respectively, in $(ZD(R))^c$).*

Proof. (1) Assume that $n = 2$. As $(2, 2)$ and $(1, 1)$ are adjacent in $(TD(R))^c$, we get that $(TD(R))^c$ admits at least one edge. Let $V_1 = \{(1, 0), (2, 0)\}$ and let $V_2 = \{(0, 1), (0, 2)\}$. Observe that $ZD(R) = \Gamma(R)$ is a complete bipartite graph with vertex partition $Z(R)^* = V_1 \cup V_2$, so $(ZD(R))^c = (\Gamma(R))^c$ has exactly two components g_1, g_2 , where g_i is the subgraph of $(ZD(R))^c$ induced by V_i for each $i \in \{1, 2\}$ and g_i is complete with two vertices for each $i \in \{1, 2\}$. Hence, it follows that $(ZD(R))^c$ does not contain any cycle.

Let $x = (x_1, x_2), y = (y_1, y_2) \in R^*$ be such that $x - y$ is an edge of $(TD(R))^c$. Then $x \cdot y = \sum_{i=1}^2 x_i y_i \neq 0$. As $R^* = Z(R)^* \cup U(R)$, it follows that either $x, y \in Z(R)^*$ or $x, y \in U(R)$ or exactly one of x and y belongs to $Z(R)^*$. Suppose that both $x, y \in Z(R)^*$. Then either both $x, y \in V_1$ or both $x, y \in V_2$. In either of these cases, it is clear that $x - y - (2, 1) - x$ is a cycle of length 3 in $(TD(R))^c$. Suppose that both $x, y \in U(R)$. Observe that $x_1 y_1, x_2 y_2, x \cdot y \in \{1, 2\}$. It is not hard to verify that the edge $x - y$ is either $(1, 1) - (2, 2)$ or $(1, 2) - (2, 1)$. Observe that $(1, 1) - (2, 2) - (2, 0) - (1, 1)$ is a cycle of length 3 in $(TD(R))^c$ and $(1, 2) - (2, 1) - (2, 0) - (1, 2)$ is a cycle of length 3 in $(TD(R))^c$. If exactly one of x and y belongs to $U(R)$, then without loss of generality, we can assume that $x \in U(R)$ and $y \in Z(R)^*$. If $y \in V_1$, then for $y' \in V_1 \setminus \{y\}$, $x - y - y' - x$ is a cycle of length 3 in $(TD(R))^c$. Similarly, if $y \in V_2$, then for $y' \in V_2 \setminus \{y\}$, it follows that $x - y - y' - x$ is a cycle of length 3 in $(TD(R))^c$. This shows that any edge of $(TD(R))^c$ is an edge of a triangle in $(TD(R))^c$.

(2) Let $n \geq 3$. Observe that $(TD(R))^c$ (respectively, $(ZD(R))^c$) admits at least one edge. Let $x = (x_1, x_2, x_3, \dots, x_n), y = (y_1, y_2, y_3, \dots, y_n) \in R^*$ be such that $x - y$ is an edge of $(TD(R))^c$. Then $x \cdot y = \sum_{i=1}^n x_i y_i \neq 0$. Hence, there exists at least one $i \in \{1, 2, 3, \dots, n\}$ such that $x_i y_i \neq 0$. Fix $i \in \{1, 2, 3, \dots, n\}$ such that $x_i y_i \neq 0$. As $R^* = Z(R)^* \cup U(R)$, either both x and y belong to $Z(R)^*$ or both x and y belong to $U(R)$ or exactly one of x and y belongs to $U(R)$.

