Algebraic Structures and Their Applications Vol. X No. X (20XX) pp XX-XX.

Research Paper

MEET-MAXIMAL FILTER GRAPH OF A LATTICE

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Abstract. In this paper, we introduce and investigate the meet-maximal filter graph of a bounded lattice. It is the (undirected) graph with all nontrivial filters as vertices, and two distinct vertices are adjacent if and only if their meet is a maximal filter. The basic properties and possible structures of the meet-maximal filter graph are investigated. Also, connectedness, clique number, split character, planarity and domination number of meet-maximal filter graph and their relations to algebraic properties of lattices are explored.

1. Introduction

Over recent years, there has been a significant surge in research associating graphs with algebraic structures, with a particular emphasis on the zero-divisor graph of commutative rings. This interdisciplinary approach allows for a deeper understanding of concepts and inherent properties of both graph theory and algebraic structures. In abstract algebra, structures composed of certain subobjects tend to have their distinct qualities by way of lattice theoretic properties (see for example [9, 10, 11, 12, 13, 14, 15]).

DOI: 10.22034/as.2026.23878.1845

MSC(2010): Primary: 97H50, 06A11; Secondary: 16G30, 06D25..

Keywords: Lattice, Maximal filter, Meet-maximal graph.

Received: 27 October 2025, Accepted: 23 September 2026.

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Beck [4], Anderson and Naseer [3], and Anderson and Livingston [2] et. al. have studied graphs on commutative rings. One of the most important graphs which have been studied is the intersection graph. Bosak [5] defined the intersection graph of semigroups. Csákány and Pollák [8] studied the graph of subgroups of a finite group. The intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [7]. The intersection minimal ideal graph of a ring, i.e. a simple graph whose vertices are nontrivial ideals of a ring R and two vertices I, J are adjacent if the intersection of corresponding ideals is a minimal ideal, was investigated by Barman and Rajkhowa in [6]. In [1] the maximal ideal graph of a commutative ring was introduced; it is the undirected graph with all non-trivial ideals as vertices and two distinct vertices are adjacent if and only if sum of them is a maximal ideal. Motivated by this graph, we intend to investigate a new graph assigning to filters of a lattice to exposing the relationship between algebra and graph theory and at advancing the application of one to the other.

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. The purpose of this paper is to investigate a graph associated to a lattice $\mathcal{L}$ called the meet-maximal filter graph of $\mathcal{L}$. This will result in characterization of lattices in terms of some specific properties of those graphs. The meet-maximal filter graph of $\mathcal{L}$ is a simple graph $\mathbb{MG}(\mathcal{L})$ whose vertices are all nontrivial filters and two distinct vertices are adjacent if and only if the meet of the corresponding filters is a maximal filter of $\mathcal{L}$. Here is a brief outline of the article. Among many results in this paper, the first, Preliminaries section contains elementary observations needed later on. In Section 3, Section 4 and Section 5, We characterize the lattices for which the meet-maximal filter graphs are connected, complete bipartite, star. The concepts of planarity, clique number, domination number and split character are also investigated.

2. Preliminaries

Let G be a simple graph with vertex set $\mathcal{V}$ (G) and edge set $\mathcal{E}$ (G). The degree of a vertex v of the graph G, denoted by $\deg_G(v)$, is the number of edges incident to v. The (open) neighborhood $N(v)$ of a vertex v of $\mathcal{V}(G)$ is the set of vertices which are adjacent to v. A graph G is said to be connected if there exists a path between any two distinct vertices, G is a complete graph if every pair of distinct vertices of G are adjacent and K_n will stand for a complete graph with n vertices. If the vertices of G can be partitioned into two disjoint sets V_1 and V_2 with every vertex of V_1 is adjacent to any vertex of V_2 and no two vertices belonging to same set are adjacent, then G is called a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. If one of the partite sets contains exactly one element, then the graph becomes a star graph. If G graph does not have K_5 or

