

Research Paper

VARIOUS SPECTRAL ASPECTS OF NCCC-GRAPHS OF CERTAIN FINITE NON-ABELIAN GROUPS

RISHABH CHAKRABORTY, FIRDOUS EE JANNAT AND RAJAT KANTI NATH*

ABSTRACT. Let G be a finite non-abelian group. The non-commuting conjugacy class graph (abbreviated as NCCC-graph) of G is a simple undirected graph whose vertex set is the set of conjugacy classes of non-central elements of G and two vertices x^G and y^G are adjacent to each other if x' and y' do not commute for all $x' \in x^G$ and $y' \in y^G$, where x^G is the conjugacy class of $x \in G$. In this paper, we compute the spectrum, Laplacian spectrum, signless Laplacian spectrum and corresponding energies of NCCC-graphs of certain families of finite non-abelian groups. We determine whether these graphs are integral, L-integral and Q-integral. Further, we compare energy, Laplacian energy and signless Laplacian energy; and determine whether these graphs are borderenergetic, L-borderenergetic, Q-borderenergetic, hyperenergetic, L-hyperenergetic or Q-hyperenergetic.

DOI: 10.22034/as.2026.23008.1785

MSC(2010): Primary: 05C25; Secondary: 20D99, 05C10, 15A18.

Keywords: Conjugacy class graph, Laplacian, Signless Laplacian, Spectrum.

Received: 17 April 2025, Accepted: 23 September 2026.

*Corresponding author

1. INTRODUCTION

Let $\mathcal{G}$ be a simple graph with vertex set $v(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$. If $A(\mathcal{G})$ and $D(\mathcal{G})$ are the adjacency and degree matrices of $\mathcal{G}$ then the Laplacian matrix and signless Laplacian matrix of $\mathcal{G}$, denoted by $L(\mathcal{G})$ and $Q(\mathcal{G})$ respectively, are given by

$$L(\mathcal{G}) := D(\mathcal{G}) - A(\mathcal{G}), \quad \text{and} \quad Q(\mathcal{G}) := D(\mathcal{G}) + A(\mathcal{G}).$$

The spectrum (denoted by $\text{Spec}(\mathcal{G})$), Laplacian spectrum (denoted by $\text{L-spec}(\mathcal{G})$) and signless Laplacian spectrum (denoted by $\text{Q-spec}(\mathcal{G})$) of $\mathcal{G}$ are sets of eigenvalues of $A(\mathcal{G})$, $L(\mathcal{G})$ and $Q(\mathcal{G})$ with multiplicities respectively. We write $\text{Spec}(\mathcal{G}) = \{[\alpha_1]^{a_1}, [\alpha_2]^{a_2}, [\alpha_3]^{a_3}, \dots, [\alpha_j]^{a_j}\}$, $\text{L-spec}(\mathcal{G}) = \{[\beta_1]^{b_1}, [\beta_2]^{b_2}, [\beta_3]^{b_3}, \dots, [\beta_k]^{b_k}\}$ and $\text{Q-spec}(\mathcal{G}) = \{[\gamma_1]^{c_1}, [\gamma_2]^{c_2}, [\gamma_3]^{c_3}, \dots, [\gamma_l]^{c_l}\}$, where $[x_i]^{y_i}$ means that the eigenvalue x_i has algebraic multiplicity y_i . A graph $\mathcal{G}$ is called integral, L-integral and Q-integral if $\text{Spec}(\mathcal{G})$, $\text{L-spec}(\mathcal{G})$ and $\text{Q-spec}(\mathcal{G})$ contain only integers. The notion of integral graph was introduced by Harary and Schwenk [15] in 1974. Grone and Merris [9] in 1994 and Simic and Stanic [25] in 2008 introduced the notions of L-integral and Q-integral graphs respectively.

The energy, Laplacian energy and signless Laplacian energy of $\mathcal{G}$, denoted by $E(\mathcal{G})$, $\text{LE}(\mathcal{G})$ and $\text{LE}^+(\mathcal{G})$ are defined as

$$E(\mathcal{G}) := \sum_{\alpha \in \text{Spec}(\mathcal{G})} |\alpha|, \quad \text{LE}(\mathcal{G}) := \sum_{\beta \in \text{L-spec}(\mathcal{G})} |\beta - \Delta(\mathcal{G})|,$$

and

$$\text{LE}^+(\mathcal{G}) := \sum_{\gamma \in \text{Q-spec}(\mathcal{G})} |\gamma - \Delta(\mathcal{G})|,$$

where $\Delta(\mathcal{G}) = \frac{|2e(\Gamma_{\mathcal{G}})|}{|v(\Gamma_{\mathcal{G}})|}$. In 1978, Gutman [10] introduced the notion of $E(\mathcal{G})$; in 2006, Gutman and Zhou [14] introduced the notion of $\text{LE}(\mathcal{G})$; and in 2008, Gutman et al. [12] introduced the notion of $\text{LE}^+(\mathcal{G})$. As noted in [13], these energies have applications in crystallography, macromolecular theory, protein sequence analysis and comparison, network analysis, satellite communication, image processing etc. Note that

$$(1) \quad E(K_n) = \text{LE}(K_n) = \text{LE}^+(K_n) = 2(n-1),$$

where K_n is a complete graph of order n . A graph $\mathcal{G}$ is called hyperenergetic if $E(\mathcal{G}) > E(K_{|v(\mathcal{G})|})$ and borderenergetic if $E(\mathcal{G}) = E(K_{|v(\mathcal{G})|})$; L-hyperenergetic if $\text{LE}(\mathcal{G}) > \text{LE}(K_{|v(\mathcal{G})|})$ and L-borderenergetic if $\text{LE}(\mathcal{G}) = \text{LE}(K_{|v(\mathcal{G})|})$ and Q-hyperenergetic if $\text{LE}^+(\mathcal{G}) > \text{LE}^+(K_{|v(\mathcal{G})|})$ and Q-borderenergetic if $\text{LE}^+(\mathcal{G}) = \text{LE}^+(K_{|v(\mathcal{G})|})$. It is worth mentioning that the concept of hyperenergetic graph was introduced by Walikar et al. [29] and Gutman [11], independently in 1999 and L-hyperenergetic and Q-hyperenergetic graphs were introduced in [7]. The concepts of borderenergetic, L-borderenergetic and Q-borderenergetic graphs were introduced by

Gong et al. [8], Tura [28] and Tao et al. [27] in the years 2015, 2017 and 2018 respectively. Throughout the paper, we write mK_n to denote the disjoint union of m copies of K_n .

Let G be a finite non-abelian group with center $Z(G)$. We write x^G to denote the conjugacy class of $x \in G$. The non-commuting conjugacy class graph (abbreviated as NCCC-graph) of G , denoted by Γ_G , is a simple undirected graph whose vertex set is $\{x^G : x \in G \setminus Z(G)\}$ and two vertices x^G and y^G are adjacent to each other if $x'y' \neq y'x'$ for all $x' \in x^G$ and $y' \in y^G$. The complement of this graph is known as commuting conjugacy class graph (in short CCC-graph) of G and it is denoted by $CCC(G)$. The study of CCC-graphs of groups was initiated by Herzog, Longobardi, and Maj [16] in 2009. In 2016, Mohammadian et al. [20] characterized finite groups whose CCC-graphs are triangle-free. Subsequently, structures of CCC-graphs of several families of finite non-abelian groups have been obtained in [21, 22, 23]. Later on, Bhowal and Nath [2, 3, 4] have computed spectrum, Laplacian spectrum, signless Laplacian spectrum and their corresponding energies. Bhowal and Nath also obtained various groups such that CCC-graphs are hyperenergetic, L-hyperenergetic and Q-hyperenergetic. Recently, Jannat and Nath [17] have computed various common neighbourhood spectra and common neighbourhood energies of CCC-graphs of several classes of finite non-abelian groups and obtained finite non-abelian groups such that their CCC-graphs are common neighbourhood hyperenergetic, common neighbourhood Laplacian hyperenergetic and common neighbourhood signless Laplacian hyperenergetic. All these results can be found in the survey article [5]. It is worth mentioning that NCCC-graphs of groups were not much studied. To our knowledge, [18] is the only paper where Jannat and Nath studied several distance spectral properties and Wiener index of NCCC-graphs.

In Section 2 of this paper, we first compute spectrum, Laplacian spectrum, signless Laplacian spectrum and their corresponding energies of Γ_G when G is the group such that $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) and the dihedral group D_{2m} (for any integer $m \geq 3$). As a consequence, we compute the above mentioned graph parameters for NCCC-graphs of the dihedral group $D_{2m} = \langle x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$ (for $m \geq 3$), the dicyclic group $T_{4m} = \langle x, y : x^{2m} = 1, x^m = y^2, y^{-1}xy = x^{-1} \rangle$ (for $m \geq 2$), the semidihedral group $SD_{8m} = \langle x, y : x^{4m} = y^2 = 1, yxy = x^{2m-1} \rangle$ (for $m \geq 2$) and the groups $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1, x^{-1}yx = y^{-1} \rangle$ (for $m \geq 3$ and $n \geq 2$), $U_{6m} = \langle x, y : x^{2m} = y^3 = 1, x^{-1}yx = y^{-1} \rangle$ (for $m \geq 2$) and $V_{8m} = \langle x, y : x^{2m} = y^4 = 1, yx = x^{-1}y^{-1}, y^{-1}x = x^{-1}y \rangle$ (for $m \geq 2$).

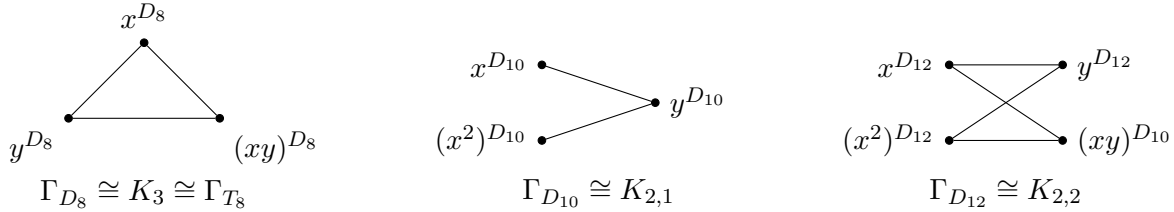


FIGURE 1. NCCC-graphs of D_8, T_8, D_{10} and D_{12} .

In the subsequent section, we compare these energies and identify conditions under which these graphs are borderenergetic, L-borderenergetic, Q-borderenergetic, hyperenergetic, L-hyperenergetic or Q-hyperenergetic. Additionally, we derive several conditions such that these graphs are integral, L-integral and Q-integral. We also provide a graphical comparison of the energy, Laplacian energy and signless Laplacian energy of NCCC-graphs of the aforementioned groups.

2. VARIOUS SPECTRA AND ENERGIES OF NCCC-GRAPHS

Let $P(\mathcal{G}, x)$, $P_L(\mathcal{G}, x)$ and $P_Q(\mathcal{G}, x)$ be the characteristic polynomials of $A(\mathcal{G})$, $L(\mathcal{G})$ and $Q(\mathcal{G})$ respectively. We write $K_{a_1 \cdot p_1, a_2 \cdot p_2}$ to denote the complete r -partite graph $K_{\underbrace{p_1, \dots, p_1}_{a_1\text{-times}}, \underbrace{p_2, \dots, p_2}_{a_2\text{-times}}}$, where $r = a_1 + a_2$. The following lemma that follows from [24, Theorem 1.6] is useful in computing spectrum, L-spectrum and Q-spectrum of the NCCC-graphs of groups.

Lemma 2.1. *If $\mathcal{G} = K_{a_1 \cdot p_1, a_2 \cdot p_2}$ on $n = a_1 p_1 + a_2 p_2$ vertices and $r = a_1 + a_2$ then*

- (1) $P(\mathcal{G}, x) = x^{n-r}(x+p_1)^{a_1-1}(x+p_2)^{a_2-1}(x^2 + (p_1(1-a_1)+p_2(1-a_2))x + p_1 p_2(1-a_1-a_2))$.
- (2) $P_L(\mathcal{G}, x) = x(x-(a_1 p_1 + p_2(a_2-1)))^{a_2(p_2-1)}(x-(a_2 p_2 + p_1(a_1-1)))^{a_1(p_1-1)}(x-n)^{a_1+a_2-1}$.
- (3) $P_Q(\mathcal{G}, x) = \prod_{i=1}^2 (x-(n-p_i))^{a_i(p_i-1)} \prod_{i=1}^2 (x-(n-2p_i))^{a_i-1} (x^2 - x(2n+p_1(a_1-2)+p_2(a_2-2)) + n^2 - 2n(p_1+p_2) + n(a_1 p_1 + a_2 p_2) + 2p_1 p_2(2-a_1-a_2))$.

