

Research Paper

A NEW STUDY ON SEMISIMPLE MODULES APPLYING A FIXED SUBMODULE

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ABSTRACT. Let M be an R -module and T a fixed submodule of M . Then we say that M is a T -semisimple module if each submodule of M is a T -direct summand of M , i.e. for each submodule N of M , there exists a submodule K of M such that $M = N + K$ and $N \cap K$ is contained in T . We also present the definition of a T -simple module in terms of simple modules. It is shown that a module M is T -semisimple if and only if M/T is a semisimple module. We prove that any homomorphic image and each submodule of a T -semisimple module inherits the property with a suitable submodule. As expected, it is proved that M is T -semisimple if and only if M is a sum of T -simple modules.

1. INTRODUCTION

All rings in this paper are associative with identity. Likewise, unless stated otherwise, all modules will be treated as right modules. A submodule K of M is considered essential in M (denoted by $N \leq_e M$) if the intersection of K with each nonzero submodule of M is nonzero.

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Dually, a submodule L of M is regarded as small in M if M cannot be expressed as the sum of any proper submodule T of M and L . A module M is termed hollow if every proper submodule is small in M . For a module M and a submodule N , the notion $N \leq^\oplus M$ indicates that N is a direct summand of M . Undefined terminologies can be found in [3, 4, 5, 8]. A module M is called local if M includes a unique maximal submodule containing all proper submodules of M .

Semisimplicity is a central structural notion in module theory, allowing a clean analysis of modules via direct sums of simple components. Recall that an R -module M is semisimple if every submodule N of M admits a complementary submodule $K \leq M$ such that $M = N \oplus K$ ([3]), i.e., $M = N + K$ and $N \cap K = 0$. Many natural modules fail to be semisimple for controlled reasons: the obstruction to splitting may be confined to a fixed submodule, rather than appearing globally across M .

Motivated by this, we introduce a relative notion of semisimplicity controlled by a fixed submodule T of M : we say that M is T -semisimple if for every submodule N of M there exists a submodule K of M such that $M = N + K$ and $N \cap K \subseteq T$, denoted by $M = N \oplus_T K$. Thus the usual requirement $N \cap K = 0$ is relaxed, the intersection of N and its complement K being allowed inside T . When $T = 0$ this recovers ordinary semisimplicity, and when $T \neq 0$ it measures how far M is from being semisimple relative to T .

Our first main result shows that this relative notion has a transparent description via quotients: M is T -semisimple if and only if M/T is semisimple. Building on this characterization, we study how T -semisimplicity behaves under homomorphic images and submodules, introduce the corresponding notion of a T -simple module, and show that M is T -semisimple if and only if M is a sum of T -simple submodules, thereby extending the familiar structural picture of classical semisimple module theory to the relative setting.

2. T -DIRECT SUMMANDS AND T -SEMISIMPLE MODULES

In this section, we shall study a new generalization of the concept of direct summands of a module applying a fixed submodule of that module.

Definition 2.1. Let M be an R -module and T a fixed submodule of M . A submodule N of M is said to be T -direct summand of M in case there exists another submodule K such that $M = N + K$ and $N \cap K \subseteq T$. It is clear that, T and M are trivial T -direct summands of M .

Based on the definition of a T -direct summand, it is natural to define a T -indecomposable module. In this direction, a module M is T -indecomposable provided M includes no nontrivial T -direct summands.

Remark 2.2. (1) Each direct summand of an R -module M , is a T -direct summand of M .

(2) Let M be a module and T a fixed submodule of M . Then a submodule N of M is a 0-direct summand of M if and only if N is a direct summand of M .

(3) Assume T is a simple submodule of M and N a T -direct summand of M . Then $N + K = M$ and $N \cap K \subseteq T$. In this direction, $N \cap K = 0$ or $N \cap K = T$. Then, $M = N \oplus K$ or T is contained in both N and K .

(4) An indecomposable module can include a T -direct summand. For instance, consider $M = \mathbb{Z}$ as a $\mathbb{Z}$ -module. It is known that M is indecomposable. Assume $\gcd(m, n) = 1$ for $m, n \in \mathbb{N}$. Then $m\mathbb{Z}$ is a $m\mathbb{Z}$ -direct summand of M .

(5) If N is a $T = (N \cap K)$ -direct summand of M , then there exists an idempotent $e : M/T \rightarrow M/T$ with image of e is equal to N/T .

(6) Suppose $M = N + K$. Then N is a $(N \cap K)$ -direct summand of M . In particular, if a submodule N of M has a supplement K in M , then N is a $N \cap K$ -direct summand of M .