$K_{3,3}$ as its subgraph, then G is planar [19]. Let u and v be elements of $\mathcal{V}\ (G)$. We say that u is a universal vertex of G if u is adjacent to all other vertices of G and write $u \backsim v$ if u and v are adjacent. The distance $d(u, v)$ is the length of the shortest path from u to v if such path exists, otherwise, $d(a, b) = \infty$. The diameter of G is $\text{diam}(G) = \sup\{d(a, b) : a, b \in \mathcal{V}(G)\}$. The girth of a graph G, denoted by $\text{gr}(G)$, is the length of a shortest cycle in G. If G has no cycles, then $\text{gr}(G) = \infty$. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the clique number of G. A subset S of $\mathcal{V}\ (G)$ is said to be an independent set if no two vertices of S are adjacent. If $\mathcal{V}\ (G)$ can be partitioned in an independent set and a clique then G is said to be split. A set $D \subseteq \mathcal{V}\ (G)$ is said to be a dominating set if every vertex not in D is adjacent to at least one of the members of D. The cardinality of smallest dominating set is the domination number of the graph G and is denoted by $\gamma(G)$. Note that a graph whose vertices set is empty is a null graph and a graph whose edge set is empty is an empty graph. For a connected graph G, x is a cut vertex of G if $G \setminus \{x\}$ is not connected [17].

A poset $(\mathcal{L}, \leq)$ is a lattice if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the meet and $\vee$ the join). A lattice $\mathcal{L}$ is complete when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a distributive lattice if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in \mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ for all $x, y, z \in \mathcal{L}$). A non-empty subset F of a lattice $\mathcal{L}$ is called a filter, if for $a \in F$, $b \in \mathcal{L}$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\mathcal{L}$ is a lattice with $0, 1$, then $1 \in F$, $0 \in F$ if and only if $F = \mathcal{L}$ and $\{1\}$ is a filter of $\mathcal{L}$). A proper filter F of $\mathcal{L}$ is said to be maximal (resp. minimal) if G is a filter in $\mathcal{L}$ with $F \subsetneqq G$ (resp. $\{1\} \subsetneqq G \subseteq F$), then $G = \mathcal{L}$ (resp. $G = F$). The set of all maximal filters (resp. minimal filters) of $\mathcal{L}$ is denoted by $\text{Max}(\mathcal{L})$ (resp. $\text{Min}(\mathcal{L})$). The Jacobson radical of $\mathcal{L}$ is the intersection of all maximal filters of $\mathcal{L}$ and is denoted as $J(\mathcal{L})$, i.e. $J(\mathcal{L}) = \bigcap_{M \in \text{Max}(\mathcal{L})} M$. A lattice $\mathcal{L}$ is called local if it has exactly on maximal filter that contains all proper filters. Let A be subset of a lattice $\mathcal{L}$. Then the filter generated by A, denoted by $T(A)$, is the intersection of all filters that is containing A.

Lemma 2.1. *Let $\mathcal{L}$ be a lattice* [9, 10, 13, 15]. *(1) A non-empty subset F of $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \mathcal{L}$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \mathcal{L}$.*

(2) If F_1, F_2 are filters of $\mathcal{L}$ and $a \in \mathcal{L}$, then $F_1 \vee F_2 = \{a_1 \vee a_2 : a_1 \in F_1, a_2 \in F_2\}$ and $a \vee F_1 = \{a \vee a_1 : a_1 \in F_1\}$ are filters of $\mathcal{L}$ and $F_1 \cap F_2 = F_1 \vee F_2 \subseteq F_1, F_2$.

(3) If $\mathcal{L}$ is distributive and F_1, F_2 are filters of $\mathcal{L}$, then $F_1 \wedge F_2 = \{a \wedge b : a \in F_1, b \in F_2\}$ is a filter of $\mathcal{L}$, $F_1, F_2 \subseteq F_1 \wedge F_2$ (for if $x \in F_1$, then $x = x \wedge 1 \in F_1 \wedge F_2$) and if $F_1 \subseteq F_2$, then $F_1 \wedge F_2 = F_2$.