Putting $a_1 = a, p_1 = b$ and $a_2 = p_2 = 0$ in Lemma 2.1, we get the following result.

Lemma 2.2. *If $\mathcal{G} = K_{a \cdot b}$ then*

- (1) $P(\mathcal{G}, x) = x^{a(b-1)}(x+b)^{a-1}(x-b(a-1))$.
- (2) $P_L(\mathcal{G}, x) = x(x-b(a-1))^{a(b-1)}(x-ab)^{a-1}$.
- (3) $P_Q(\mathcal{G}, x) = (x-b(a-1))^{a(b-1)}(x-b(a-2))^{a-1}(x-2b(a-1))$.

2.1. Groups G such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. In this subsection, we consider NCCC-graphs of the groups whose central quotients are isomorphic to the group $\mathbb{Z}_p \times \mathbb{Z}_p$ for any prime p .

Theorem 2.3. *Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is any prime. Then*

$$\begin{aligned} \text{Spec}(\Gamma_G) &= \{[0]^{(p+1)(n-1)}, [-n]^p, [np]^1\}, \quad \text{L-spec}(\Gamma_G) = \{[0]^1, [np]^{(p+1)(n-1)}, [(p+1)n]^p\}, \\ \text{Q-spec}(\Gamma_G) &= \{[np]^{(p+1)(n-1)}, [n(p-1)]^p, [2np]^1\} \text{ and } E(\Gamma_G) = \text{LE}(\Gamma_G) = \text{LE}^+(\Gamma_G) = 2np, \\ \text{where } |Z(G)| &= z \text{ and } n = \frac{(p-1)z}{p}. \end{aligned}$$

Proof. By [21, Theorem 3.1], we have $CCC(G) \cong (p+1)K_n$, where $n = \frac{(p-1)z}{p}$. Therefore, $\Gamma_G \cong K_{(p+1) \cdot n}$. Since $|v(\Gamma_G)| = n(p+1)$, by Lemma 2.2(a), we get

$$P(\Gamma_G, x) = x^{(p+1)(n-1)}(x+n)^p(x-np).$$

Hence, $\text{Spec}(\Gamma_G) = \{[0]^{(p+1)(n-1)}, [-n]^p, [np]^1\}$ and so

$$E(\Gamma_G) = (p+1)(n-1) \times 0 + p \times |-n| + |np| = 2np.$$

By using Lemma 2.2(b), we get

$$P_L(\Gamma_G, x) = x(x-np)^{(p+1)(n-1)}(x-(p+1)n)^p.$$

Therefore, $\text{L-spec}(\Gamma_G) = \{[0]^1, [np]^{(p+1)(n-1)}, [(p+1)n]^p\}$.

We have $|2e(\Gamma_G)| = p(p+1)n^2$ and so $\Delta(\Gamma_G) = \frac{|2e(\Gamma_G)|}{|v(\Gamma_G)|} = pn$. Therefore,

$$\begin{aligned} \text{LE}(\Gamma_G) &= 1 \times |0 - \Delta(\Gamma_G)| + (p+1)(n-1) \times |np - \Delta(\Gamma_G)| + p \times |(p+1)n - \Delta(\Gamma_G)| \\ &= 1 \times |-pn| + (p+1)(n-1) \times |0| + p \times |n| = 2np. \end{aligned}$$

By Lemma 2.2(c), we get

$$P_Q(\Gamma_G, x) = (x-np)^{(p+1)(n-1)}(x-n(p-1))^p(x-2np).$$

Therefore, $\text{Q-spec}(\Gamma_G) = \{[np]^{(p+1)(n-1)}, [n(p-1)]^p, [2np]^1\}$ and so

$$\begin{aligned} \text{LE}^+(\Gamma_G) &= (p+1)(n-1) \times |np - \Delta(\Gamma_G)| + p \times |n(p-1) - \Delta(\Gamma_G)| + 1 \times |2np - \Delta(\Gamma_G)| \\ &= (p+1)(n-1) \times |0| + p \times |-n| + 1 \times |pn| = 2np. \end{aligned}$$

This completes the proof. $\square$

Corollary 2.4. *Let G be a non-abelian group of order p^m and $|Z(G)| = p^{m-2}$. Then*

$$\begin{aligned} \text{Spec}(\Gamma_G) &= \left\{ [0]^{(p+1)((p-1)p^{m-3}-1)}, [-(p-1)p^{m-3}]^p, [(p-1)p^{m-2}]^1 \right\}, \\ \text{L-spec}(\Gamma_G) &= \left\{ [0]^1, [(p-1)p^{m-2}]^{(p+1)((p-1)p^{m-3}-1)}, [(p^2-1)p^{m-3}]^p \right\}, \\ \text{Q-spec}(\Gamma_G) &= \left\{ [(p-1)p^{m-2}]^{(p+1)((p-1)p^{m-3}-1)}, [(p-1)^2p^{m-3}]^p, [2(p-1)p^{m-2}]^1 \right\}. \end{aligned}$$

and $E(\Gamma_G) = \text{LE}(\Gamma_G) = \text{LE}^+(\Gamma_G) = 2(p-1)p^{m-2}$.

Proof. Here $\left| \frac{G}{Z(G)} \right| = p^2$. Since G is non-abelian, $\frac{G}{Z(G)}$ is non-cyclic. Therefore, $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, the result follows from Theorem 2.3. $\square$

2.2. Groups G such that $\frac{G}{Z(G)} \cong D_{2m}$. In this subsection, we consider NCCC-graphs of the groups whose central quotients are isomorphic to the group D_{2m} .

Theorem 2.5. *Let G be a finite group and $\frac{G}{Z(G)} \cong D_{2m}$, where $m \geq 2$. Suppose that $|Z(G)| = z$. Then we have the following.*

(1) For m is even

$$(i) \text{ Spec}(\Gamma_G) = \left\{ [0]^{\frac{z(m+1)}{2}-3}, \left[\frac{-z}{2}\right]^1, \left[\frac{z(1+\sqrt{8m-7})}{4}\right]^1, \left[\frac{z(1-\sqrt{8m-7})}{4}\right]^1 \right\} \text{ and}$$

$$E(\Gamma_G) = \frac{z(1 + \sqrt{8m-7})}{2}.$$

$$(ii) \text{ L-spec}(\Gamma_G) = \left\{ [0]^1, [z]^{\frac{(m-1)z}{2}-1}, \left[\frac{zm}{2}\right]^{z-2}, \left[\frac{z(m+1)}{2}\right]^2 \right\} \text{ and } \text{LE}(\Gamma_G) = \frac{z^2(m-1)(m-2)}{(m+1)} + 2z.$$

$$(iii) \text{ Q-spec}(\Gamma_G) = \left\{ [z]^{\frac{(m-1)z}{2}-1}, \left[\frac{zm}{2}\right]^{z-2}, \left[\frac{z(m-1)}{2}\right]^1, \left[\frac{z(m+3+\sqrt{(m-1)(m+7)})}{4}\right]^1, \left[\frac{z(m+3-\sqrt{(m-1)(m+7)})}{4}\right]^1 \right\},$$

and

$$\text{LE}^+(\Gamma_G) = \frac{z^2(m-1)(m-2)}{m+1} + \frac{z(m-1)(\sqrt{1+\frac{8}{m-1}}-1)}{2}.$$

(2) For m is odd

$$(i) \text{ Spec}(\Gamma_G) = \left\{ [0]^{\frac{z(m+1)}{2}-2}, \left[z\sqrt{\frac{m-1}{2}}\right]^1, \left[-z\sqrt{\frac{m-1}{2}}\right]^1 \right\} \text{ and } E(\Gamma_G) = 2z\sqrt{\frac{m-1}{2}}.$$

$$(ii) \text{ L-spec}(\Gamma_G) = \left\{ [0]^1, [z]^{\frac{z(m-1)}{2}-1}, \left[\frac{z(m-1)}{2}\right]^{z-1}, \left[\frac{z(m+1)}{2}\right]^1 \right\} \text{ and}$$

$$\text{LE}(\Gamma_G) = \left(\frac{z^2(m-1)(m-3)}{(m+1)} \right) + 2z.$$

$$(iii) \text{ Q-spec}(\Gamma_G) = \left\{ [0]^1, [z]^{\frac{z(m-1)}{2}-1}, \left[\frac{z(m-1)}{2}\right]^{z-1}, \left[\frac{z(m+1)}{2}\right]^1 \right\} \text{ and}$$

$$\text{LE}^+(\Gamma_G) = \left(\frac{z^2(m-1)(m-3)}{(m+1)} \right) + 2z.$$

Proof. (a) If m is even then by [23, Theorem 1.2], we have $CCC(G) \cong 2K_{\frac{z}{2}} \cup K_{\frac{(m-1)z}{2}}$.

Therefore, $\Gamma_G \cong K_{2, \frac{z}{2}, 1, \frac{(m-1)z}{2}}$. Here $|v(\Gamma_G)| = \frac{z(m+1)}{2}$.

(i) Using Lemma 2.1(a), we get

$$P(\Gamma_G, x) = x^{\frac{z(m+1)}{2}-3} \left(x + \frac{z}{2} \right) \left(x^2 - x \left(\frac{z}{2} \right) - \frac{z^2(m-1)}{2} \right).$$

Thus, $\text{Spec}(\Gamma_G) = \left\{ [0]^{\frac{z(m+1)}{2}-3}, \left[\frac{-z}{2}\right]^1, \left[\frac{z(1+\sqrt{1+8(m-1)})}{4}\right]^1, \left[\frac{z(1-\sqrt{1+8(m-1)})}{4}\right]^1 \right\}$ and so

$$\begin{aligned} E(\Gamma_G) &= \left(\frac{z(m+1)}{2} - 3 \right) \times 0 + \left| -\frac{z}{2} \right| + \left| \frac{z(1+\sqrt{1+8(m-1)})}{4} \right| + \left| \frac{z(1-\sqrt{1+8(m-1)})}{4} \right| \\ &= \frac{z}{2} + \frac{z(1+\sqrt{1+8(m-1)})}{4} + \frac{z(\sqrt{1+8(m-1)}-1)}{4} \\ &= \frac{z(1+\sqrt{1+8(m-1)})}{2}. \end{aligned}$$

(ii) Using Lemma 2.1(b), we get

$$P_L(\Gamma_G, x) = x(x-z)^{\frac{(m-1)z}{2}-1} \left(x - \frac{zm}{2} \right)^{z-2} \left(x - \frac{z(m+1)}{2} \right)^2.$$

Therefore, $\text{L-spec}(\Gamma_G) = \left\{ [0]^1, [z]^{\frac{(m-1)z}{2}-1}, \left[\frac{zm}{2}\right]^{z-2}, \left[\frac{z(m+1)}{2}\right]^2 \right\}$.