(7) Let N be a simple submodule of M . If N is a T -direct summand of M , then $N \cap K = 0$ or $K = M$. In fact, either N is a direct summand of M or N is contained in T .

(8) Suppose that M is a semisimple module with length 2. Then each submodule of M is a T -direct summand of M for each $T \leq M$.

(9) Assume N is a T -direct summand of M and $T \subseteq Y$. Then N is also a Y -direct summand of M .

Example 2.3. (1) Consider $\mathbb{Z}_{12}$ as a $\mathbb{Z}$ -module. Set $T = \{0, 4, 8\} = \langle 4 \rangle$ which is a simple submodule of M . Then by Remark 2.2(7), M has no nontrivial T -direct summands.

(2) Let $M = \mathbb{Z}_{20}$ as an $\mathbb{Z}$ -module. If we set $T = \langle 4 \rangle$ which is a simple submodule of M , we can conclude that M is a T -indecomposable $\mathbb{Z}$ -module (see Remark 2.2(7)).

Definition 2.4. Let M be a module and T a fixed submodule of M . Then we call M a T -semisimple module in case each submodule of M is a T -direct summand of M . Any semisimple module indeed is a T -semisimple module.

The following is a useful characterization of T -semisimple modules.

Theorem 2.5. Let M be a module and $T \leq M$. Then M is T -semisimple if and only if M/T is semisimple.

Proof. First, assume M is T -semisimple and $N/T \leq M/T$. By assumption, there exists a submodule K of M such that $M = N + K$ and $N \cap K \subseteq T$. Therefore, $M/T = N/T + (K+T)/T$ and $N/T \cap (K+T)/T = \frac{T+(N \cap K)}{T} = \frac{T}{T}$. Hence, M/T is a semisimple module. For the converse, let $U \leq M$. Then $(U + T)/T$ is a direct summand of M/T . Then $(U + T)/T \oplus Y/T = M/T$ and $(U + T) \cap Y = T$ for a submodule Y of M containing T . Therefore, $U + Y = M$ and $U \cap Y \subseteq T$, as desired. $\square$

Corollary 2.6. *Let M be a module and $T \leq M$. Then M is T -semisimple if and only if $\text{Soc}(M/T) = M/T$. In other words, M is not T -semisimple if and only if there exists a submodule N of M properly containing T such that $N/T \leq_e M/T$.*

Lemma 2.7. *Let M be a T -semisimple module and $T \subseteq L$. Then M is L -semisimple.*

Proof. As M is T -semisimple, M/T is semisimple by Theorem 2.5. Now, the result follows from the fact that $\frac{M}{L} \cong \frac{M/T}{L/T}$ is a semisimple module as well as a homomorphic image of M/T .

□

Lemma 2.8. *Let M be a module and $T_1, T_2, \dots$ be submodules of M . Then M is T_i -semisimple for $i \in I$ if and only if M is $\sum_{i \in I} T_i$ -semisimple.*

Proof. ($\Rightarrow$) As M is T_i -semisimple, we conclude that M/T_i is a semisimple module. Note that $M/\sum_{i \in I} T_i$ is a homomorphic image of M/T_i , since $\frac{M}{\sum_{i \in I} T_i} \cong \frac{M/T_i}{\sum_{i \in I} T_i/T_i}$. Therefore, $M/\sum_{i \in I} T_i$ is a semisimple module. It means M is $\sum_{i \in I} T_i$ -semisimple from Theorem 2.5.

($\Leftarrow$) Follows from Lemma 2.7. □

Example 2.9. (1) Consider the $\mathbb{Z}$ -module $M = \mathbb{Z}$. Then for each $n \in \mathbb{Z}$, we have $nM + (n+1)M = M$ and $nM \cap (n+1)M = n(n+1)M \subseteq 2M$. Hence, M is $T = 2\mathbb{Z}$ -semisimple. Also, note that $\mathbb{Z}/2\mathbb{Z}$ is a simple $\mathbb{Z}$ -module. In fact, M is T -semisimple via Theorem 2.5.

(2) Assume $T = 3\mathbb{Z}$ as a submodule of $M = \mathbb{Z}$. Then for each $n \in \mathbb{Z}$, either $n = 3k$, $n = 3k+1$ or $n = 3k+2$. Then $(3k)M + (3k+1)M = M$ and $(3k)M \cap (3k+1)M \subseteq T$. Also in the second case, $(3k+1)M + 3M = M$ and $(3k+1)M \cap 3M \subseteq T$. For the last case, $(3k+2)M + 3M = M$ and $(3k+2)M \cap 3M \subseteq T$.