Suppose that both $x, y \in Z(R)^*$. Then at least one coordinate of x (respectively, y) equals 0. Either there exists $j \in \{1, 2, 3, \dots, n\}$ such that $x_j = y_j = 0$ or there exist $j, k \in \{1, 2, 3, \dots, n\}$ such that $x_j = 0, y_j \neq 0, y_k = 0$, and $x_k \neq 0$. If there exists $j \in \{1, 2, 3, \dots, n\}$ such that $x_j = y_j = 0$, then $j \neq i$ and with $z = e_i + e_j$, we get that $z \in Z(R)^* \setminus \{x, y\}$ and $x - y - z - x$ is a cycle of length 3 in $(ZD(R))^c$ and hence, in $(TD(R))^c$. If there exist $j, k \in \{1, 2, 3, \dots, n\}$ such that $x_j = 0, y_j \neq 0, y_k = 0$, and $x_k \neq 0$. Then $j \neq k, j \neq i$, and $k \neq i$. Then with $z = e_j + e_k$, we obtain that $z \in Z(R)^* \setminus \{x, y\}$ and $x - y - z - x$ is a cycle of length 3 in $(ZD(R))^c$ and hence, in $(TD(R))^c$. Suppose that $x, y \in U(R)$. Then $x - y - e_1 - x$ is a cycle of length 3 in $(TD(R))^c$. Suppose that exactly one of x and y belongs to $U(R)$. We can assume without loss of generality that $x \in U(R)$ and $y \in Z(R)^*$. There exists $j \in \{1, 2, 3, \dots, n\} \setminus \{i\}$ such that $y_j = 0$. Define $z = (z_1, z_2, z_3, \dots, z_n) \in R^*$ as follows: $z_i = x_i, z_j = x_j$ and $z_t = 0$ for all $t \in \{1, 2, 3, \dots, n\} \setminus \{i, j\}$. Then it is clear that $z \in Z(R)^* \setminus \{x, y\}$ and as $x_i^2 + x_j^2 \neq 0, x_i y_i \neq 0$, we get that $x - y - z - x$ is a cycle of length 3 in $(TD(R))^c$.

Thus if $n \geq 3$, it follows from the proof given above that any edge of $(TD(R))^c$ (respectively, $(ZD(R))^c$) is an edge of a triangle in $(TD(R))^c$ (respectively, in $(ZD(R))^c$). $\square$

Proposition 2.6. *Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2$ and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \dots \times A$ (n times). Then any edge of $(TD(R))^c$ (respectively, $(ZD(R))^c$) is an edge of a triangle in $(TD(R))^c$ (respectively, in $(ZD(R))^c$).*

Proof. By hypothesis, $n \in \mathbb{N}$ with $n \geq 2$ and $R = A \times A \times \dots \times A$ (n times), where $A = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2 \times \mathbb{Z}_2$ ($2n$ times) as rings. Since $2n \geq 4$, it is clear that $(ZD(R))^c$ admits at least one edge, so does $(TD(R))^c$. It follows from Proposition 2.4(2) and (3) that any edge of $(TD(R))^c$ (respectively, $(ZD(R))^c$) is an edge of a triangle in $(TD(R))^c$ (respectively, in $(ZD(R))^c$). $\square$

3. ON THE GIRTH OF $(TD(R))^c$ (RESPECTIVELY, $(ZD(R))^c$)

Let A be a ring and let $n \in \mathbb{N}$. Let $R = A \times \dots \times A$ (n times). In this section, we determine $gr((TD(R))^c)$ (respectively, $gr((ZD(R))^c)$) and moreover, we characterize R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) does not contain any cycle. First, we consider $n = 1$. In such a case, $R = A$.

Proposition 3.1. *Let A be a ring. Then $gr((TD(A))^c) \in \{3, \infty\}$ and $gr((TD(A))^c) = \infty$ if and only if A is ring-isomorphic to one of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$.*

Proof. We have noted in the proof of Proposition 2.1 that $(TD(A))^c$ admits at least one edge if and only if $|A^*| \geq 2$. Assume that $|A^*| \geq 2$. Let $x - y$ be an edge of $(TD(A))^c$. If A is not ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, then we know from Proposition 2.1 that the edge $x - y$ is an edge of a triangle in $(TD(A))^c$. Hence, $gr((TD(A))^c) = 3$. As $(TD(\mathbb{Z}_2))^c$ consists

of a single vertex, $(TD(\mathbb{Z}_3))^c$ consists of the single edge $1 - 2$ and $(TD(\mathbb{Z}_2 \times \mathbb{Z}_2))^c$ is the path of length 2 given by $(1, 0) - (1, 1) - (0, 1)$, we obtain that if A is ring-isomorphic to one of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, then $gr((TD(A))^c) = \infty$. This proves that $gr((TD(A))^c) \in \{3, \infty\}$ and $gr((TD(A))^c) = \infty$ if and only if A is ring-isomorphic to one of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$. $\square$

Let T be a ring. Let $n \in \mathbb{N}$ and let $t_1, \dots, t_n \in T$. We denote the ideal of T generated by $t_1, \dots, t_n$ by $(t_1, \dots, t_n)$. We denote by $T[X]$ (respectively, $T[X, Y]$) the polynomial ring in one variable X (respectively, in two variables X, Y) over T .