(4) Let A be an arbitrary non-empty subset of $\mathcal{L}$ and $a \in \mathcal{L}$. Then $T(\{a\}) = T(a) = \{a \vee x : x \in \mathcal{L}\}$ and

$$T(A) = \{x \in \mathcal{L} : \; a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x \text{ for some } a_i \in A \; (1 \leq i \leq n), n \in \mathbb{N}\}.$$

The undefined terms related to lattice theory are taken from [9] and terms related to graph theory are taken from [17].

3. Connectedness $\mathbb{MG}(\mathcal{L})$

In this section, we collect some basic properties concerning the meet-maximal filter graph $\mathbb{MG}(\mathcal{L})$ of $\mathcal{L}$. We remind the reader with the following definition.

Definition 3.1. The meet-maximal filter graph $\mathbb{MG}(\mathcal{L})$ of $\mathcal{L}$ is simple undirected graph whose vertices are all nontrivial filters of $\mathcal{L}$ (i.e. different from $\{1\}$ and $\mathcal{L}$) and any two distinct vertices F and G are adjacent if and only if $F \wedge G$ is a maximal filter of $\mathcal{L}$.

Lemma 3.2. *The following hold in $\mathbb{MG}(\mathcal{L})$:*

(1) Every non-maximal vertex of the graph $\mathbb{MG}(\mathcal{L})$ is adjacent to at least one of the maximal filters of $\mathcal{L}$.

(2) If $\mathbb{MG}(\mathcal{L})$ has a non-maximal universal vertex F, then F is a minimal filter.

(3) If $J(\mathcal{L}) \neq \{1\}$, then every element of $\mathrm{Max}(\mathcal{L})$ is adjacent to $J(\mathcal{L})$.

Proof. (1), (2) and (3) are straightforward. □

Proposition 3.3. *The subgraph induced by the $\mathrm{Max}(\mathcal{L})$ is an empty graph.*

Proof. Let $M, M' \in \mathrm{Max}(\mathcal{L})$ such that $M \neq M'$. If $M \wedge M'$ is maximal, then $M, M' \subseteq M \wedge M'$ gives $M = M \wedge M' = M'$ which is a contradiction. Therefore, M is not adjacent to M'. □

Proposition 3.4. *The following hold in $\mathbb{MG}(\mathcal{L})$:*

(1) If $\mathbb{MG}(\mathcal{L})$ has a maximal universal vertex M, then $\mathcal{L}$ is a local lattice with unique maximal filter M.

(2) If $\mathcal{L}$ is a local lattice with unique maximal filter M, then M is a universal vertex.

(3) If $\mathbb{MG}(\mathcal{L})$ is a complete graph, then $\mathcal{L}$ is a local lattice.

Proof. (1) Let N be any maximal filter of $\mathcal{L}$. Since M is not adjacent to N by Proposition 3.3, we conclude that $M = N$, as M is a universal vertex of $\mathbb{MG}(\mathcal{L})$. Therefore, $\mathcal{L}$ is a local lattice.

(2) If N is a nontrivial filter of $\mathcal{L}$, then $N \subseteq M$ by [15, Lemma 2.1], which gives $N \wedge M = M$ and so M is a universal vertex.

(3) If $M, N \in \mathrm{Max}(\mathcal{L})$ with $M \neq N$, then M and N are not adjacent in $\mathbb{MG}(\mathcal{L})$ by Proposition 3.3 which is impossible, as $\mathbb{MG}(\mathcal{L})$ is complete. Thus, $\mathcal{L}$ is a local lattice. □

The following example shows that, in general, the converse of Proposition 3.4(3) is not true.