Now $|2e(\Gamma_G)| = \frac{z^2(2m-1)}{2}$ and so $\Delta(\Gamma_G) = \frac{|2e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{z(2m-1)}{(m+1)}$. We have

$$\begin{aligned} |0 - \Delta(\Gamma_G)| &= \left| \frac{-z(2m-1)}{(m+1)} \right| = \frac{z(2m-1)}{(m+1)}, \quad |z - \Delta(\Gamma_G)| = \left| \frac{-z(m-2)}{(m+1)} \right| = \frac{z(m-2)}{(m+1)}, \\ \left| \frac{zm}{2} - \Delta(\Gamma_G) \right| &= \left| \frac{z(m-2)(m-1)}{2(m+1)} \right| = \frac{z(m-2)(m-1)}{2(m+1)}, \end{aligned}$$

and

$$\left| \frac{z(m+1)}{2} - \Delta(\Gamma_G) \right| = \left| \frac{z((m-1)^2+2)}{2(m+1)} \right| = \frac{z((m-1)^2+2)}{2(m+1)}.$$

Therefore,

$$\begin{aligned} \text{LE}(\Gamma_G) &= 1 \times \left| \frac{-z(m-2)}{(m+1)} \right| + \left(\frac{z(m-1)}{2} - 1 \right) \times \left| \frac{-z(m-2)}{(m+1)} \right| \\ &\quad + (z-2) \times \left| \frac{z(m-2)(m-1)}{2(m+1)} \right| + 2 \times \left| \frac{z((m-1)^2+2)}{2(m+1)} \right| \\ &= \frac{z^2(m-1)(m-2)}{(m+1)} + 2z. \end{aligned}$$

(iii) Using Lemma 2.1(c), we get

$$\begin{aligned} P_Q(\Gamma_G, x) &= (x-z)^{\frac{(m-1)z}{2}-1} \left(x - \frac{zm}{2} \right)^{z-2} \left(x - \frac{z(m-1)}{2} \right) \\ &\quad \times \left(x^2 - x \left(2\frac{z(m+1)}{2} - \frac{(m-1)z}{2} \right) + \left(\frac{z(m+1)}{2} \right)^2 \right. \\ &\quad \left. - 2\frac{z(m+1)}{2} \left(\frac{z}{2} + \frac{(m-1)z}{2} \right) + \frac{z(m+1)}{2} \left(2\frac{z}{2} + \frac{(m-1)z}{2} \right) - 2\frac{z}{2} \frac{(m-1)z}{2} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Q-spec}(\Gamma_G) = & \left\{ [z]^{\frac{(m-1)z}{2}-1}, \left[\frac{zm}{2}\right]^{z-2}, \left[\frac{z(m-1)}{2}\right]^1, \right. \\ & \left. \left[\frac{z\left(m+3+\sqrt{(m-1)(m+7)}\right)}{4}\right]^1, \left[\frac{z\left(m+3-\sqrt{(m-1)(m+7)}\right)}{4}\right]^1 \right\}. \end{aligned}$$

We have

$$\begin{aligned} \left| \frac{z(m-1)}{2} - \Delta(\Gamma_G) \right| &= \left| \frac{z((m-2)^2-3)}{2(m+1)} \right| = \frac{z((m-2)^2-3)}{2(m+1)}, \\ \left| \frac{z\left(m+3+(m-1)\sqrt{1+\frac{8}{m-1}}\right)}{4} - \Delta(\Gamma_G) \right| &= \left| \frac{z\left((m-2)^2+3+(m+1)\sqrt{(m-1)(m+7)}\right)}{4(m+1)} \right| \\ &= \frac{z\left((m-2)^2+3+(m+1)\sqrt{(m-1)(m+7)}\right)}{4(m+1)}, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{z\left(m+3-(m-1)\sqrt{1+\frac{8}{m-1}}\right)}{4} - \Delta(\Gamma_G) \right| &= \left| \frac{z\left((m-2)^2+3-(m+1)\sqrt{(m-1)(m+7)}\right)}{4(m+1)} \right| \\ &= \frac{-z\left((m-2)^2+3-(m+1)\sqrt{(m-1)(m+7)}\right)}{4(m+1)}, \end{aligned}$$

since for $m \geq 4$, $((m-2)^2+3)^2 - (m+1)^2(m-1)(m+7) = -8m^2(2m-3) - 48m + 56 < 0$ and so $(m-2)^2+3 < (m+1)\sqrt{(m-1)(m+7)}$. Therefore,

$$\begin{aligned} \text{LE}^+(\Gamma_G) &= \left(\frac{(m-1)z}{2} - 1 \right) \times \left| \frac{-z(m-2)}{(m+1)} \right| + (z-2) \times \left| \frac{z(m-2)(m-1)}{2(m+1)} \right| \\ &\quad + 1 \times \left| \frac{z((m-2)^2-3)}{2(m+1)} \right| + \left| \frac{z\left((m-2)^2+3+(m+1)\sqrt{(m-1)(m+7)}\right)}{4(m+1)} \right| \\ &\quad + \left| \frac{z\left((m-2)^2+3-(m+1)\sqrt{(m-1)(m+7)}\right)}{4(m+1)} \right| \\ &= \left(\frac{z^2(m-1)(m-2)}{m+1} \right) + \left(\frac{z(m-1)\left(\sqrt{1+\frac{8}{m-1}}-1\right)}{2} \right). \end{aligned}$$

Hence the result follows.

(b) If m is odd then by [23, Theorem 1.2], we have $CCC(G) \cong K_z \cup K_{\frac{(m-1)z}{2}}$. Therefore, $\Gamma_G \cong K_{1,z,1,\frac{(m-1)z}{2}}$. Here $|v(\Gamma_G)| = \frac{z(m+1)}{2}$ and $r = 2$.

(i) Using Lemma 2.1(a), we get

$$P(\Gamma_G, x) = x^{\frac{z(m+1)}{2}-2} \left(x^2 - \frac{z^2(m-1)}{2} \right).$$

Thus, $\text{Spec}(\Gamma_G) = \left\{ [0]^{\frac{z(m+1)}{2}-2}, \left[z\sqrt{\frac{m-1}{2}} \right]^1, \left[-z\sqrt{\frac{m-1}{2}} \right]^1 \right\}$ and so

$$E(\Gamma_G) = \left(\frac{z(m+1)}{2} - 2 \right) \times 0 + 1 \times \left| z\sqrt{\frac{m-1}{2}} \right| + 1 \times \left| -z\sqrt{\frac{m-1}{2}} \right| = 2z\sqrt{\frac{m-1}{2}}.$$

(ii) Using Lemma 2.1(b), we get

$$P_L(\Gamma_G, x) = x(x-z)^{\frac{z(m-1)}{2}-1} \left(x - \frac{z(m-1)}{2} \right)^{z-1} \left(x - \frac{z(m+1)}{2} \right)^1.$$

So, $\text{L-spec}(\Gamma_G) = \left\{ [0]^1, [z]^{\frac{z(m-1)}{2}-1}, \left[\frac{z(m-1)}{2} \right]^{z-1}, \left[\frac{z(m+1)}{2} \right]^1 \right\}.$

Now $|2e(\Gamma_G)| = (z^2(m-1))$ and so $\Delta(\Gamma_G) = \frac{|2e(\Gamma_G)|}{|v(\Gamma_G)|} = \frac{2z(m-1)}{(m+1)}$. We have

$$\begin{aligned} |0 - \Delta(\Gamma_G)| &= \left| \frac{-2z(m-1)}{(m+1)} \right| = \frac{2z(m-1)}{(m+1)}, & |z - \Delta(\Gamma_G)| &= \left| \frac{-z(m-3)}{(m+1)} \right| = \frac{z(m-3)}{(m+1)}, \\ \left| \frac{z(m-1)}{2} - \Delta(\Gamma_G) \right| &= \left| \frac{z(m-1)(m-3)}{2(m+1)} \right| = \frac{z(m-1)(m-3)}{2(m+1)}, \end{aligned}$$

and

$$\left| \frac{z(m+1)}{2} - \Delta(\Gamma_G) \right| = \left| \frac{z((m-1)^2 + 4)}{2(m+1)} \right| = \frac{z((m-1)^2 + 4)}{2(m+1)}.$$

Therefore,

$$\begin{aligned} \text{LE}(\Gamma_G) &= 1 \times \left| \frac{-2z(m-1)}{(m+1)} \right| + \left(\frac{z(m-1)}{2} - 1 \right) \times \left| \frac{-z(m-3)}{(m+1)} \right| \\ &\quad + (z-1) \times \left| \frac{z(m-1)(m-3)}{2(m+1)} \right| + 1 \times \left| \frac{z((m-1)^2 + 4)}{2(m+1)} \right| \\ &= \frac{z^2(m-1)(m-3)}{(m+1)} + 2z. \end{aligned}$$

(iii) Using Lemma 2.1(c), we get

$$\begin{aligned} P_Q(\Gamma_G, x) &= (x-z)^{\frac{z(m-1)}{2}-1} \left(x - \left(\frac{z(m-1)}{2} \right) \right)^{z-1} \left(x^2 - x \left(2\frac{z(m+1)}{2} - z - \frac{z(m-1)}{2} \right) \right. \\ &\quad \left. + \left(\frac{z(m+1)}{2} \right)^2 - 2\frac{z(m+1)}{2} \left(z + \frac{z(m-1)}{2} \right) + \frac{z(m+1)}{2} \left(z + \frac{z(m-1)}{2} \right) \right). \end{aligned}$$

Therefore, $\text{Q-spec}(\Gamma_G) = \left\{ [0]^1, [z]^{\frac{z(m-1)}{2}-1}, \left[\frac{z(m-1)}{2} \right]^{z-1}, \left[\frac{z(m+1)}{2} \right]^1 \right\}.$ Since $\text{L-spec}(\Gamma_G) = \text{Q-spec}(\Gamma_G)$, so $\text{LE}(\Gamma_G) = \text{LE}^+(\Gamma_G).$ $\square$

Corollary 2.6. Consider the dihedral group D_{2m} , where $m \geq 3$.

(1) For m is odd

(i) $\text{Spec}(\Gamma_{D_{2m}}) = \left\{ [0]^{\frac{(m-3)}{2}}, \left[\sqrt{\frac{m-1}{2}} \right]^1, \left[-\sqrt{\frac{m-1}{2}} \right]^1 \right\}$ and $E(\Gamma_{D_{2m}}) = 2\sqrt{\frac{m-1}{2}}.$

(ii) $\text{L-spec}(\Gamma_{D_{2m}}) = \left\{ [0]^1, [1]^{\frac{m-3}{2}}, \left[\frac{m+1}{2} \right]^1 \right\}$ and $\text{LE}(\Gamma_{D_{2m}}) = \frac{(m-1)(m-3)}{(m+1)} + 2.$

(iii) $\text{Q-spec}(\Gamma_{D_{2m}}) = \left\{ [0]^1, [1]^{\frac{(m-3)}{2}}, \left[\frac{m+1}{2} \right]^1 \right\}$ and $\text{LE}^+(\Gamma_{D_{2m}}) = \frac{(m-1)(m-3)}{(m+1)} + 2.$

- (2) For m is even and $\frac{m}{2}$ is odd
- (i) $\text{Spec}(\Gamma_{D_{2m}}) = \left\{ [0]^{\frac{(m-2)}{2}}, [\sqrt{m-2}]^1, [-\sqrt{m-2}]^1 \right\}$ and $E(\Gamma_{D_{2m}}) = 2\sqrt{m-2}$.
 - (ii) $\text{L-spec}(\Gamma_{D_{2m}}) = \left\{ [0]^1, [2]^{\frac{m-4}{2}}, \left[\frac{m-2}{2}\right]^1, \left[\frac{m+2}{2}\right]^1 \right\}$ and $\text{LE}(\Gamma_{D_{2m}}) = \frac{2(m-2)(m-6)}{(m+2)} + 4$.
 - (iii) $\text{Q-spec}(\Gamma_{D_{2m}}) = \left\{ [0]^1, [2]^{\frac{(m-4)}{2}}, \left[\frac{m-2}{2}\right]^1, \left[\frac{m+2}{2}\right]^1 \right\}$ and $\text{LE}^+(\Gamma_{D_{2m}}) = \frac{2(m-2)(m-6)}{(m+2)} + 4$.
- (3) For m and $\frac{m}{2}$ are even
- (i) $\text{Spec}(\Gamma_{D_{2m}}) = \left\{ [0]^{\frac{(m-4)}{2}}, [-1]^1, \left[\frac{1+\sqrt{4m-7}}{2}\right]^1, \left[\frac{1-\sqrt{4m-7}}{2}\right]^1 \right\}$ and $E(\Gamma_{D_{2m}}) = 1 + \sqrt{4m-7}$.
 - (ii) $\text{L-spec}(\Gamma_{D_{2m}}) = \left\{ [0]^1, [2]^{\frac{m-4}{2}}, \left[\frac{m+2}{2}\right]^2 \right\}$ and $\text{LE}(\Gamma_{D_{2m}}) = \frac{2(m-2)(m-4)}{(m+2)} + 4$.
 - (iii) $\text{Q-spec}(\Gamma_{D_{2m}}) = \left\{ [2]^{\frac{m-4}{2}}, \left[\frac{m-2}{2}\right]^1, \left[\frac{m+6+\sqrt{(m-2)(m+14)}}{4}\right]^1, \left[\frac{m+6-\sqrt{(m-2)(m+14)}}{4}\right]^1 \right\}$
- and

$$\text{LE}^+(\Gamma_{D_{2m}}) = \begin{cases} 4, & \text{for } m = 4, \\ \frac{2(m-2)(m-4)}{(m+2)} + \frac{(m-2)\left(\sqrt{1+\frac{16}{m-2}}-1\right)}{2}, & \text{for } m \neq 4. \end{cases}$$

Proof. We know that $\frac{D_{2m}}{Z(D_{2m})} \cong D_{2 \times \frac{m}{2}}$ or D_{2m} according as m is even or odd. Therefore, by Theorem 2.3 and Theorem 2.5, we get the required result. $\square$

Corollary 2.7. Consider the dicyclic group T_{4m} , where $m \geq 2$.