Let A be a ring such that $Z(A)^* \neq \emptyset$. If $|Z(A)^*| = 1$, then $gr((\Gamma(A))^c) = \infty$ and A is ring-isomorphic to either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[X]}{(X^2)}$ by [6, Example 2.1(a)]. We assume that $|Z(A)^*| \geq 2$ and in the following proposition, we list known results about $gr((\Gamma(A))^c)$ from [16, Section 3]. First, it is useful to recall the following definitions and notations from commutative ring theory.

Let T be a ring. We denote the set of all prime ideals of T by $Spec(T)$. Let I be an ideal of T with $I \neq T$. Recall that $\mathfrak{p} \in Spec(T)$ is said to be a *maximal N-prime* of I if $\mathfrak{p}$ is maximal with respect to being contained in $Z_T(\frac{T}{I}) = \{t \in T \mid tx \in I \text{ for some } x \in T \setminus I\}$ [10]. Thus $\mathfrak{p} \in Spec(T)$ is a maximal N-prime of (0) if $\mathfrak{p}$ is maximal with respect to being contained in $Z(T)$. We denote the set of all maximal N-primes of (0) in T by $MNP(T)$. It is clear that $S = T \setminus Z(T)$ is a multiplicatively closed subset of T . If I is an ideal of T with $I \cap S = \emptyset$, then it follows from Zorn's lemma and [11, Theorem 1] that there exists $\mathfrak{p} \in MNP(T)$ with $I \subseteq \mathfrak{p}$. In particular, if $x \in Z(T)$, then $Tx \cap S = \emptyset$, so there exists $\mathfrak{p} \in MNP(T)$ such that $x \in \mathfrak{p}$. It follows from the above argument that if $MNP(T) = \{\mathfrak{p}_\alpha\}_{\alpha \in \Lambda}$, then $Z(T) = \bigcup_{\alpha \in \Lambda} \mathfrak{p}_\alpha$. Observe that $Z(T)$ is an ideal of T if and only if $|MNP(T)| = 1$. Let I be an ideal of T with $I \neq T$. Recall that $\mathfrak{p} \in Spec(T)$ is said to be an *associated prime of I in the sense of Bourbaki* if $\mathfrak{p} = (I :_T x)$ for some $x \in T$ [9]. In such a case, we say that $\mathfrak{p}$ is a B-prime of I . For a prime number p and $n \in \mathbb{N}$, we denote the finite field containing exactly p^n elements by $\mathbb{F}_{p^n}$.

Proposition 3.2. *Let A be a ring with $|Z(A)^*| \geq 2$. Then $gr((\Gamma(A))^c) \in \{3, \infty\}$ and moreover, $gr((\Gamma(A))^c) = \infty$ if and only if either $MNP(A) = \{\mathfrak{p}\}$ with $|\mathfrak{p}| \geq 5$ and $\mathfrak{p}^2 = (0)$ or A is ring-isomorphic to one of the rings from the collection $\{\mathbb{Z}_9, \frac{\mathbb{Z}_3[X]}{(X^2)}, \mathbb{Z}_8, \frac{\mathbb{Z}_2[X]}{(X^3)}, \frac{\mathbb{Z}_2[X, Y]}{(X^2, XY, Y^2)}, \frac{\mathbb{F}_4[X]}{(X^2)}\} \cup \{\mathbb{F}_p \times \mathbb{F}_q \mid p, q \in \{2, 3\}\}$.*

Proof. By hypothesis, A is a ring with $|Z(A)^*| \geq 2$. Either $(\Gamma(A))^c$ is connected or $(\Gamma(A))^c$ is not connected. If $(\Gamma(A))^c$ is connected, then $gr((\Gamma(A))^c) = 3$ by [16, Theorem 3.6]. We know from [15, Theorem 1.1] that $(\Gamma(A))^c$ is not connected if and only if either $Z(A) = \mathfrak{p}$ is an ideal of A with $\mathfrak{p}$ is a B-prime of (0) in A or $|MNP(A)| = 2$ with $\bigcap_{i=1}^2 \mathfrak{p}_i = (0)$, where $MNP(A) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Assume that $(\Gamma(A))^c$ is not connected. Let $MNP(A) = \{\mathfrak{p}\}$. Then $gr((\Gamma(A))^c) \in \{3, \infty\}$ and $gr((\Gamma(A))^c) = \infty$ if and only if either $\mathfrak{p}^2 = (0)$ or $|\mathfrak{p}| \leq 4$