Example 3.5. The collection of ideals of $\mathbb{Z}$, the ring of integers, form a lattice under set inclusion which we shall denote by $\mathcal{L}$ with respect to the following definitions: $m\mathbb{Z} \vee n\mathbb{Z} = (m,n)\mathbb{Z}$ and $m\mathbb{Z} \wedge n\mathbb{Z} = [m,n]\mathbb{Z}$, for all ideals $m\mathbb{Z}$ and $n\mathbb{Z}$ of $\mathbb{Z}$, where (m,n) and $[m,n]$ are greatest common divisor and least common multiple of m, n, respectively. Note that $\mathcal{L}$ is a distributive complete lattice with least element the zero ideal and the greatest element $\mathbb{Z}$. By [10, Theorem 2.9], $M = \mathcal{L} \setminus \{0\}$ is the only maximal filter of $\mathcal{L}$, i.e. $\mathcal{L}$ is a local lattice. Since $\{\mathbb{Z}, 2\mathbb{Z}\} \wedge \{\mathbb{Z}, 3\mathbb{Z}\}$ is not a maximal filter, we infer that $\mathbb{MG}(\mathcal{L})$ is not a complete graph.

The following theorem, we give a condition under which $\mathbb{MG}(\mathcal{L})$ is an empty graph.

Theorem 3.6. *Every nontrivial filter of a lattice $\mathcal{L}$ is a maximal filter if and only if $\mathbb{MG}(\mathcal{L})$ is an empty graph.*

Proof. If every nontrivial filter of $\mathcal{L}$ is maximal, then $\mathbb{MG}(\mathcal{L})$ is an empty graph by Proposition 3.3. Conversely, assume that $\mathbb{MG}(\mathcal{L}$ is empty and let F be any vertex of the graph of $\mathbb{MG}(\mathcal{L}$ such that F is not a maximal filter. Then by Lemma 3.2, there exists a maximal filter M of $\mathcal{L}$ such that M adjacent to F which is impossible, i.e. the result holds. □

Theorem 3.7. *If $J(\mathcal{L}) \neq \{1\}$, then the subgraph induced by the non-maximal filters of $\mathcal{L}$ is a connected graph of diameter not bigger than* 4.

Proof. Let F and G be distinct non-maximal vertices of the graph $\mathbb{MG}(\mathcal{L})$. If F adjacent to G, then $F \backsim G$ is a path. So we may assume that $F \wedge G$ is not a maximal filter. Then there exist $M, M' \in \mathrm{Max}(\mathcal{L})$ such that $F \subsetneqq M$ and $G \subsetneqq M'$. If $M = M'$, then $F \backsim M \backsim G$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(F,G) = 2$. If $M \neq M'$, then $\{1\} \subsetneqq J(\mathcal{L})) \subseteq M \cap M' \subseteq M \neq \mathcal{L}$ gives $M \cap M'$ is a vertex of the graph $\mathbb{MG}(\mathcal{L})$ and so $F \backsim M \backsim M \cap M' \backsim M' \backsim G$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(F,G) = 4$, as needed. □

The next theorem gives a more explicit description of the diameter of $\mathbb{MG}(\mathcal{L})$.

Theorem 3.8. *The graph* $\mathbb{MG}(\mathcal{L})$ *is connected with* $\mathrm{diam}(\mathbb{MG}(\mathcal{L})) \leq 4$ *if and only if the join of any two distinct maximal filters of* $\mathcal{L}$ *is not* $\{1\}$ *or* $\mathcal{L}$ *is a local lattice.*