- (1) For m is even
- (i) $\text{Spec}(\Gamma_{T_{4m}}) = \left\{ [0]^{m-2}, [-1]^1, \left[\frac{1+\sqrt{8m-7}}{2}\right]^1, \left[\frac{1-\sqrt{8m-7}}{2}\right]^1 \right\}$ and $E(\Gamma_{T_{4m}}) = 1 + \sqrt{8m-7}$.
 - (ii) $\text{L-spec}(\Gamma_{T_{4m}}) = \left\{ [0]^1, [2]^{m-2}, [m+1]^2 \right\}$ and $\text{LE}(\Gamma_{T_{4m}}) = \frac{4(m-1)(m-2)}{(m+1)} + 4$.
 - (iii) $\text{Q-spec}(\Gamma_{T_{4m}}) = \left\{ [2]^{m-2}, [m-1]^1, \left[\frac{m+3+\sqrt{(m-1)(m+7)}}{2}\right]^1, \left[\frac{m+3-\sqrt{(m-1)(m+7)}}{2}\right]^1 \right\}$
- and

$$\text{LE}^+(\Gamma_{T_{4m}}) = \begin{cases} 4, & \text{for } m = 2, \\ \frac{4(m-1)(m-2)}{(m+1)} + (m-1) \left(\sqrt{1 + \frac{8}{m-1}} - 1 \right), & \text{for } m \neq 2. \end{cases}$$

- (2) For m is odd
- (i) $\text{Spec}(\Gamma_{T_{4m}}) = \left\{ [0]^{m-1}, \left[2\sqrt{\frac{m-1}{2}}\right]^1, \left[-2\sqrt{\frac{m-1}{2}}\right]^1 \right\}$ and $E(\Gamma_{T_{4m}}) = 4\sqrt{\frac{m-1}{2}}$.
 - (ii) $\text{L-spec}(\Gamma_{T_{4m}}) = \left\{ [0]^1, [2]^{m-2}, [m-1]^1, [m+1]^1 \right\}$ and $\text{LE}(\Gamma_{T_{4m}}) = \frac{4(m-1)(m-3)}{(m+1)} + 4$.

$$(iii) \quad \begin{aligned} \text{Q-spec}(\Gamma_{T_{4m}}) &= \left\{ [0]^1, [2]^{m-2}, [m-1]^1, [m+1]^1 \right\} \quad \text{and} \\ \text{LE}^+(\Gamma_{T_{4m}}) &= \frac{4(m-1)(m-3)}{(m+1)} + 4. \end{aligned}$$

Proof. We know that $\frac{T_{4m}}{Z(T_{4m})} \cong D_{2m}$. Therefore, by Theorem 2.5 and Theorem 2.3, we get the required result. $\square$

Corollary 2.8. *Consider the group U_{6m} , where $m \geq 2$. Then*

$$\begin{aligned} \text{Spec}(\Gamma_{U_{6m}}) &= \left\{ [0]^{2m-2}, [m]^1, [-m]^1 \right\}, \text{L-spec}(\Gamma_{U_{6m}}) = \left\{ [0]^1, [m]^{2m-2}, [2m]^1 \right\}, \\ \text{Q-spec}(\Gamma_{U_{6m}}) &= \left\{ [0]^1, [m]^{2m-2}, [2m]^1 \right\} \text{ and } \text{E}(\Gamma_{U_{6m}}) = \text{LE}(\Gamma_{U_{6m}}) = \text{LE}^+(\Gamma_{U_{6m}}) = 2m. \end{aligned}$$

Proof. We know that $\frac{U_{6m}}{Z(U_{6m})} \cong D_{2 \times 3}$. Therefore, by Theorem 2.5, we get the required result.

$\square$

Corollary 2.9. *Consider the group $U_{(n,m)}$, where $m \geq 3$ and $n \geq 2$.*

(1) *For m is odd*

$$(i) \quad \text{Spec}(\Gamma_{U_{(n,m)}}) = \left\{ [0]^{\frac{n(m+1)}{2}-2}, \left[n\sqrt{\frac{m-1}{2}} \right]^1, \left[-n\sqrt{\frac{m-1}{2}} \right]^1 \right\} \quad \text{and} \quad \text{E}(\Gamma_{U_{(n,m)}}) = 2n\sqrt{\frac{m-1}{2}}.$$

$$(ii) \quad \text{L-spec}(\Gamma_{U_{(n,m)}}) = \left\{ [0]^1, [n]^{\frac{n(m-1)}{2}-1}, \left[\frac{n(m-1)}{2} \right]^{n-1}, \left[\frac{n(m+1)}{2} \right]^1 \right\} \quad \text{and}$$

$$\text{LE}(\Gamma_{U_{(n,m)}}) = \frac{n^2(m-1)(m-3)}{(m+1)} + 2n.$$

$$(iii) \quad \text{Q-spec}(\Gamma_{U_{(n,m)}}) = \left\{ [0]^1, [n]^{\frac{n(m-1)}{2}-1}, \left[\frac{n(m-1)}{2} \right]^{n-1}, \left[\frac{n(m+1)}{2} \right]^1 \right\} \quad \text{and}$$

$$\text{LE}^+(\Gamma_{U_{(n,m)}}) = \frac{n^2(m-1)(m-3)}{(m+1)} + 2n.$$

(2) *For m is even and $\frac{m}{2}$ is odd*

$$(i) \quad \text{Spec}(\Gamma_{U_{(n,m)}}) = \left\{ [0]^{\frac{n(m+2)}{2}-2}, \left[n\sqrt{m-2} \right]^1, \left[-n\sqrt{m-2} \right]^1 \right\} \quad \text{and} \quad \text{E}(\Gamma_{U_{(n,m)}}) = 2n\sqrt{m-2}.$$

$$(ii) \quad \text{L-spec}(\Gamma_{U_{(n,m)}}) = \left\{ [0]^1, [2n]^{\frac{n(m-2)}{2}-1}, \left[\frac{n(m-2)}{2} \right]^{2n-1}, \left[\frac{n(m+2)}{2} \right]^1 \right\} \quad \text{and}$$

$$\text{LE}(\Gamma_{U_{(n,m)}}) = \frac{2n^2(m-2)(m-6)}{(m+2)} + 4n.$$

$$(iii) \quad \text{Q-spec}(\Gamma_{U_{(n,m)}}) = \left\{ [0]^1, [2n]^{\frac{n(m-2)}{2}-1}, \left[\frac{n(m-2)}{2} \right]^{2n-1}, \left[\frac{n(m+2)}{2} \right]^1 \right\} \quad \text{and}$$

$$\text{LE}^+(\Gamma_{U_{(n,m)}}) = \frac{2n^2(m-2)(m-6)}{(m+2)} + 4n.$$

(3) For m is even and $\frac{m}{2}$ is even

$$(i) \text{ Spec}(\Gamma_{U_{(n,m)}}) = \left\{ [0]^{\frac{n(m+2)}{2}-3}, [-n]^1, \left[\frac{n(1+\sqrt{4m-7})}{2} \right]^1, \left[\frac{n(1-\sqrt{4m-7})}{2} \right]^1 \right\} \text{ and}$$

$$E(\Gamma_{U_{(n,m)}}) = n(1 + \sqrt{4m-7}).$$

$$(ii) \text{ L-spec}(\Gamma_{U_{(n,m)}}) = \left\{ [0]^1, [2n]^{\frac{n(m-2)}{2}-1}, \left[\frac{nm}{2} \right]^{2n-2}, \left[\frac{n(m+2)}{2} \right]^2 \right\} \text{ and}$$

$$\text{LE}(\Gamma_{U_{(n,m)}}) = \frac{2n^2(m-2)(m-4)}{(m+2)} + 4n.$$

$$(iii) \text{ Q-spec}(\Gamma_{U_{(n,m)}}) = \left\{ [2n]^{\frac{n(m-2)}{2}-1}, \left[\frac{nm}{2} \right]^{2n-2}, \left[\frac{n(m-2)}{2} \right]^1, \left[\frac{n(m+6+\sqrt{(m-2)(m+14)})}{4} \right]^1, \left[\frac{n(m+6-\sqrt{(m-2)(m+14)})}{4} \right]^1 \right\} \text{ and}$$

$$\text{LE}^+(\Gamma_{U_{(n,m)}}) = \begin{cases} 4n, & \text{for } m = 4, \\ \frac{2n^2(m-2)(m-4)}{(m+2)} + \frac{n(m-2)(\sqrt{1+\frac{16}{m-2}}-1)}{2}, & \text{for } m \neq 4. \end{cases}$$

Proof. We know that $\frac{U_{(n,m)}}{Z(U_{(n,m)})} \cong D_{2 \times \frac{m}{2}}$ or D_{2m} according as m is even or odd. Therefore, by Theorem 2.5 and Theorem 2.3, we get the required result. $\square$

Corollary 2.10. Consider the group SD_{8m} , where $m \geq 2$.

(1) For m is even

$$(i) \text{ Spec}(\Gamma_{SD_{8m}}) = \left\{ [0]^{2m-2}, [-1]^1, \left[\frac{1+\sqrt{16m-7}}{2} \right]^1, \left[\frac{1-\sqrt{16m-7}}{2} \right]^1 \right\} \text{ and}$$

$$E(\Gamma_{SD_{8m}}) = 1 + \sqrt{16m-7}.$$

$$(ii) \text{ L-spec}(\Gamma_{SD_{8m}}) = \left\{ [0]^1, [2]^{2m-2}, [2m+1]^2 \right\} \text{ and } \text{LE}(\Gamma_{SD_{8m}}) = \frac{4(2m-1)(2m-2)}{(2m+1)} + 4.$$

$$(iii) \text{ Q-spec}(\Gamma_{SD_{8m}}) = \left\{ [2]^{2m-2}, [2m-1]^1, \left[\frac{(2m+3+\sqrt{(2m-1)(2m+7)})}{2} \right]^1, \left[\frac{(2m+3-\sqrt{(2m-1)(2m+7)})}{2} \right]^1 \right\} \text{ and}$$

$$\text{LE}^+(\Gamma_{SD_{8m}}) = \frac{4(2m-1)(2m-2)}{2m+1} + (2m-1) \left(\sqrt{1 + \frac{8}{2m-1}} - 1 \right).$$

(2) For m is odd

$$(i) \text{ Spec}(\Gamma_{SD_{8m}}) = \left\{ [0]^{2m}, \left[4\sqrt{\frac{m-1}{2}} \right]^1, \left[-4\sqrt{\frac{m-1}{2}} \right]^1 \right\} \text{ and } E(\Gamma_{SD_{8m}}) = 8\sqrt{\frac{m-1}{2}}.$$

$$(ii) \text{ L-spec}(\Gamma_{SD_{8m}}) = \left\{ [0]^1, [4]^{2m-3}, [2(m-1)]^3, [2(m+1)]^1 \right\} \text{ and}$$

$$\text{LE}(\Gamma_{SD_{8m}}) = \frac{16(m-1)(m-3)}{(m+1)} + 8.$$

$$(iii) \text{ Q-spec}(\Gamma_{SD_{8m}}) = \left\{ [0]^1, [4]^{2m-3}, [2(m-1)]^3, [2(m+1)]^1 \right\} \text{ and}$$

$$\text{LE}^+(\Gamma_{SD_{8m}}) = \frac{16(m-1)(m-3)}{(m+1)} + 8.$$

Proof. We know that $\frac{SD_{8m}}{Z(SD_{8m})} \cong D_{2 \times 2m}$ or D_{2m} according as m is even or odd. Therefore, by Theorem 2.5, we get the required result. $\square$

Theorem 2.11. Consider the group V_{8m} , where $m \geq 2$.