by [16, Proposition 3.7]. If $|\mathfrak{p}| \leq 4$, then $|\mathfrak{p}| \in \{3, 4\}$, since $|Z(A)^*| \geq 2$ by assumption. If $|\mathfrak{p}| = 3$, then A is ring-isomorphic to either $\mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[X]}{(X^2)}$, see [4, p.17]. If $|\mathfrak{p}| = 4$, then A is ring-isomorphic to one of the rings from $\{\mathbb{Z}_8, \frac{\mathbb{Z}_2[X]}{(X^3)}, \frac{\mathbb{F}_4[X]}{(X^2)}, \frac{\mathbb{Z}_2[X, Y]}{(X^2, XY, Y^2)}\}$, see [4, p.17]. Assume that $\bigcap_{i=1}^2 \mathfrak{p}_i = (0)$, where $MNP(A) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Then it follows from [16, Remark 3.9] that $gr((\Gamma(A))^c) \in \{3, \infty\}$ and $gr((\Gamma(A))^c) = \infty$ if and only if A is ring-isomorphic to one of the rings from the collection $\{\mathbb{F}_p \times \mathbb{F}_q \mid p, q \in \{2, 3\}\}$.

From the above discussion, it is clear that the proof of this proposition is complete. $\square$

Let A, n, R be as in the beginning of this section. We assume $n \geq 2$ and in the following corollaries, we determine $gr((TD(R))^c)$ (respectively, $gr((ZD(R))^c)$).

Corollary 3.3. *Let A be a ring and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). If A is not ring-isomorphic to any of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, then $gr((TD(R))^c) = 3 = gr((ZD(R))^c)$.*

Proof. It is clear that $(ZD(R))^c$ admits at least one edge, so does $(TD(R))^c$. We know from Proposition 2.3 that any edge of $(TD(R))^c$ (respectively, $(ZD(R))^c$) is an edge of a triangle in $(TD(R))^c$ (respectively, in $(ZD(R))^c$). Hence, $gr((TD(R))^c) = 3 = gr((ZD(R))^c)$. $\square$

We next consider $A \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, $n \in \mathbb{N} \setminus \{1\}$, $R = A \times A \times \cdots \times A$ (n times), and determine $gr((TD(R))^c)$ (respectively, $gr((ZD(R))^c)$).

Corollary 3.4. *Let $A = \mathbb{Z}_2$ and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). Then $gr((TD(R))^c) = gr((ZD(R))^c) \in \{3, \infty\}$ and $gr((TD(R))^c) = gr((ZD(R))^c) = \infty$ if and only if $n = 2$.*

Proof. If $n = 2$, then by Proposition 2.4(1), $|E((TD(R))^c)| = 2$ and $(ZD(R))^c$ has no edges. Therefore, $gr((TD(R))^c) = gr((ZD(R))^c) = \infty$. If $n \geq 3$, then it is clear that $(ZD(R))^c$ admits at least one edge and it follows from Proposition 2.4(2) that $gr((ZD(R))^c) = 3$ and hence, $gr((TD(R))^c) = 3$. $\square$

Corollary 3.5. *Let $A = \mathbb{Z}_3$ and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). Then the following statements hold:*

- (1) $gr((TD(R))^c) = 3$.
- (2) $gr((ZD(R))^c) \in \{3, \infty\}$ and $gr((ZD(R))^c) = \infty$ if and only if $n = 2$.

Proof. (1) It is clear that $(TD(R))^c$ admits at least one edge and it follows from Proposition 2.5(1) and (2) that $gr((TD(R))^c) = 3$.

(2) If $n = 2$, then it follows from Proposition 2.5(1) that $gr((ZD(R))^c) = \infty$. If $n \geq 3$, then it is clear that $(ZD(R))^c$ admits at least one edge and we obtain from Proposition 2.5(2) that $gr((ZD(R))^c) = 3$. $\square$

Corollary 3.6. *Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2$ and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). Then $gr((TD(R))^c) = 3 = gr((ZD(R))^c)$.*

Proof. It is clear that $(ZD(R))^c$ admits at least one edge and it follows from Proposition 2.6 that $gr((TD(R))^c) = 3 = gr((ZD(R))^c)$. $\square$

4. WHEN IS $(TD(R))^c$ (RESPECTIVELY, $(ZD(R))^c$) COMPLEMENTED?

Let A be a ring and let $n \in \mathbb{N}$. Let $R = A \times \cdots \times A$ (n times). The aim of this section is to characterize R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) is complemented. First, we consider $n = 1$.