Proof. Suppose that $\mathcal{L}$ is a local lattice with unique maximal filter M and let F and G be distinct non-maximal vertices of the graph $\mathbb{MG}(\mathcal{L})$. Then $F \backsim M \backsim G$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(F, G) = 2$. So suppose that $|\mathrm{Max}(\mathcal{L})| \neq 1$ and the join of any two distinct maximal filters of $\mathcal{L}$ is not $\{1\}$. Consider two distinct vertices B and C of $\mathbb{MG}(\mathcal{L})$. If B adjacent to C, then $B \backsim C$ is a path. So we may assume that $B \wedge C$ is not a maximal filter of $\mathcal{L}$. Then either $B \wedge C = \mathcal{L}$ or $B \wedge C \subsetneqq M$ for some $M \in \mathrm{Max}(\mathcal{L})$. If $B \wedge C \subsetneqq M$ (so $B \subsetneqq M$ and $C \subsetneqq M$), then $B \backsim M \backsim C$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(B, C) = 2$. If $B \wedge C = \mathcal{L}$, we split the proof into three cases:

Case 1. $B, C \in \mathrm{Max}(\mathcal{L})$. Since $\{1\} \subsetneqq B \vee C \subseteq B \neq \mathcal{L}$, we conclude that $B \vee C$ is a vertex and so $B \backsim B \vee C \backsim C$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(B, C) = 2$.

Case 2. If exactly one of B and C is maximal, then without loss of generality, assume that $B \in \mathrm{Max}(\mathcal{L})$ and $C \notin \mathrm{Max}(\mathcal{L})$. By [15, Lemma 2.1], there is a maximal filter M of $\mathcal{L}$ such that $C \subsetneqq M$ which implies that $B \backsim M \vee B \backsim M \backsim C$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(B, C) = 3$.

Case 3. $B, C \notin \mathrm{Max}(\mathcal{L})$. Then there exist $M, M' \in \mathrm{Max}(\mathcal{L})$ such that $B \subsetneqq M$ and $C \subsetneqq M'$. If $M = M'$, then $B \backsim M \backsim C$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(C, D) = 2$. If $M \neq M'$, then $B \backsim M \backsim M \vee M' \backsim M' \backsim C$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(B, C) = 4$. Hence, we infer that $\mathbb{MG}(\mathcal{L})$ is connected with $\mathrm{diam}(\mathbb{MG}(\mathcal{L})) \leq 4$.

Conversely, assume that $\mathbb{MG}(\mathcal{L})$ is connected. If $\mathcal{L}$ is a local lattice, then we are done. So we may assume that $|\mathrm{Max}(\mathcal{L})| \neq 1$. On the contrary, assume that there are two maximal filters M_1 and M_2 of $\mathcal{L}$ such that $M_1 \vee M_2 = \{1\}$. We claim that M_2 is a minimal filter of $\mathcal{L}$. Assume to the contrary, $\{1\} \subsetneqq N \subseteq M_2$ for some filter N of $\mathcal{L}$. Then $M_1 \vee N \subseteq M_2 \vee M_1 = \{1\}$ gives $N \vee M_1 = \{1\}$. If $N \wedge M_1 \neq \mathcal{L}$, then $M_1 \subseteq M_1 \wedge N \neq \mathcal{L}$ gives $N \subseteq M_1$; so $\{1\} = N \vee M_1 = N$ which is impossible. Thus, $N \wedge M_1 = \mathcal{L}$. Hence $0 = n \wedge m_1$, for some $n \in N$ and $m_1 \in M_1$. Let $m_2 \in M_2$. Then $m_2 = m_2 \vee 0 = m_2 \vee (n \wedge m_1) = (m_2 \vee n) \wedge (m_2 \vee m_1)$. As $m_2 \vee m_1 \in M_1 \vee M_2 = \{1\}$, $m_2 = m_2 \vee n \in N$. Therefore $M_2 \subseteq N$ and so $M_2 = N$. So M_2 is a minimal filter. Similarly, M_1 is a minimal filter. By the hypothesis, $M_1 \backsim K$ for some vertex K of $\mathbb{MG}(\mathcal{L})$. Then $M_1 \wedge K = M_1$ which gives $K \subsetneqq M_1$, a contradiction, because M_1 is a minimal filter, as required. $\square$

Theorem 3.9. *If* $J(\mathcal{L}) \neq \{1\}$ *and* $\mathbb{MG}(\mathcal{L})$ *is a graph which contains a cycle, then* $\mathrm{gr}(\mathbb{MG}(\mathcal{L})) = 3, 4$.