(1) For m is even

$$(i) \text{ Spec}(\Gamma_{V_{8m}}) = \left\{ [0]^{2m-1}, [-2]^1, [1 + \sqrt{8m-7}]^1, [1 - \sqrt{8m-7}]^1 \right\} \text{ and}$$

$$\text{E}(\Gamma_{V_{8m}}) = 2(1 + \sqrt{8m-7}).$$

$$(ii) \text{ L-spec}(\Gamma_{V_{8m}}) = \left\{ [0]^1, [4]^{2m-3}, [2m]^2, [2(m+1)]^2 \right\} \text{ and } \text{LE}(\Gamma_{V_{8m}}) = \frac{16(m-1)(m-2)}{(m+1)} + 8.$$

$$(iii) \text{ Q-spec}(\Gamma_{V_{8m}}) = \left\{ [4]^{2m-3}, [2m]^2, [2(m-1)]^1, \left[m+3 + \sqrt{(m-1)(m+7)} \right]^1, \right. \\ \left. \left[m+3 - \sqrt{(m-1)(m+7)} \right]^1 \right\} \text{ and}$$

$$\text{LE}^+(\Gamma_{V_{8m}}) = \begin{cases} 8, & \text{for } m = 2, \\ \frac{16(m-1)(m-2)}{m+1} + 2(m-1) \left(\sqrt{1 + \frac{8}{m-1}} - 1 \right), & \text{for } m \neq 2. \end{cases}$$

(2) For m is odd

$$(i) \text{ Spec}(\Gamma_{V_{8m}}) = \left\{ [0]^{2m-2}, [-1]^1, \left[\frac{1+\sqrt{16m-7}}{2} \right]^1, \left[\frac{1-\sqrt{16m-7}}{2} \right]^1 \right\} \text{ and}$$

$$\text{E}(\Gamma_{V_{8m}}) = 1 + \sqrt{16m-7}.$$

$$(ii) \text{ L-spec}(\Gamma_{V_{8m}}) = \left\{ [0]^1, [2]^{2m-2}, [2m+1]^2 \right\} \text{ and } \text{LE}(\Gamma_{V_{8m}}) = \frac{(4m-2)^2+8}{(2m+1)}.$$

$$(iii) \text{ Q-spec}(\Gamma_{V_{8m}}) = \left\{ [2]^{2m-2}, [2m-1]^1, \left[\frac{2m+3+\sqrt{(2m-1)(2m+7)}}{2} \right]^1, \right. \\ \left. \left[\frac{2m+3-\sqrt{(2m-1)(2m+7)}}{2} \right]^1 \right\} \text{ and}$$

$$\text{LE}^+(\Gamma_{V_{8m}}) = \frac{3(2m-1)(2m-3)}{2m+1} + (2m-1) \sqrt{1 + \frac{8}{(2m-1)}}.$$

Proof. (a) If m is even then by [22, Proposition 2.4], we have $CCC(V_{8m}) = 2K_2 \cup K_{2m-2}$. Therefore, $\Gamma_{V_{8m}} \cong K_{2,2,1,2m-2}$. Here $|v(\Gamma_{V_{8m}})| = 2(m+1)$ and $r = 3$.

(i) Using Lemma 2.1(a), we get

$$P(\Gamma_{V_{8m}}, x) = x^{2(m+1)-3} (x+2) (x^2 - 2x - 8(m-1)).$$

Thus, $\text{Spec}(\Gamma_{V_{8m}}) = \left\{ [0]^{2m-1}, [-2]^1, \left[1 + \sqrt{1 + 8(m-1)}\right]^1, \left[1 - \sqrt{1 + 8(m-1)}\right]^1 \right\}$ and so

$$\begin{aligned} E(\Gamma_{V_{8m}}) &= (2m-1) \times 0 + 1 \times |-2| + 1 \times \left|1 + \sqrt{1 + 8(m-1)}\right| + 1 \times \left|1 - \sqrt{1 + 8(m-1)}\right| \\ &= 2 \left(1 + \sqrt{1 + 8(m-1)}\right). \end{aligned}$$

(ii) Using Lemma 2.1(b), we get

$$P_L(\Gamma_{V_{8m}}, x) = x(x-4)^{2m-3} (x-2m)^2 (x-2(m+1))^2.$$

So, $\text{L-spec}(\Gamma_{V_{8m}}) = \left\{ [0]^1, [4]^{2m-3}, [2m]^2, [2(m+1)]^2 \right\}.$

Now $|2e(\Gamma_{V_{8m}})| = 8(2m-1)$ and so $\Delta(\Gamma_{V_{8m}}) = \frac{|2e(\Gamma_{V_{8m}})|}{|v(\Gamma_{V_{8m}})|} = \frac{4(2m-1)}{(m+1)}.$ We have

$$\begin{aligned} |0 - \Delta(\Gamma_{V_{8m}})| &= \left| \frac{-4(2m-1)}{(m+1)} \right| = \frac{4(2m-1)}{(m+1)}, \quad |4 - \Delta(\Gamma_{V_{8m}})| = \left| \frac{-4(m-2)}{(m+1)} \right| = \frac{4(m-2)}{(m+1)}, \\ |2m - \Delta(\Gamma_{V_{8m}})| &= \left| \frac{2(m-2)(m-1)}{(m+1)} \right| = \frac{2(m-2)(m-1)}{(m+1)}, \end{aligned}$$

and

$$|(2m+2) - \Delta(\Gamma_{V_{8m}})| = \left| \frac{2((m-1)^2 + 2)}{(m+1)} \right| = \frac{2((m-1)^2 + 2)}{(m+1)}.$$

Therefore,

$$\begin{aligned} \text{LE}(\Gamma_{V_{8m}}) &= 1 \times \left| \frac{-4(2m-1)}{(m+1)} \right| + (2m-3) \times \left| \frac{-4(m-2)}{(m+1)} \right| + 2 \times \left| \frac{2(m-2)(m-1)}{(m+1)} \right| \\ &\quad + 2 \times \left| \frac{2((m-1)^2 + 2)}{(m+1)} \right| \\ &= \frac{16(m-1)(m-2)}{(m+1)} + 8. \end{aligned}$$

(iii) Using Lemma 2.1(c), we get

$$P_Q(\Gamma_{V_{8m}}, x) = (x-4)^{2m-3} (x-2m)^2 (x-2(m-1)) (x^2 - (2m+6)x + 4m^2 + 4m + 16)$$

Therefore,

$$\begin{aligned} \text{Q-spec}(\Gamma_{V_{8m}}) &= \left\{ [4]^{2m-3}, [2m]^2, [2(m-1)]^1, \left[m + 3 + \sqrt{(m-1)(m+7)} \right]^1, \right. \\ &\quad \left. \left[m + 3 - \sqrt{(m-1)(m+7)} \right]^1 \right\}. \end{aligned}$$

We have

$$|2(m-1) - \Delta(\Gamma_{V_{8m}})| = \left| \frac{2((m-2)^2 - 3)}{(m+1)} \right| = \begin{cases} 2, & \text{for } m = 2, \\ \frac{2((m-2)^2 - 3)}{(m+1)}, & \text{for } m \geq 4, \end{cases}$$

$$\begin{aligned} \left| \left(m + 3 + (m-1)\sqrt{1 + \frac{8}{m-1}} \right) - \Delta(\Gamma_{V_{8m}}) \right| &= \left| \frac{\left((m-2)^2 + 3 + (m+1)\sqrt{(m-1)(m+7)} \right)}{(m+1)} \right| \\ &= \frac{\left((m-2)^2 + 3 + (m+1)\sqrt{(m-1)(m+7)} \right)}{(m+1)}, \end{aligned}$$

and

$$\left| \left(m + 3 - (m-1)\sqrt{1 + \frac{8}{m-1}} \right) - \Delta(\Gamma_{V_{8m}}) \right| = \left| \frac{\left((m-2)^2 + 3 - (m+1)\sqrt{(m-1)(m+7)} \right)}{(m+1)} \right|.$$

Let $f_1(m) := (m-2)^2 + 3 - (m+1)\sqrt{(m-1)(m+7)}$. Then for $m \geq 4$, $((m-2)^2 + 3)^2 - (m+1)^2(m-1)(m+7) = -8m^2(2m-3) - 48m + 56 < 0$ and so $(m-2)^2 + 3 < (m+1)\sqrt{(m-1)(m+7)}$ which gives $f_1(m) < 0$. Also we have, $f_1(2) = -6$. Therefore,

$$\left| \left(m + 3 - (m-1)\sqrt{1 + \frac{8}{m-1}} \right) - \Delta(\Gamma_{V_{8m}}) \right| = \frac{\left((m-2)^2 + 3 - (m+1)\sqrt{(m-1)(m+7)} \right)}{(m+1)}.$$

$$\begin{aligned} \text{LE}^+(\Gamma_{V_{8m}}) &= (2m-3) \times \left| \frac{-4(m-2)}{(m+1)} \right| + 2 \times \left| \frac{2(m-2)(m-1)}{(m+1)} \right| \\ &\quad + 1 \times \left| \frac{2((m-2)^2 - 3)}{(m+1)} \right| + \left| \frac{\left((m-2)^2 + 3 + (m+1)\sqrt{(m-1)(m+7)} \right)}{(m+1)} \right| \\ &\quad + \left| \frac{\left((m-2)^2 + 3 - (m+1)\sqrt{(m-1)(m+7)} \right)}{(m+1)} \right| \\ &= \begin{cases} 8, & \text{for } m = 2, \\ \frac{16(m-1)(m-2)}{(m+1)} + 2(m-1) \left(\sqrt{1 + \frac{8}{m-1}} - 1 \right), & \text{for } m \geq 4. \end{cases} \end{aligned}$$

Hence the result follows.

(b) If m is odd, then by [22, Proposition 2.4], we have $\Gamma_{V_{8m}} \cong K_{2 \cdot 1, 2m-1} \cong \Gamma_{D_{2 \times 4m}}$. Hence, the result follows from Corollary 2.6. $\square$

3. GROUPS WHOSE NCCC-GRAPHS ARE INTEGRAL, L-INTEGRAL AND Q-INTEGRAL

In this section, we determine whether NCCC-graphs of the groups considered in Section 2 are integral, L-integral and Q-integral. In view of Theorem 2.3–Theorem 2.11, we get the following results.

Theorem 3.1. *Let G be a finite non-abelian group with center $Z(G)$.*

- (1) *Suppose that $G = D_{2m}, T_{4m}, U_{(n,m)}, SD_{8m}$, where m is odd. Then Γ_G is integral if and only if $m = 2a^2 + 1$, where $a \geq 1$.*
- (2) *Suppose that $G = D_{2m}, U_{(n,m)}$, where m is even and $\frac{m}{2}$ is odd. Then Γ_G is integral if and only if $m = a^2 + 2$, where $a \geq 2$.*
- (3) *Suppose that $G = D_{2m}$, where m and $\frac{m}{2}$ are even. Then Γ_G is integral if and only if $4m - 7$ is a perfect square.*
- (4) *Suppose that m is an even integer. Then $\Gamma_{T_{4m}}$ and $\Gamma_{SD_{8m}}$ are integral if and only if $8m - 7$ and $16m - 7$ are perfect squares respectively.*
- (5) *For m is even, $\Gamma_{V_{8m}}$ is integral if and only if $8m - 7$ is a perfect square. For m is odd, $\Gamma_{V_{8m}}$ is integral if and only if $16m - 7$ is a perfect square.*

Theorem 3.2. *Let G be a finite non-abelian group with center $Z(G)$. Then the NCCC-graph of G is*

- (1) *integral, L-integral and Q-integral if $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ or $G \cong U_{6m}$.*
- (2) *L-integral if $G \cong D_{2m}, T_{4m}, SD_{8m}, U_{(n,m)}$ or V_{8m} . In general, Γ_G is L-integral if $\frac{G}{Z(G)} \cong D_{2m}$.*
- (3) *Q-integral if $G \cong D_{2m}, T_{4m}, SD_{8m}$ or $U_{(n,m)}$, where m is odd.*
- (4) *Q-integral if $G \cong D_{2m}$ or $U_{(n,m)}$, where m is even and $\frac{m}{2}$ is odd.*

The following lemma is useful for the cases when m and $\frac{m}{2}$ are both even.

Lemma 3.3. *Let m be any positive integer. Then*

- (1) *$m^2 + 6m - 7$ is a perfect square if and only if $m = 1, 2$.*
- (2) *$m^2 + 12m - 28$ is a perfect square if and only if $m = 2, 4, 11$.*
- (3) *$4m^2 + 12m - 7$ is a perfect square if and only if $m = 1$.*

Proof. (a) Let $m^2 + 6m - 7$ be a perfect square. Then there exist integers k such that $m^2 + 6m - 7 = k^2$ which in turn gives $(m + 3 + k)(m + 3 - k) = 16$. Therefore, we have the following cases.

Case 1. $m + 3 + k = 16$ and $m + 3 - k = 1$.

In this case, we have $m + k = 13$ and $m - k = -2$ which gives $m = \frac{11}{2}$ and $k = \frac{15}{2}$.

Case 2. $m + 3 + k = -16$ and $m + 3 - k = -1$.

In this case, we have $m + k = -19$ and $m - k = -4$ which gives $m = \frac{-23}{2}$ and $k = \frac{-15}{2}$.

Case 3. $m + 3 + k = 1$ and $m + 3 - k = 16$.

In this case, we have $m + k = -2$ and $m - k = 13$ which gives $m = \frac{11}{2}$ and $k = \frac{-15}{2}$.

Case 4. $m + 3 + k = -1$ and $m + 3 - k = -16$.

In this case, we have $m + k = -4$ and $m - k = -19$ which gives $m = \frac{-23}{2}$ and $k = \frac{15}{2}$.

Case 5. $m + 3 + k = 2$ and $m + 3 - k = 8$.

In this case, we have $m + k = -1$ and $m - k = 5$ which gives $m = 2$ and $k = -3$.

Case 6. $m + 3 + k = -2$ and $m + 3 - k = -8$.

In this case, we have $m + k = -5$ and $m - k = -11$ which gives $m = -8$ and $k = 3$.