Proposition 4.1. *Let A be a ring such that $|A^*| \geq 2$. Then $(TD(A))^c$ is complemented if and only if A is ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. If A is not ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, then each edge of $(TD(A))^c$ is an edge of a triangle in $(TD(A))^c$ by Proposition 2.1, so no vertex of $(TD(A))^c$ admits a complement in $(TD(A))^c$. Since $(TD(\mathbb{Z}_3))^c$ is the path of length 1 given by $1 - 2$, we obtain that $1 \perp 2$ in $(TD(\mathbb{Z}_3))^c$. Hence, $(TD(\mathbb{Z}_3))^c$ is complemented. It is clear that $(TD(\mathbb{Z}_2 \times \mathbb{Z}_2))^c$ is the path of length 2 given by $(1, 0) - (1, 1) - (0, 1)$. As $(1, 0) \perp (1, 1)$ and $(0, 1) \perp (1, 1)$ in $(TD(\mathbb{Z}_2 \times \mathbb{Z}_2))^c$, it follows that $(TD(\mathbb{Z}_2 \times \mathbb{Z}_2))^c$ is complemented. This shows that $(TD(A))^c$ is complemented if and only if A is ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. $\square$

Remark 4.2. Let A be a ring such that $|Z(A)^*| \geq 2$. Observe that $(ZD(A))^c = (\Gamma(A))^c$. Hence, from the results proved in [17], we obtain that $(\Gamma(A))^c$ is complemented if and only if A is ring-isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Let A be a ring and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). In the following proposition, we characterize R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) is complemented.

Proposition 4.3. *Let A be a ring and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). Then $(TD(R))^c$ (respectively, $(ZD(R))^c$) is complemented if and only if R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (respectively, $\mathbb{Z}_3 \times \mathbb{Z}_3$).*

Proof. If A is not ring-isomorphic to any of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, then we obtain from Proposition 2.3 that any edge of $(TD(R))^c$ (respectively, $(ZD(R))^c$) is an edge of a triangle in $(TD(R))^c$ (respectively, in $(ZD(R))^c$). Hence, no vertex of $(TD(R))^c$ (respectively, $(ZD(R))^c$) admits a complement in $(TD(R))^c$ (respectively, in $(ZD(R))^c$).

Assume that $A \cong \mathbb{Z}_2$ as rings. Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times). Then $R \cong T$ as rings. So, $(TD(R))^c$ (respectively, $(ZD(R))^c$) is complemented if and only if $(TD(T))^c$ (respectively, $(ZD(T))^c$) is complemented. If $n = 2$, then we have noted in the proof of Proposition 4.1 that

$(TD(T))^c$ is complemented. If $n = 2$, then it is verified in the proof of Proposition 2.4(1) that $(ZD(T))^c$ has no edges, so no vertex of $(ZD(T))^c$ admits a complement in $(ZD(T))^c$. If $n \geq 3$, then we know from Proposition 2.4(2) that any edge of $(ZD(T))^c$ is an edge of a triangle in $(ZD(T))^c$, so no vertex of $(ZD(T))^c$ admits a complement in $(ZD(T))^c$. If $n \geq 4$, then each edge of $(TD(T))^c$ is an edge of a triangle in $(TD(T))^c$ by Proposition 2.4(3), so no vertex of $(TD(T))^c$ admits a complement in $(TD(T))^c$. Let $n = 3$ and let $a = (1, 1, 0)$. If $b \in T^*$ is such that $a - b$ is an edge of $(TD(T))^c$, then $b \in \{(1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1)\}$. The edge $a - b$ of $(ZD(T))^c$ is an edge of a triangle in $(ZD(T))^c$ by Proposition 2.4(2) and hence, an edge of a triangle in $(TD(T))^c$. This shows that a does not admit any complement in $(TD(T))^c$, so $(TD(T))^c$ is not complemented. Thus if A is ring-isomorphic to $\mathbb{Z}_2$, then $(TD(R))^c$ is complemented if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings and $(ZD(R))^c$ is not complemented.