Proof. Since $J(\mathcal{L}) \neq \{1\}$ and $J(\mathcal{L}) \subseteq M \neq \mathcal{L}$ for some maximal filter M of $\mathcal{L}$, we conclude that $J(\mathcal{L})$ is a vertex of $\mathbb{MG}(\mathcal{L})$. Let $F \backsim G$. By Proposition 3.3, either $F \notin \mathrm{Max}(\mathcal{L})$ or

$G \notin \mathrm{Max}(\mathcal{L})$. If $F, G \notin \mathrm{Max}(\mathcal{L})$, then $F \backsim F \wedge G \backsim G \backsim F$ is a cycle (so $\mathrm{gr}(\mathbb{MG}(\mathcal{L})) = 3$). Suppose that one of F or G is a maximal filter. We can assume that $F \in \mathrm{Max}(\mathcal{L})$ and $G \notin \mathrm{Max}(\mathcal{L})$. By Lemma 3.2, there is a maximal filter M such that $G \subsetneqq M$. Therefore, we obtain the cycle $F \backsim G \backsim M \backsim J(\mathcal{L}) \backsim F$ which implies that $\mathrm{gr}(\mathbb{MG}(\mathcal{L})) = 4$, as needed. □

Assume that $(\mathcal{L}_1, \leq_1), (\mathcal{L}_2, \leq_2), \cdots, (\mathcal{L}_n, \leq_n)$ are lattices ($n \geq 2$) and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2, \cdots, n\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$. In this case, we say that $\mathcal{L}$ is a decomposable lattice.

Theorem 3.10. *Let* $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ *be a decomposable lattice such that* $\mathrm{Max}(\mathcal{L}_i) = \{M_i\}$ *for* $i \in \{1, 2, \cdots, n\}$. *Then* $\mathrm{diam}(\mathbb{MG}(\mathcal{L})) \leq 2$.

Proof. By [11, Proposition 3.17], $\mathrm{Max}(\mathcal{L}) = \{M'_1, \cdots, M'_n\}$, where we have that $M'_k = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_{k-1} \times M_k \times \mathcal{L}_{k+1} \times \cdots \times \mathcal{L}_n$ for $k \in \{1, 2, \cdots, n\}$. Let F and G be two vertices of $\mathbb{MG}(\mathcal{L})$ such that they are not adjacent. If $F \subseteq M'_i$ and $G \subseteq M'_i$ for some M'_i, then $F \backsim M'_i \backsim G$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(F, G) = 2$. Otherwise, there are $M'_i, M'_j \in \mathrm{Max}(\mathcal{L})$ such that $F \subseteq M'_i$, $G \nsubseteq M'_i$, $G \subseteq M'_j$ and $F \nsubseteq M'_j$. Without loss of generality, We may assume that $i < j$. Consider the following filter:

$$H = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_{i-1} \times M_i \times \mathcal{L}_{i+1} \times \cdots \times \mathcal{L}_{j-1} \times M_j \times \mathcal{L}_{j+1} \times \cdots \times \mathcal{L}_n.$$

Since $F \nsubseteq M'_j$ and $G \nsubseteq M'_i$, we conclude that $H \wedge F = M'_i$ and $H \wedge G = M'_j$. This shows that $F \backsim H \backsim G$ is a path in $\mathbb{MG}(\mathcal{L})$ with $\mathrm{d}(F, G) = 2$. Thus, $\mathrm{diam}(\mathbb{MG}(\mathcal{L})) \leq 2$. □

Theorem 3.11. *Let* $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ *be a decomposable lattice such that* $\mathrm{Max}(\mathcal{L}_i) = \{M_i\}$ *for* $i \in \{1, 2, \cdots, n\}$. *Then* $\mathrm{gr}(\mathbb{MG}(\mathcal{L})) = 3$.