Case 7. $m + 3 + k = 8$ and $m + 3 - k = 2$.

In this case, we have $m + k = 5$ and $m - k = -1$ which gives $m = 2$ and $k = 3$.

Case 8. $m + 3 + k = -8$ and $m + 3 - k = -2$.

In this case, we have $m + k = -11$ and $m - k = -5$ which gives $m = -8$ and $k = -3$.

Case 9. $m + 3 + k = 4$ and $m + 3 - k = 4$.

In this case, we have $m + k = 1$ and $m - k = 1$ which gives $m = 1$ and $k = 0$.

Case 10. $m + 3 + k = -4$ and $m + 3 - k = -4$.

In this case, we have $m + k = -7$ and $m - k = -7$ which gives $m = -7$ and $k = 0$.

Hence the result follows.

(b) If $m^2 + 12m - 28$ is a perfect square. Then there exist integers k such that $m^2 + 12m - 28 = k^2$ which in turn gives $(m + 6 + k)(m + 6 - k) = 64$. By considering different cases as above we get $m = 2, 4, 11$. Hence, the result follows.

(c) If $4m^2 + 12m - 7$ is a perfect square. Then there exist integers k such that $4m^2 + 12m - 7 = k^2$ which in turn gives $(2m + 3 + k)(2m + 3 - k) = 16$. By considering different cases as above we get $m = 1$. Hence, the result follows. $\square$

Theorem 3.4. *Let G be a finite non-abelian group.*

- (1) *If $G \cong D_{2m}$ or $U_{(n,m)}$, where m and $\frac{m}{2}$ are both even, then Γ_G is Q -integral if and only if $m = 4$.*
- (2) *If $G \cong T_{4m}$, where m is even, then Γ_G is Q -integral if and only if $m = 2$.*
- (3) *If $G \cong V_{8m}$ then Γ_G is Q -integral if and only if $m = 2$.*
- (4) *If $G \cong SD_{8m}$, where m is even, then Γ_G is not Q -integral.*

Proof. (a) Suppose that m and $\frac{m}{2}$ are both even. If $G \cong D_{2m}$ (where $m \geq 4$) then, in view of Corollary 2.6(c), it is sufficient to show that $\frac{m+6+\sqrt{m^2+12m-28}}{4}$ and $\frac{m+6-\sqrt{(m-2)(m+14)}}{4}$ are integers. By Lemma 3.3(b), we have $m = 4$. Note that $\frac{m+6+\sqrt{m^2+12m-28}}{4} = 4$ and $\frac{m+6-\sqrt{m^2+12m-28}}{4} = 1$.

If $G \cong U_{(n,m)}$ (where $m \geq 4$ and $n \geq 2$) then, in view of Corollary 2.9(c), it is sufficient to show that $\frac{n(m+6+\sqrt{m^2+12m-28})}{4}$ and $\frac{n(m+6-\sqrt{m^2+12m-28})}{4}$ are integers. By Lemma 3.3(b), we have $m = 4$. Note that $\frac{n(m+6+\sqrt{m^2+12m-28})}{4} = 4n$ and $\frac{n(m+6-\sqrt{m^2+12m-28})}{4} = n$.

(b) Suppose that m is even. If $G \cong T_{4m}$ (where $m \geq 2$) then, in view of Corollary 2.7(a), it is sufficient to show that $\frac{m+3+\sqrt{m^2+6m-7}}{2}$ and $\frac{m+3-\sqrt{m^2+6m-7}}{2}$ are integers. By Lemma 3.3(a), we have $m = 2$. Note that $\frac{m+3+\sqrt{m^2+6m-7}}{2} = 4$ and $\frac{m+3-\sqrt{m^2+6m-7}}{2} = 1$.

(c) Suppose that m is even. If $G \cong V_{8m}$ (where $m \geq 2$) then, in view of Theorem 2.11(a), it is sufficient to show that $m+3+\sqrt{m^2+6m-7}$ and $m+3-\sqrt{m^2+6m-7}$ are integers. By Lemma 3.3(a), we have $m = 2$. Note that $m+3+\sqrt{m^2+6m-7} = 8$ and $m+3-\sqrt{m^2+6m-7} = 2$.

If m is odd then, by Theorem 2.11(b), it is sufficient to show that $\frac{2m+3+\sqrt{4m^2+12m-7}}{2}$ and $\frac{2m+3-\sqrt{4m^2+12m-7}}{2}$ are integers. By Lemma 3.3(c), we have $m = 1$ but $m \geq 3$. Thus, Γ_G is not Q-integral.

(d) If m is even and $G = SD_{8m}$ (where $m \geq 2$) then, by Corollary 2.10(a), it is sufficient to show that $\frac{2m+3+\sqrt{4m^2+12m-7}}{2}$ and $\frac{2m+3-\sqrt{4m^2+12m-7}}{2}$ are integers. By Lemma 3.3(c), we have $m = 1$. Thus, Γ_G is not Q-integral. $\square$

In view of Theorem 2.5 and Lemma 3.3(a), we also have the following result.

Theorem 3.5. *Let G be a finite group and $\frac{G}{Z(G)} \cong D_{2m}$, where $m \geq 3$. Then Γ_G is Q-integral if and only if m is odd.*

The integrality results for the NCCC-graphs of the discussed groups are summarized in Tables 1-2.

TABLE 1. Integrality conditions for Γ_G for given $G/Z(G)$.

Groups with given central factor	Condition for integrality	Condition for L-integrality	Condition for Q-integrality
$\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for any prime p	Always integral	Always L-integral	Always Q-integral
$\frac{G}{Z(G)} \cong D_{2m}$, $m \geq 3$	Undetermined	Always L-integral	m is odd

4. GROUPS WHOSE NCCC-GRAPHS ARE HYPERENERGETIC, L-HYPERENERGETIC AND Q-HYPERENERGETIC

In this section, we consider NCCC-graphs of the groups considered in Section 2 and determine whether those graphs are hyperenergetic, L-hyperenergetic and Q-hyperenergetic.

TABLE 2. Integrality conditions for Γ_G .

Groups	Condition for integrality	Condition for L-integrality	Condition for Q-integrality
D_{2m}	(i) m is odd and $m = 2a^2 + 1$ for some integer a (ii) m is even but $\frac{m}{2}$ is odd and $m = a^2 + 2$ for some integer $a \geq 2$ (iii) m and $\frac{m}{2}$ are both even and $4m - 7$ is a perfect square	Always L-integral	Always Q-integral
$U_{(n,m)}$	(i) m is odd and $m = 2a^2 + 1$ for some integer $a \geq 1$ (ii) m is even but $\frac{m}{2}$ is odd and $m = a^2 + 2$ for some integer $a \geq 2$ (iii) m and $\frac{m}{2}$ both are even and $4m - 7$ is a perfect square	Always L-integral	(i) m is odd (ii) m is even but $\frac{m}{2}$ is odd (iii) $m = 4$
T_{4m}	(i) m is odd and $m = 2a^2 + 1$ for some integer $a \geq 1$ (ii) m is even and $8m - 7$ is a perfect square	Always L-integral	(i) m is even (ii) $m = 2$
SD_{8m}	(i) m is odd and $m = 2a^2 + 1$ for some integer $a \geq 1$ (ii) m is even and $16m - 7$ is a perfect square	Always L-integral	m is odd
V_{8m}	(i) m is even and $8m - 7$ is a perfect square (ii) m is odd and $16m - 7$ is a perfect square	Always L-integral	$m = 2$

Theorem 4.1. *Let G be a finite non-abelian group with center $Z(G)$ such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is any prime. Then the NCCC-graph of G is neither hyperenergetic, L-hyperenergetic nor Q-hyperenergetic.*

Proof. Let $|Z(G)| = z$ and $n = \frac{(p-1)z}{p}$. Then $|v(\Gamma_G)| = n(p+1)$ and, by Theorem 2.3 and (1), we get

$$E(\Gamma_G) - E(K_{|v(\Gamma_G)|}) = -2(n-1).$$

Hence, $E(\Gamma_G) - E(K_{|v(\Gamma_G)|}) \leq 0$. By Theorem 2.3, we also have $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$ and so the NCCC-graph of G is neither hyperenergetic, L-hyperenergetic nor Q-hyperenergetic.

□

The following corollary follows immediately.

Corollary 4.2. *Let G be a non-abelian group of order p^m with $|Z(G)| = p^{m-2}$. Then the NCCC-graph of G is neither hyperenergetic, L-hyperenergetic nor Q-hyperenergetic.*

Theorem 4.3. *Let G be a finite group with center $Z(G)$ and $|Z(G)| = z$. If $\frac{G}{Z(G)}$ is isomorphic to D_{2m} (where $m \geq 3$). Then the NCCC-graph of G is*

- (1) *borderenergetic if $m = 3$ and $z = 1$.*
- (2) *not hyperenergetic.*
- (3) *L-borderenergetic if $m = 3$ and $z = 1$.*
- (4) *L-hyperenergetic except for $m \geq 5$ and $z = 1$; $m = 3$ and $z \geq 2$; $m = 5$ and $z = 1, 2$ and $m = 7$ and $z = 1$.*
- (5) *Q-borderenergetic if $m = 3$ and $z = 1$.*
- (6) *Q-hyperenergetic except for $m = 4$ and $z = 2$; $m \geq 5$ and $z = 1$; $m = 3$ and $z \geq 2$; $m = 5$ and $z = 1, 2$ and $m = 7$ and $z = 1$.*

Proof. We have $|v(\Gamma_G)| = \frac{z(m+1)}{2}$. Consider the following two cases.

Case 1. m is even.

In this case, $m \geq 4$ and $z \geq 2$. By Theorem 2.5 and (1), we have

$$E(\Gamma_G) - E(K_{|v(\Gamma_G)|}) = \frac{z(-1 + \sqrt{8m-7} - 2m) + 4}{2} < 0,$$

since $(-1 + \sqrt{8m-7} - 2m) \leq -4$ for $m \geq 4$.

By Theorem 2.5 and (1), we have

$$LE(\Gamma_G) - LE(K_{|v(\Gamma_G)|}) = \frac{f_1(m, z)}{m+1},$$

where $f_1(m, z) = m^2z^2 - m^2z - 3mz^2 + 2m + 2z^2 + z + 2$. Then $f_1(m, z) = \frac{zm^2(z-2)}{2} + \frac{z^2m(m-6)}{2} + 2m + 2z^2 + z + 2 > 0$ for $m \geq 6$ and $z \geq 2$. Now, $f_1(4, z) = 6z^2 - 15z + 10 > 0$ for $z \geq 2$. Hence,

$$LE(\Gamma_G) - LE(K_{|v(\Gamma_G)|}) > 0.$$

By Theorem 2.5 and (1), we have

$$LE^+(\Gamma_G) - LE^+(K_{|v(\Gamma_G)|}) = \frac{f_2(m, z)}{2(m+1)},$$

where $f_2(m, z) = 2z^2m^2 - 6z^2m - 3zm^2 - z - 4mz + 4z^2 + 4m + 4 + z(m+1)\sqrt{(m-1)(m+7)}$. Then $f_2(m, z) = z^2m(m-6) + zm^2(z-4) + zm(m-4) + \frac{z(z-4)}{4} + 4m + 4 + z(m+1)\sqrt{(m-1)(m+7)} > 0$ for $m \geq 6$ and $z \geq 4$.

We have $f_2(4, z) = 12z^2 + (5\sqrt{33} - 65)z + 20 > 0$ for $z \geq 3$. Also $f_2(4, 2) = -4.55$. Therefore $f_2(4, z) < 0$ or > 0 according as $z = 2$ or $z \geq 3$.

We have $f_2(m, 2) = 2m^2 - 28m + 18 + 2(m+1)\sqrt{(m-1)(m+7)} < 0$ or > 0 according as $m = 4$ or $m \geq 6$. Also, $f_2(m, 3) = 9m^2 - 62m + 37 + 3(m+1)\sqrt{(m-1)(m+7)} > 0$ for $m \geq 4$. Hence,

$$\text{LE}^+(\Gamma_G) - \text{LE}^+(K_{|v(\Gamma_G)|}) \begin{cases} < 0, & \text{if } m = 4 \text{ and } z = 2, \\ > 0, & \text{otherwise.} \end{cases}$$

Case 2. m is odd.

By Theorem 2.5 and (1), we have

$$\text{E}(\Gamma_G) - \text{E}(K_{|v(\Gamma_G)|}) = z \left(\sqrt{2(m-1)} - m - 1 \right) + 2.$$

We have $\left(\sqrt{2(m-1)} - m - 1 \right) < -2$ for $m \geq 5$. Hence, $\text{E}(\Gamma_G) - \text{E}(K_{|v(\Gamma_G)|}) = 0$ for $m = 3$ and $z = 1$; otherwise $\text{E}(\Gamma_G) - \text{E}(K_{|v(\Gamma_G)|}) < 0$.