Assume that $A \cong \mathbb{Z}_3$ as rings. Let $T_1 = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$ (n times). Then R is ring-isomorphic to T_1 . So, $(TD(R))^c$ (respectively, $(ZD(R))^c$) is complemented if and only if $(TD(T_1))^c$ (respectively, $(ZD(T_1))^c$) is complemented. If $n \geq 3$, then any edge of $(TD(T_1))^c$ (respectively, $(ZD(T_1))^c$) is an edge of a triangle in $(TD(T_1))^c$ (respectively, in $(ZD(T_1))^c$) by Proposition 2.5(2). Hence, no vertex of $(TD(T_1))^c$ (respectively, $(ZD(T_1))^c$) admits a complement in $(TD(T_1))^c$ (respectively, in $(ZD(T_1))^c$). If $n = 2$, then any edge of $(TD(T_1))^c$ is an edge of a triangle in $(TD(T_1))^c$ by Proposition 2.5(1), so no vertex of $(TD(T_1))^c$ admits a complement in $(TD(T_1))^c$. Thus if A is ring-isomorphic to $\mathbb{Z}_3$, then no vertex of $(TD(R))^c$ admits a complement in $(TD(R))^c$. If $n = 2$, then we know from the proof of Proposition 2.5(1) that $(ZD(T_1))^c$ has exactly two components g_1, g_2 , where g_1 is the subgraph of $(ZD(T_1))^c$ induced by $\{(1, 0), (2, 0)\}$, g_2 is the subgraph of $(ZD(T_1))^c$ induced by $\{(0, 1), (0, 2)\}$, and g_i is complete for each $i \in \{1, 2\}$. Observe that $(1, 0) \perp (2, 0)$ and $(0, 1) \perp (0, 2)$ in $(ZD(T_1))^c$. Thus if $n = 2$, then $(ZD(T_1))^c$ is complemented. Thus if A is ring-isomorphic to $\mathbb{Z}_3$, then $(TD(R))^c$ is not complemented and $(ZD(R))^c$ is complemented if and only if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Assume that $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings. Let $T_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_2$ ($2n$ times). Then R is ring-isomorphic to T_2 . As any edge of $(TD(T_2))^c$ (respectively, $(ZD(T_2))^c$) is an edge of a triangle in $(TD(T_2))^c$ (respectively, in $(ZD(T_2))^c$) by Proposition 2.6, we obtain that no vertex of $(TD(T_2))^c$ (respectively, $(ZD(T_2))^c$) admits a complement in $(TD(T_2))^c$ (respectively, in $(ZD(T_2))^c$). Thus if A is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $(TD(R))^c$ (respectively, $(ZD(R))^c$) is not complemented.

Therefore, $(TD(R))^c$ (respectively, $(ZD(R))^c$) is complemented if and only if R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (respectively, $\mathbb{Z}_3 \times \mathbb{Z}_3$). $\square$

5. CHARACTERIZATION OF R SUCH THAT $(TD(R))^c$ (RESPECTIVELY, $(ZD(R))^c$) IS
CONNECTED AND ADMITS AT LEAST ONE CUT EDGE

Let A be a ring and let $n \in \mathbb{N}$. Let $R = A \times \cdots \times A$ (n times). The aim of this section is to characterize R such that $(TD(R))^c$ (respectively, $(ZD(R))^c$) is connected and admits at least one cut edge. We use the following remark in our discussion.

Remark 5.1. Let $G = (V, E)$ be a connected graph. If G admits a cut edge, then it follows that $|V| \geq 2$. If $|V| \geq 3$ and if G admits a cut edge, then G admits a cut vertex.

First, we consider $n = 1$. Then $R = A$. In view of Remark 5.1, we assume that $|A^*| \geq 2$ and in the following proposition, we characterize A such that $(TD(A))^c$ is connected and admits at least one cut edge.

Proposition 5.2. *Let A be a ring with $|A^*| \geq 2$. Then $(TD(A))^c$ is connected. Moreover, it admits at least one cut edge if and only if A is ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. By hypothesis, $|A^*| \geq 2$. Hence, $(TD(A))^c$ is connected by [19, Proposition 2.1(1)].

If A is not ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, then any edge of $(TD(A))^c$ is an edge of a triangle in $(TD(A))^c$ by Proposition 2.1. Thus if $(TD(A))^c$ admits a cut edge, then A is ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Assume that A is ring-isomorphic to $\mathbb{Z}_3$. As $(TD(\mathbb{Z}_3))^c$ is the path of length 1 given by $1 - 2$, we get that the edge $1 - 2$ is a cut edge of $(TD(\mathbb{Z}_3))^c$.