Proof. By [11, Proposition 3.17], $\mathrm{Max}(\mathcal{L}) = \{M'_1, \cdots, M'_n\}$, where we have that $M'_k = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_{k-1} \times M_k \times \mathcal{L}_{k+1} \times \cdots \times \mathcal{L}_n$ for $k \in \{1, 2, \cdots, n\}$. Now we consider the filters $F = M_1 \times M_2 \times \mathcal{L}_3 \times \cdots \times \mathcal{L}_n$,

$$G = M_1 \times \mathcal{L}_2 \times M_3 \times \mathcal{L}_4 \cdots \times \mathcal{L}_n,$$

and $H = \mathcal{L}_1 \times M_2 \times M_3 \times \mathcal{L}_4 \cdots \times \mathcal{L}_n$. Since $F \wedge G = M'_1$, $F \wedge H = M'_2$ and $G \wedge H = M'_3$, we obtain the cycle $F \backsim G \backsim H \backsim F$. This shows that $\mathrm{gr}(\mathbb{MG}(\mathcal{L})) = 3$. □

4. Further results in related lattices

In the following theorem, we describe the relation between cut vertex filters and maximal filters.

Theorem 4.1. *Suppose that the join of any two distinct maximal filters of $\mathcal{L}$ is not $\{1\}$ and let C be a cut vertex of $\mathbb{MG}(\mathcal{L})$. Then there exist maximal filters M, M' of $\mathcal{L}$ such that $C = M \vee M' = M \cap M'$.*

Proof. If C is a maximal filter, then $C = C \vee C$. So we may assume that C is not a maximal filter. Let A and B be two vertices of $\mathbb{MG}(\mathcal{L})$ such that $A \in V_1$ and $B \in V_2$, where V_1 and V_2 are the distinct components $\mathbb{MG}(\mathcal{L}) \setminus \{C\}$. We split the proof into four cases:

Case 1. $A, B \in \mathrm{Max}(\mathcal{L})$. Since $A \backsim A \vee B \backsim B$ is a path in $\mathbb{MG}(\mathcal{L})$ and C is a cut vertex, we infer that $C = A \vee B$.

Case 2. $A \in \mathrm{Max}(\mathcal{L})$ and $B \notin \mathrm{Max}(\mathcal{L})$. Since B is not a maximal filter, we conclude that $B \backsim M$ for some maximal filter M of $\mathcal{L}$ by Lemma 3.2(1); hence $M \in V_2$. Since $A \backsim A \vee M \backsim M$ is a path in $\mathbb{MG}(\mathcal{L})$, $A \in V_1$ and C is a cut vertex, we infer that $C = A \vee M$.

Case 3. $A \notin \mathrm{Max}(\mathcal{L})$ and $B \in \mathrm{Max}(\mathcal{L})$. By an argument like that in Case 2, $C = B \vee M'$ for some maximal filter M' of $\mathcal{L}$ with $A \backsim M'$.

Case 4. $A, B \notin \mathrm{Max}(\mathcal{L})$. By assumption, $A \backsim M$ and $B \backsim M'$ for some maximal filters M and M' of $\mathcal{L}$ by Lemma 3.2(1); so $M \in V_1$ and $M' \in V_2$. Since $M \backsim M \vee M' \backsim M'$ is a path in $\mathbb{MG}(\mathcal{L})$ and C is a cut vertex, we infer that $C = M \vee M'$. □

The following theorem provides some condition under which the graph $\mathbb{MG}(\mathcal{L})$ is star.