By Theorem 2.5 and (1), we have

$$\text{LE}(\Gamma_G) - \text{LE}(K_{|v(\Gamma_G)|}) = \frac{f_3(m, z)}{m+1},$$

where $f_3(m, z) = m^2z^2 - m^2z - 4mz^2 + 2m + 3z^2 + z + 2$. Then $f_3(m, z) = \frac{zm^2(z-2)}{2} + \frac{z^2m(m-8)}{2} + 2m + 3z^2 + z + 2 > 0$ for $m \geq 8$ and $z \geq 2$. Now, $f_3(m, 1) = -2m + 6 < 0$ for $m \geq 5$. Therefore, $f_3(m, 1) = 0$ or < 0 according as $m = 3$ or $m \geq 5$. We have $f_3(3, z) = -8z + 8 < 0$ for $z \geq 2$. Also $f_3(5, z) = 8z^2 - 24z + 12 > 0$ for $z \geq 3$ and $f_3(5, 1) = f_3(5, 2) = -4$. So, $f_3(5, z) > 0$ or < 0 according as $z \geq 3$ or $z = 1, 2$. Again $f_3(7, z) = 24z^2 - 48z + 16 > 0$ for $z \geq 2$ and $f_3(7, 1) = -8$. Therefore, $f_3(7, z) > 0$ or < 0 according as $z \geq 2$ or $z = 1$. Hence,

$$\text{LE}(\Gamma_G) - \text{LE}(K_{|v(\Gamma_G)|}) \begin{cases} = 0, & \text{if } m = 3 \& z = 1, \\ < 0, & \text{if } m \geq 5 \& z = 1; \ m = 3 \& z \geq 2; \\ & m = 5 \& z = 1, 2 \text{ and } m = 7 \& z = 1, \\ > 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.5, we also have $\text{LE}(\Gamma_G) = \text{LE}^+(\Gamma_G)$. Hence the result follows. $\square$

As a corollary of the above theorem we have the following result.

Corollary 4.4. *Let $G = D_{2m}, T_{4m}, SD_{8m}$ and $U_{(n,m)}$. Then*

- (1) Γ_G is borderenergetic if and only if $G = D_6, D_8, T_8$.

- (2) Γ_G is not hyperenergetic.
- (3) Γ_G is L-borderenergetic if and only if $G = D_6, D_8, T_8$.
- (4) Γ_G is not L-hyperenergetic if and only if $G = D_{2m}$ for m is odd and $m \geq 5$, D_{12} , D_{20} , T_{12}, T_{20} , SD_{24} , $U_{(n,m)}$ for $m = 3, 4, 6$ and $n \geq 2$ and $m = 5$ and $n = 2$.
- (5) Γ_G is Q-borderenergetic if and only if $G = D_6, D_8, T_8$.
- (6) Γ_G is not Q-hyperenergetic if and only if $G = D_{2m}$ for m is odd and $m \geq 5$, D_{12}, D_{16}, D_{20} , T_{12}, T_{16}, T_{20} , SD_{16}, SD_{24} , $U_{(n,m)}$ for $m = 3, 4, 6$ and $n \geq 2$ and $m = 5$ and $n = 2$.

We conclude this section with the following result.

Theorem 4.5. *The NCCC-graph of V_{8m} ($m \geq 2$) is L-hyperenergetic and Q-hyperenergetic except for $m = 2$ but not hyperenergetic.*

Proof. If $m = 2$ then we have $|v(\Gamma_G)| = 6$. By Theorem 2.11 and (1), we get

$$E(\Gamma_G) - E(K_{|v(\Gamma_G)|}) = -2 < 0.$$

Also, $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$. Thus, Γ_G is neither hyperenergetic, L-hyperenergetic nor Q-hyperenergetic.

If m is even and $m \geq 4$ then we have $|v(\Gamma_G)| = 2m + 2$. By Theorem 2.11 and (1), we get

$$\begin{aligned} E(\Gamma_G) - E(K_{|v(\Gamma_G)|}) &= 2(\sqrt{8m-7} - 2m) < 0, \\ LE(\Gamma_G) - LE(K_{|v(\Gamma_G)|}) &= \frac{12m^2 - 46m + 38}{m+1} > 0, \end{aligned}$$

and

$$LE^+(\Gamma_G) - LE^+(K_{|v(\Gamma_G)|}) = \frac{10m^2 - 54m + 32 + 2(m+1)\sqrt{(m-1)(m+7)}}{m+1} > 0.$$

Thus, Γ_G is not hyperenergetic but L-hyperenergetic and Q-hyperenergetic.

If m is odd then we have $|v(\Gamma_G)| = 2m + 1$. By Theorem 2.11 and (1), we get

$$\begin{aligned} E(\Gamma_G) - E(K_{|v(\Gamma_G)|}) &= 1 + \sqrt{16m-7} - 4m < 0, \\ LE(\Gamma_G) - LE(K_{|v(\Gamma_G)|}) &= \frac{8m^2 - 20m + 12}{2m+1} > 0, \end{aligned}$$

and

$$LE^+(\Gamma_G) - LE^+(K_{|v(\Gamma_G)|}) = \frac{4m^2 - 28m + 9 + (2m+1)\sqrt{(2m-1)(2m+7)}}{2m+1} > 0.$$

Thus, Γ_G is not hyperenergetic but L-hyperenergetic and Q-hyperenergetic. $\square$

5. COMPARING DIFFERENT ENERGIES

Comparison among various energies become popular because of the E-LE conjecture: $E(\mathcal{G}) \leq LE(\mathcal{G})$ of Gutman et al. [12] which is eventually proved to be false in [19, 26]. Das et al. [6] posed the following problem comparing $LE(\mathcal{G})$ and $LE^+(\mathcal{G})$.

Problem 5.1. [6, Problem 1] Characterize all the graphs for which $LE(\mathcal{G}) > LE^+(\mathcal{G})$, $LE(\mathcal{G}) < LE^+(\mathcal{G})$ or $LE(\mathcal{G}) = LE^+(\mathcal{G})$.

Das et al. [6] also posed the following problem where $E_D(\mathcal{G})$, $E_{DL}(\mathcal{G})$ and $E_{DQ}(\mathcal{G})$ denote the distance energy, distance Laplacian energy and distance signless Laplacian energy of $\mathcal{G}$.

Problem 5.2. [6, Problem 4] Is there any connected graph $\mathcal{G} (\not\cong K_n)$ such that $E(\mathcal{G}) = LE(\mathcal{G}) = LE^+(\mathcal{G}) = E_D(\mathcal{G}) = E_{DL}(\mathcal{G}) = E_{DQ}(\mathcal{G})$?

Recently, Jannat and Nath [18] have computed several distance energies of NCCC-graphs of the groups considered in this paper and discussed relations (comparison) among them. In Theorem 2.3 and Corollary 2.8, we have seen that $E(\mathcal{G}) = LE(\mathcal{G}) = LE^+(\mathcal{G})$. In this section, we consider the groups D_{2m} , T_{4m} , SD_{8m} , V_{8m} and compare energy, Laplacian energy and signless Laplacian energy of NCCC-graphs of these groups graphically.

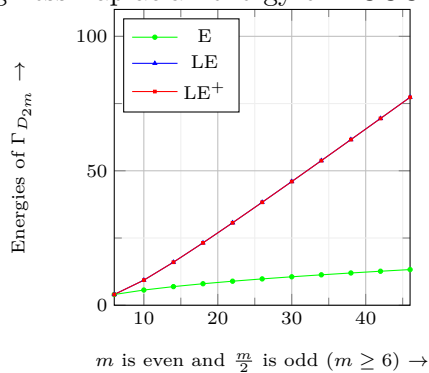


FIGURE 2. Energies of $\Gamma_{D_{2m}}$, m is even and $\frac{m}{2}$ is odd.

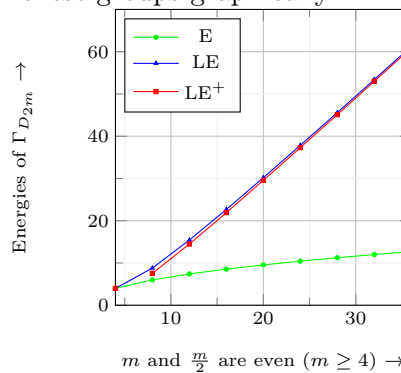


FIGURE 3. Energies of $\Gamma_{D_{2m}}$, m and $\frac{m}{2}$ are even.

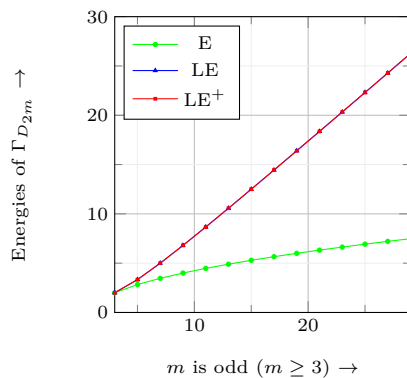


FIGURE 4. Energies of $\Gamma_{D_{2m}}$, m is odd.

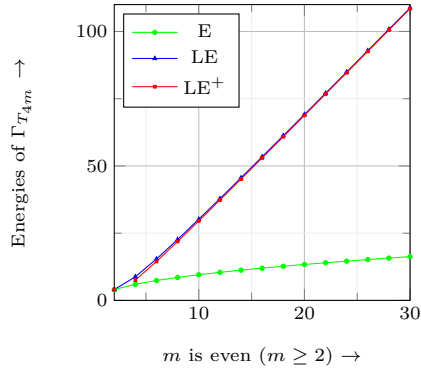


FIGURE 5. Energies of $\Gamma_{T_{4m}}$, m is even.

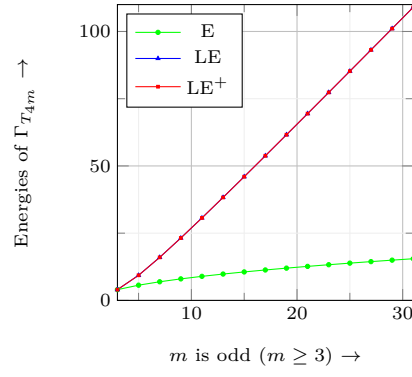


FIGURE 6. Energies of $\Gamma_{T_{4m}}$, m is odd.

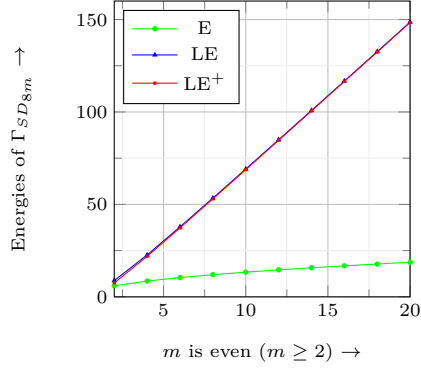


FIGURE 7. Energies of $\Gamma_{SD_{8m}}$, m is even.

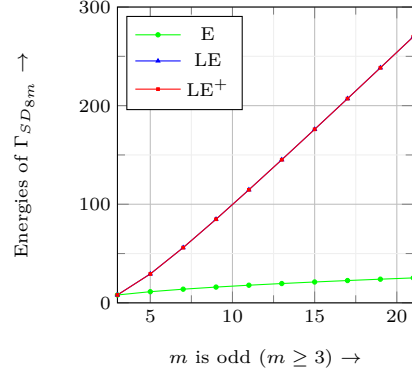


FIGURE 8. Energies of $\Gamma_{SD_{8m}}$, m is odd.

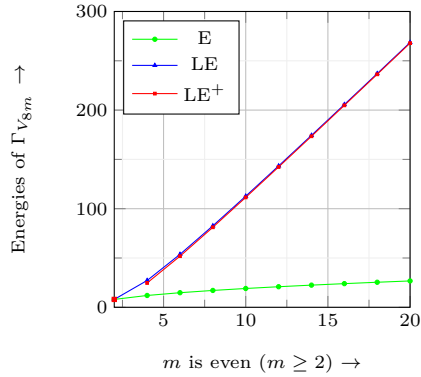


FIGURE 9. Energies of $\Gamma_{V_{8m}}$, m is even.

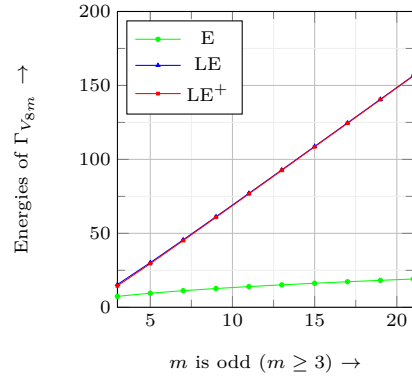


FIGURE 10. Energies of $\Gamma_{V_{8m}}$, m is odd.