Assume that A is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. As $(TD(\mathbb{Z}_2 \times \mathbb{Z}_2))^c$ is the path of length 2 given by $(1, 0) - (1, 1) - (0, 1)$, we obtain that both the edges $(1, 0) - (1, 1)$ and $(1, 1) - (0, 1)$ are cut edges of $(TD(\mathbb{Z}_2 \times \mathbb{Z}_2))^c$.

This proves that for a ring A with $|A^*| \geq 2$, $(TD(A))^c$ is connected, and it admits at least one cut edge if and only if A is ring-isomorphic to either $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. $\square$

Let A be a ring such that $|Z(A)^*| \geq 2$. Note that $(ZD(A))^c = (\Gamma(A))^c$. In Proposition 5.4, we characterize A such that $(\Gamma(A))^c$ is connected and admits at least one cut edge. The following remark is used in its proof.

Remark 5.3. Let A be a ring such that $|Z(A)^*| \geq 2$. It is known that $(\Gamma(A))^c$ is connected if and only if one of the following conditions is satisfied: (1) $|MNP(A)| = 1$ and the unique member $\mathfrak{p}$ of $MNP(A)$ is not a B-prime of (0) in A ; (2) $|MNP(A)| = 2$ with $\bigcap_{i=1}^2 \mathfrak{p}_i \neq (0)$, where $MNP(A) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$; (3) $|MNP(A)| \geq 3$ [15, Theorem 1.1]. Thus if $(\Gamma(A))^c$ is connected, then it is not hard to verify that $|Z(A)^*| \geq 3$. Hence, if $(\Gamma(A))^c$ is connected and admits at least one cut edge, then it admits a cut vertex by Remark 5.1.

Proposition 5.4. *Let A be a ring such that $|Z(A)^*| \geq 2$. Then $(\Gamma(A))^c$ is connected and admits at least one cut edge if and only if A is ring-isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]} \times \mathbb{Z}_2$.*

Proof. Assume that $|Z(A)^*| \geq 2$ and $(\Gamma(A))^c$ is connected and admits at least one cut edge. Then $|Z(A)^*| \geq 3$ and $(\Gamma(A))^c$ admits a cut vertex by Remark 5.3. Therefore, A is ring-isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]} \times \mathbb{Z}_2$ by [18, Theorem 5.1].

Assume that A is ring-isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]} \times \mathbb{Z}_2$. Let us denote the ring $\mathbb{Z}_4 \times \mathbb{Z}_2$ by T . Then $MNP(T) = \{\mathfrak{P}_1 = 2\mathbb{Z}_4 \times \mathbb{Z}_2, \mathfrak{P}_2 = \mathbb{Z}_4 \times (0)\}$ and $Z(T)^* = \{(2,0), (2,1), (0,1), (1,0), (3,0)\}$. As $(2,0) \in \bigcap_{i=1}^2 \mathfrak{P}_i$, it follows from [15, Theorem 1.1] that $(\Gamma(T))^c$ is connected. It is clear that $(0,1)$ is adjacent to $(2,1)$ in $(\Gamma(T))^c$ but it is not adjacent to any other vertex in $(\Gamma(T))^c$. Hence, the edge $(0,1) - (2,1)$ is a cut edge of $(\Gamma(T))^c$. Let us denote the ring $\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]} \times \mathbb{Z}_2$ by T_1 . Then $MNP(T_1) = \{\mathfrak{P}'_1 = (X + X^2\mathbb{Z}_2[X]) \times \mathbb{Z}_2, \mathfrak{P}'_2 = \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]} \times (0)\}$ and $Z(T_1)^* = \{(X + X^2\mathbb{Z}_2[X], 0), (X + X^2\mathbb{Z}_2[X], 1), (0 + X^2\mathbb{Z}_2[X], 1), (1 + X^2\mathbb{Z}_2[X], 0), (1 + X + X^2\mathbb{Z}_2[X], 0)\}$. Since $(X + X^2\mathbb{Z}_2[X], 0) \in \bigcap_{i=1}^2 \mathfrak{P}'_i$, it follows that $(\Gamma(T_1))^c$ is connected. As $(0 + X^2\mathbb{Z}_2[X], 1)$ is adjacent to $(X + X^2\mathbb{Z}_2[X], 1)$ in $(\Gamma(T_1))^c$ but it is not adjacent to any other vertex in $(\Gamma(T_1))^c$, we obtain that the edge $(0 + X^2\mathbb{Z}_2[X], 1) - (X + X^2\mathbb{Z}_2[X], 1)$ is a cut edge of $(\Gamma(T_1))^c$. Since A is ring-isomorphic to either T or T_1 , it follows that $(\Gamma(A))^c$ is connected and admits at least one cut edge. $\square$

Let A, n, R be as mentioned in the beginning of this section. We consider $n \neq 1$ and in Theorem 5.5, we observe that $(TD(R))^c$ is connected and we characterize R such that $(TD(R))^c$ admits at least one cut edge and we show that if $(ZD(R))^c$ is connected, then it does not admit any cut edge.