(3) Consider $\mathbb{Z}$ as an $\mathbb{Z}$ -module and $T = 4\mathbb{Z}$. Assume $6\mathbb{Z}$ is a T -direct summand of M . Then there exists a submodule $n\mathbb{Z}$ of M such that $6\mathbb{Z} + n\mathbb{Z} = M$ and $6\mathbb{Z} \cap n\mathbb{Z} \subseteq T$. It follows that $\gcd(6, n) = 1$ and $4 \mid 6n$. In fact, $2 \mid n$ which implies $\gcd(6, n) \geq 2$. This causes a contradiction. Therefore, $6\mathbb{Z}$ is not a T -direct summand of M . Note that $9\mathbb{Z}$ is a T -direct summand of M .

Lemma 2.10. *Let M be a hollow module and $T < M$. Then M is T -semisimple if and only if M is a local module and $T = \text{Rad}(M)$.*

Proof. Let M be T -semisimple. Since M is hollow, every proper submodule of M is small in M . As M is T -semisimple, for each proper submodule N of M , there exists a submodule K of M such that $M = N + K$ and $N \cap K \subseteq T$. Since every proper submodule of M is small, K cannot be a proper submodule of M ; hence $K = M$. Now $N \cap K = N \subseteq T$, which means T is a maximal submodule of M containing all proper submodules of M . Therefore, $T = \text{Rad}(M)$ and, in addition, M is local.

Conversely, assume M is local and $T = \text{Rad}(M)$. Then T is the unique maximal submodule of M , so every proper submodule N of M satisfies $N \subseteq T$. Thus $N + T = T$ and $N \cap T = N \subseteq T$, so N is a T -direct summand of M . Therefore, M is T -semisimple. $\square$

Note that, if M is a hollow T -semisimple module with $\text{Rad}(M) = M$, then $T = M$. It follows from the fact that M/T is semisimple via Theorem 2.5. Then for each $L/T \leq M/T$, we have $L/T \ll M/T$ as well as $L \ll M$. Therefore, $L = T$ which implies $M = T$.

Proposition 2.11. *Let M be a T -semisimple module and $K \leq M$. Then the following hold:*

- (1) M/K is $(T + K)/K$ -semisimple.
- (2) K is $T \cap K$ -semisimple.

Proof. (1) Assume $L/K \leq M/K$ where L is a submodule of M containing K . By assumption, there exists a submodule U of M such that $M = L + U$ and $L \cap U \subseteq T$. Then $M/K = L/K + (U + K)/K$. Note that $L/K \cap (U + K)/K = \frac{L \cap (U + K)}{K} = \frac{(L \cap U) + K}{K} \subseteq \frac{T + K}{K}$, where the middle equality follows from modularity.

(2) Let U be a submodule of K . Then there exists a submodule N of M such that $U + N = M$ and $U \cap N \subseteq T$. Now, modularity implies $U + (N \cap K) = K$. Also we have $U \cap N \subseteq T \cap K$. This completes the proof. $\square$

Corollary 2.12. *Any homomorphic image and each submodule of a semisimple module inherit the property.*

The following, says that any direct sum of T_i -semisimple modules is T -semisimple for an appropriate T .

Proposition 2.13. *Let $M = M_1 \oplus M_2 \oplus \dots$ be a direct sum of T_i -semisimple module M_i for $i \in I$ where I is an indexed set. Then M is $T_1 \oplus T_2 \oplus \dots$ -semisimple.*

Proof. Set $T = T_1 \oplus T_2 \oplus \dots$. Then $\frac{M}{T} = \frac{M_1 \oplus M_2 \oplus \dots}{T_1 \oplus \dots \oplus T_2 \oplus \dots} \cong \frac{M_1}{T_1} \oplus \frac{M_2}{T_2} \oplus \dots$ is a direct sum of semisimple modules. It follows that M/T is semisimple. Hence, M is T -semisimple by Theorem 2.5. $\square$

Definition 2.14. Let M be a nonzero R -module. Then we call M a T -simple R -module in case M/T is a simple R -module. In fact, M is T -simple, provided there is no submodules of M properly including T . In other words, M is T -simple if and only if T is a maximal submodule of M .

Lemma 2.15. (1) *Every T -simple module is T -semisimple.*

(2) *Let $f : M \rightarrow N$ be an R -homomorphism and L a T -simple submodule of M such that $\text{Ker } f \not\subseteq L$. Then $f(L)$ is a $f(T)$ -simple submodule of N .*

Proof. (1) Assume M is T -simple and $N \leq M$. Then $N + T = T$ or $N + T = M$. In the first case, N is contained in T so that N is a T -direct summand of M . The second equality implies $N + T = M$ and $N \cap T \subseteq T$. Hence N is a T -direct summand of M .