Based on Figures 2–10, we establish the following result.

Theorem 5.3. *Let $G = D_{2m}, T_{4m}, SD_{8m}$ or V_{8m} . Then*

- (1) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G)$ if and only if $G = D_6, D_8, D_{12}, T_8, T_{12}, SD_{24}$ or V_{16} .
- (2) $E(\Gamma_G) < LE(\Gamma_G) = LE^+(\Gamma_G)$ if and only if $G = D_{2m}$, where m is odd or $m \geq 10$ is even and $\frac{m}{2}$ is odd; T_{4m} , where $m \geq 5$ is odd; SD_{8m} , where $m \geq 5$ is odd.

- (3) $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G)$ if and only if $G = D_{2m}$, where $m \geq 8$ and $\frac{m}{2}$ is even;
 T_{4m} , where $m \geq 4$ is even; SD_{8m} , where m is even; V_{8m} , where $m \geq 3$.

In view of Corollary 2.8, Theorem 5.3 and [18, Theorem 4.1], we obtain the following result.

Theorem 5.4. *Let $G = D_{2m}, T_{4m}, SD_{8m}, U_{6m}$ or V_{8m} . Then*

- (1) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = E_D(\Gamma_G) = E_{DL}(\Gamma_G) = E_{DQ}(\Gamma_G)$ if and only if $G = D_8$ or T_8 .
- (2) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = E_{DL}(\Gamma_G) = E_{DQ}(\Gamma_G) < E_D(\Gamma_G)$ if and only if $G = D_6$.
- (3) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) < E_D(\Gamma_G) = E_{DL}(\Gamma_G) = E_{DQ}(\Gamma_G)$ if and only if $G = V_{16}$.
- (4) $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) < E_{DL}(\Gamma_G) = E_{DQ}(\Gamma_G) < E_D(\Gamma_G)$ if and only if $G = D_{12}$ or T_{12} or SD_{24} or U_{6m} , where $m \geq 2$.
- (5) $E(\Gamma_G) < LE(\Gamma_G) = LE^+(\Gamma_G) < E_{DQ}(\Gamma_G) < E_D(\Gamma_G) = E_{DL}(\Gamma_G)$ if and only if $G = D_{28}$ or T_{28} .
- (6) $E(\Gamma_G) < LE(\Gamma_G) = LE^+(\Gamma_G) < E_{DL}(\Gamma_G) < E_{DQ}(\Gamma_G) < E_D(\Gamma_G)$ if and only if $G = D_{10}$.
- (7) $E(\Gamma_G) < LE(\Gamma_G) = LE^+(\Gamma_G) < E_{DQ}(\Gamma_G) < E_{DL}(\Gamma_G) < E_D(\Gamma_G)$ if and only if $G = D_{14}$ or D_{20} or T_{20} or SD_{40} .
- (8) $E(\Gamma_G) < E_{DQ}(\Gamma_G) < LE(\Gamma_G) = LE^+(\Gamma_G) < E_{DL}(\Gamma_G) < E_D(\Gamma_G)$ if and only if $G = D_{18}$ or D_{22} or D_{26} .
- (9) $E(\Gamma_G) < E_{DQ}(\Gamma_G) < LE(\Gamma_G) = LE^+(\Gamma_G) < E_D(\Gamma_G) < E_{DL}(\Gamma_G)$ if and only if $G = D_{2m}$, where $m \geq 15$ is odd; $m \geq 18$ is even and $\frac{m}{2}$ is odd or T_{4m} , where $m \geq 9$ is odd or SD_{56} or SD_{72} .
- (10) $E(\Gamma_G) < E_{DQ}(\Gamma_G) < E_D(\Gamma_G) < LE(\Gamma_G) = LE^+(\Gamma_G) < E_{DL}(\Gamma_G)$ if and only if $G = SD_{88}$.
- (11) $E(\Gamma_G) < E_D(\Gamma_G) < E_{DQ}(\Gamma_G) < LE(\Gamma_G) = LE^+(\Gamma_G) < E_{DL}(\Gamma_G)$ if and only if $G = SD_{8m}$, where $m \geq 13$ is odd.
- (12) $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G) = E_{DQ}(\Gamma_G) < E_D(\Gamma_G) < E_{DL}(\Gamma_G)$ if and only if $G = V_{32}$.
- (13) $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G) < E_D(\Gamma_G) < E_{DL}(\Gamma_G) < E_{DQ}(\Gamma_G)$ if and only if $G = D_{16}$ or T_{16} or SD_{16} .
- (14) $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G) < E_D(\Gamma_G) < E_{DQ}(\Gamma_G) < E_{DL}(\Gamma_G)$ if and only if $G = D_{2m}$, where $m \geq 12$ is even and $\frac{m}{2}$ is even or T_{4m} , where $m \geq 6$ is even or SD_{8m} , where $m \geq 4$ is even or V_{8m} , where $m \geq 3$ is odd.
- (15) $E(\Gamma_G) < E_D(\Gamma_G) < E_{DQ}(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G) < E_{DL}(\Gamma_G)$ if and only if $G = V_{8m}$, where $m \geq 6$ is even.

Note that $\Gamma_G \cong K_3$ if $G = D_8$ or T_8 . Therefore, Theorem 5.4(a) does not solve Problem 5.2, and the following problem remains open.

Problem 5.5. Does there exist a finite non-abelian group G such that Γ_G is not a complete graph and $E(\Gamma_G) = LE(\Gamma_G) = LE^+(\Gamma_G) = E_D(\Gamma_G) = E_{DL}(\Gamma_G) = E_{DQ}(\Gamma_G)$?

We remark that the method adopted in this paper is applicable to all group-related graphs. However, it does not fully characterize the finite non-abelian groups satisfying the inequalities in Problem 5.1. Consequently, we propose the following problem for future study.

Problem 5.6. Characterize all finite non-abelian groups for which $LE(\Gamma_G) > LE^+(\Gamma_G)$, $LE(\Gamma_G) < LE^+(\Gamma_G)$ or $LE(\Gamma_G) = LE^+(\Gamma_G)$.

6. ACKNOWLEDGMENTS

The authors would like to thank the referee for his/her valuable comments and suggestions which improve the quality of the paper.

REFERENCES

- [1] O. Ahmadi, N. Alon, I. F. Blake and I. E. Shaparlinski, *Graphs with integral spectrum*, Linear Algebra Appl., **430** No. 1 (2009) 547-552.
- [2] P. Bhowal and R. K. Nath, *Spectral aspects of commuting conjugacy class graph of finite groups*, Algebr. Struct. their Appl., **8** No. 2 (2021) 67-118.
- [3] P. Bhowal and R. K. Nath, *Genus of commuting conjugacy class graph of certain finite groups*, Algebr. Struct. their Appl., **9** No. 1 (2022) 93-108.
- [4] P. Bhowal and R. K. Nath, *Spectrum and energies of commuting conjugacy class graph of a finite group*, Algebr. Struct. their Appl., **12** No. 1 33-49, 2025.
- [5] P. J. Cameron, F. E. Jannat, R. K. Nath and R. Sharafadini, *A survey on conjugacy class graphs of groups*, Expo. Math., **42** No. 4 (2024) 125585.
- [6] K. C. Das, M. Aouchiche and P. Hansen, *On (distance) Laplacian energy and (distance) signless Laplacian energy of graphs*, Discrete Appl. Math., **243** (2018) 172-185.
- [7] W. N. T. Fafous, R. K. Nath and R. Sharafadini, *Various spectra and energies of commuting graphs of finite rings*, Hacet. J. Math. Stat., **49** No. 6 (2020) 1915-1925.
- [8] S. Gong, X. Li, G. Xu, I. Gutman and B. Furtula, *Borderenergetic Graphs*, MATCH Commun. Math. Comput. Chem., **74** (2015) 321-332.
- [9] R. Grone and R. Merris, *The Laplacian spectrum of a graph II*, SIAM J. Discrete Math. **7** No. 2 (1994) 221-299.
- [10] I. Gutman, *The energy of a graph*, Ber. Math.-Statist. Sect. Forschungsz. Graz, **103** (1978) 1-22.
- [11] I. Gutman, *Hyperenergetic molecular graphs*, J. Serb. Chem. Soc., **64** (1999) 199-205.
- [12] I. Gutman, N. M. M. Abreu, C. T. M. Vinagre, A. S. Bonifacio and S. Radenkovic, *Relation between energy and Laplacian energy*, MATCH Commun. Math. Comput. Chem., **59** (2008) 343-354.

- [13] I. Gutman and B. Furtula, *Graph energies and their applications*, Bulletin (Académie Serbe Des Sciences Et Des Arts. Classe Des Sciences Mathématiques Et Naturelles. Sciences Mathématiques, **44** (2019) 29-45.
- [14] I. Gutman and B. Zhou, *Laplacian energy of a graph*, Linear Algebra Appl. **414** (2006) 29-37.
- [15] F. Harary and A. J. Schwenk, *Which graphs have integral spectra?*, Graphs and Combinatorics: Lect. Notes Maths., **406**, 45-51, 1974.
- [16] M. Herzog, M. Longobardi and M. Maj, *On a commuting graph on conjugacy classes of groups*, Commun. Algebra, **37** No. 10 (2009) 3369-3387.
- [17] F. E. Jannat and R. K. Nath, *Common neighbourhood spectrum and energy of commuting conjugacy class graph*, J. Algebr. Syst., **12** No. 2 (2025) 301-326.
- [18] F. E. Jannat and R. K. Nath, *On distance spectra, energies and Wiener index of non-commuting conjugacy class graphs*, J. Algebra Comb. Discrete Struct. Appl., **13** No. 1 (2026) 43-73.
- [19] J. Liu and B. Liu, *On relation between energy and Laplacian energy*, MATCH Commun. Math. Comput. Chem., **61** (2009) 403-406.
- [20] A. Mohammadian, A. Erfanian, D. G. M. Farrokhi and B. Wilkens, *Triangle-free commuting conjugacy class graphs*, J. Group Theory, **19** (2016) 1049-1061.
- [21] M. A. Salahshour and A. R. Ashrafi, *Commuting conjugacy class graphs of finite groups*, Algebr. Struct. their Appl., **7** No. 2 (2020) 135-145.
- [22] M. A. Salahshour and A. R. Ashrafi, *Commuting conjugacy class graph of finite CA-groups*, Khayyam J. Math., **6** No. 1 (2020) 108-118.
- [23] M. A. Salahshour, *Commuting conjugacy class graph of G when $\frac{G}{Z(G)} \cong D_{2n}$* , Math. Interdiscip. Res., **1** (2020) 379-385.
- [24] M. Sharma and R. K. Nath, *Signless Laplacian energies of non-commuting graphs of finite groups and related results*, Discrete Math. Algorithms Appl. **17** No. 4 (2025) 2450060.
- [25] S. K. Simic and Z. Stanic, *Q -integral graphs with edge-degrees at most five*, Discrete Math., **308** (2008) 4625-4634.
- [26] D. Stevanović, I. Stevanović and M. Milosević, *More on the relation between energy and Laplacian energy of graphs*, MATCH Commun. Math. Comput. Chem., **61** (2009) 395-401.
- [27] Q. Tao and Y. Hou, *Q -borderenergetic graphs*, AKCE Int. J. Graphs Comb., **17** No. 1 (2020) 38-44.
- [28] F. Tura, *L -borderenergetic graphs*, MATCH Commun. Math. Comput. Chem., **77** (2017) 37-44.
- [29] H. B. Walikar, H. S. Ramane and P. R. Hampiholi, *On the energy of a graph*, In: *Graph connections*, Eds. R. Balakrishnan, H. M. Mulder and A. Vijayakumar, Allied publishers, New Delhi, 120-123, 1999.

Rishabh Chakraborty

Department of Mathematical Science,

Tezpur University, Napaam-784028,

Sonitpur, Assam, India.

Department of Mathematics,

Assam Engineering Institute, Chandmari,

Guwahati-781003, Assam, India.

`rchakraborty2101@gmail.com`

Firdous Ee Jannat

Department of Mathematical Science,

Tezpur University, Napaam-784028,

Sonitpur, Assam, India.

Discipline of Mathematics,

K. K. Handiqui State Open University,

Khanapara, Guwahati-781022, India.

`firdusej@gmail.com`

Rajat Kanti Nath

Department of Mathematical Science,

Tezpur University, Napaam-784028,

Sonitpur, Assam, India.

`rajatkantinath@yahoo.com`