Theorem 5.5. *Let A be a ring and let $n \in \mathbb{N} \setminus \{1\}$. Let $R = A \times A \times \cdots \times A$ (n times). Then the following statements hold:*

- (1) *$(TD(R))^c$ is connected. It admits at least one cut edge if and only if R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*
- (2) *If $n = 2$, then $(ZD(R))^c$ is connected if and only if A is not an integral domain. If $n = 2$ and A is not an integral domain, then $(ZD(R))^c$ does not admit any cut edge. If $n \geq 3$, then $(ZD(R))^c$ is connected and it does not admit any cut edge.*

Proof. (1) By [19, Proposition 2.1(3)], $(TD(R))^c$ is connected. If A is not ring-isomorphic to any of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, then any edge of $(TD(R))^c$ is an edge of a triangle in $(TD(R))^c$ by Proposition 2.3. Thus if $(TD(R))^c$ admits at least one cut edge, then A is ring-isomorphic to one of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$.

Assume that A is ring-isomorphic to $\mathbb{Z}_2$. Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times). Then R is ring-isomorphic to T . If $n = 2$, then we have noted in the proof of Proposition 5.2 that $(TD(T))^c$ has exactly two edges and both of them are cut edges of $(TD(T))^c$. Therefore, $(TD(R))^c$ admits at least one cut edge. If $n \geq 3$, then any edge of $(TD(T))^c$ is an edge of a cycle of length 3 or 4 by Proposition 2.4(3), so it does not admit any cut edge. Hence,

$(TD(R))^c$ does not admit any cut edge. Thus if A is ring-isomorphic to $\mathbb{Z}_2$, then $(TD(R))^c$ admits at least one cut edge if and only if R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Assume that A is ring-isomorphic to $\mathbb{Z}_3$. Let $T_1 = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$ (n times). Then R is ring-isomorphic to T_1 . If $n \geq 2$, then any edge of $(TD(T_1))^c$ is an edge of a triangle in $(TD(T_1))^c$ by Proposition 2.5(1) and (2). Hence, $(TD(T_1))^c$ does not admit any cut edge, so $(TD(R))^c$ does not admit any cut edge.

Assume that A is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $T_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times \mathbb{Z}_2$ ($2n$ times). Since any edge of $(TD(T_2))^c$ is an edge of a triangle in $(TD(T_2))^c$ by Proposition 2.6, $(TD(T_2))^c$ does not admit any cut edge. So, $(TD(R))^c$ does not admit any cut edge, since R is ring-isomorphic to T_2 .

Thus if $n \geq 2$, then $(TD(R))^c$ is connected, and it admits at least one cut edge if and only if R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) Let $n = 2$. Then $(ZD(R))^c$ is connected if and only if A is not an integral domain by [19, Proposition 2.4(1)]. Thus if $(ZD(R))^c$ is connected, then A cannot be ring-isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. If A is not ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then any edge of $(ZD(R))^c$ is an edge of a triangle in $(ZD(R))^c$ by Proposition 2.3 and if A is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then any edge of $(ZD(R))^c$ is an edge of a triangle in $(ZD(R))^c$ by Proposition 2.6. Therefore, if $n = 2$ and $(ZD(R))^c$ is connected, then $(ZD(R))^c$ does not admit any cut edge.

Let $n \geq 3$. Then $(ZD(R))^c$ is connected by [19, Proposition 2.4(2)]. If $(ZD(R))^c$ admits a cut edge, then A must be ring-isomorphic to one of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$ by Proposition 2.3. Assume that A is ring-isomorphic to one of the rings from the collection $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$. Then any edge of $(ZD(R))^c$ is an edge of a triangle in $(ZD(R))^c$ by Propositions 2.4(2), 2.5(2), and 2.6. Therefore, $(ZD(R))^c$ does not admit any cut edge. $\square$

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