(2) Suppose $f : M \rightarrow N$ is an R -homomorphism and L a T -simple submodule of M . Then L/T is a simple module. As $\text{Ker } f \not\subseteq L$, we conclude that $f(T) \subset f(L)$. Hence f induces an isomorphism $\bar{f} : L/T \rightarrow f(L)/f(T)$ defined by $\bar{f}(x + T) = f(x) + f(T)$ for each $x \in L$. Naturally $\bar{f}$ is onto. Therefore, $f(L)/f(T)$ is a simple module. $\square$

By definition, a module M with $\text{Rad}(M) = M$ (a radical module) cannot be T -simple. As an example, the $\mathbb{Z}$ -module $\mathbb{Z}$ is $p\mathbb{Z}$ -simple for each prime p .

Since $\mathbb{Q}$ as a $\mathbb{Z}$ -module has no simple quotient, we conclude that there is no $T \leq \mathbb{Q}$ such that $\mathbb{Q}$ is T -simple.

The following is immediate from Lemma 2.15 and Proposition 2.13.

Corollary 2.16. *Let M_i be a T_i -simple module for $i \in I$ where I is an arbitrary indexed set. Then $M = \bigoplus_{i \in I} M_i$ is a $\bigoplus_{i \in I} T_i$ -semisimple module.*

Proposition 2.17. *Let $M = \sum_{i \in I} M_i$ be a sum of T -simple submodules M_i where I is an index set. Then M is T -semisimple.*

Proof. Let M_i be a T -simple module. In fact, M_i/T is simple. Note that $\frac{M}{T} = \frac{\sum_{i \in I} M_i}{T} = \sum_{i \in I} \frac{M_i}{T}$. As a sum of simple modules, we conclude that M/T is a semisimple module. Hence, M is T -semisimple. $\square$

Theorem 2.18. *Let M be a module. Then M is T -semisimple if and only if M is a sum of T -simple submodules.*

Proof. Assume M is a T -semisimple module. Then M/T is a semisimple module by Theorem 2.5. It follows that $M/T = \bigoplus_{i \in I} M_i/T$ is a direct sum of simple modules M_i/T for $i \in I$. Hence each M_i is T -simple. Therefore, $M = \sum_{i \in I} M_i$ is a sum of T -simple submodules. The converse follows from Proposition 2.17. $\square$

Definition 2.19. Let M be a module and T a submodule of M . Then we define $Soc_T(M)$ to be the sum of all T -simple submodules of M . If M has no T -simple submodules, we define $Soc_T(M)$ to be zero module. From the Proposition 2.17, the submodule $Soc_T(M)$ is a T -semisimple module.

Lemma 2.20. (1) *Let $f : M \rightarrow K$ be an R -homomorphism. Then $f(Soc_T(M))$ is contained in $Soc_{f(T)}(K)$.*

(2) *M is T -semisimple if and only if $Soc_T(M) = M$.*

Proof. (1) To verify the inclusion, assume $y = f(x)$ is an element of $f(Soc_T(M))$ where $x \in Soc_T(M)$. From the definition, there exist T -simple submodules $N_1, \dots, N_m$ of M such that $x \in N_1 + \dots + N_m$. It follows that $x = x_1 + \dots + x_m$ where $x_i \in N_i$ for $i = 1, \dots, m$. Now, $y = f(x) = f(x_1 + \dots + x_m) = f(x_1) + \dots + f(x_m)$ which belongs to $f(N_1) + \dots + f(N_m)$. From Lemma 2.15, each $f(N_i)$ is a $f(T)$ -simple submodule of K . Therefore, y belongs to $Soc_{f(T)}(K)$.

(2) Suppose that M is T -semisimple. Then by Theorem 2.18, M is a sum of T -simple submodules. As $Soc_T(M)$ is the sum of all T -simple submodules of M we conclude that $M \subseteq Soc_T(M)$. The converse is clear as $Soc_T(M)$ is T -semisimple. $\square$

3. SOME SUGGESTIONS AND PROBLEMS

(1) A module M is said to be virtually semisimple in case each submodule of M is isomorphic to a direct summand of M and a nonzero module M is virtually simple provided each submodule is isomorphic to M (see [2]). Accordingly, define virtually T -(semi)simple modules and verify their relation with the non-virtual case.

(2) Is there an analogue for Wedderburn-Artin Theorem based on T -semisimple case?

(3) A module M is virtually (dual) Rickart in case the (image) kernel of each endomorphism of M is isomorphic to a direct summand of M (see [1]). Consider an analogue for T -direct summand case.

(4) Can we provide T -versions of modules defined in [6] and [7]?

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