Theorem 4.2. *If $\mathcal{V}\ (\mathbb{MG}(\mathcal{L}))$ is a totally ordered set with set inclusion, then $\mathbb{MG}(\mathcal{L})$ is a star graph.*

Proof. By the hypothesis, there exists a maximal filter M of $\mathcal{L}$ such that $M \backsim K$ for every $K \in \mathcal{V}\ (\mathbb{MG}(\mathcal{L}))$. If $A \neq M$ and $B \neq M$ are two distinct vertices of $\mathbb{MG}(\mathcal{L})$, then either $A \subseteq B$ or $B \subseteq A$. For both cases, A and B are not adjacent vertices and so $\mathbb{MG}(\mathcal{L})$ is a star graph with center M. □

Note that, the converse of Theorem 4.2 is not correct. Let $\mathcal{L}$ be the lattice as in Example 3.5. Then $\mathbb{MG}(\mathcal{L})$ is a star graph. However the $\mathcal{V}\ (\mathbb{MG}(\mathcal{L}))$ is not a totally ordered set.

A filter F of $\mathcal{L}$ is called small in $\mathcal{L}$, written $F \ll \mathcal{L}$, if, for every filter G of $\mathcal{L}$, the equality $F \wedge G = \mathcal{L}$ implies $F = G$. The proof of the following lemma can be found in [16] (with some different proof and notions), but we give the details for convenience.

Lemma 4.3. *Let F, G be filters of $\mathcal{L}$. Then the following hold:*

(1) If $F \subseteq G$ and $G \ll \mathcal{L}$, then $F \ll \mathcal{L}$.

(2) If $F \ll \mathcal{L}$ and $G \ll \mathcal{L}$, then $F \wedge G, F \vee G \ll \mathcal{L}$.

(3) Suppose that F is a proper filter of $\mathcal{L}$ and let $a \in \mathcal{L} \setminus F$. If $F \wedge T(a) = \mathcal{L}$, then $\mathcal{L}$ has a maximal filter M with $F \subseteq M$ and $a \notin M$.

(4) $J(\mathcal{L}) = \bigwedge_{X \ll \mathcal{L}} X$.

Proof. (1) Let $F \wedge K = \mathcal{L}$ for some filter K of $\mathcal{L}$. Then $\mathcal{L} = F \wedge K \subseteq G \wedge K \subseteq \mathcal{L}$ gives $G \wedge K = \mathcal{L}$ and so so $K = \mathcal{L}$.

(2) Since $F \vee G \subseteq F$, we conclude that $F \vee G \ll \mathcal{L}$ by (1). Let $(F \wedge G) \wedge H = \mathcal{L}$ for some filter H. Then $G \ll \mathcal{L}$ gives $F \wedge H = \mathcal{L}$; hence $H = \mathcal{L}$, as $F \ll \mathcal{L}$.

(3) At first, we show that $\mathcal{L}$ has a filter M maximal with respect to $F \subseteq M$ and $a \notin M$. Consider the set Ω of all filters H of $\mathcal{L}$ such that $F \subseteq H$ and $a \notin H$. This set is not empty since $F \in \Omega$. Clearly, Ω is closed under taking unions of chains and so the result follows by Zorn's Lemma. Let M be the maximal element of Ω. Let U be a filter of $\mathcal{L}$ such that $M \subsetneqq U \subseteq \mathcal{L}$. Then $a \in U$ by maximality of M, and so $F \wedge T(a) = \mathcal{L} \subseteq U$; hence $\mathcal{L} = U$, i.e. the result holds.

(4) Let $G \ll \mathcal{L}$. If M is a maximal filter of $\mathcal{L}$ and $G \nsubseteq M$, then $G \wedge M = \mathcal{L}$; but since $G \ll \mathcal{L}$, we have $M = \mathcal{L}$, a contradiction. We infer that every small filter of $\mathcal{L}$ is contained in $J(\mathcal{L})$; hence $\bigwedge_{X \ll \mathcal{L}} X \subseteq J(\mathcal{L})$. On the other hand, let $x \in \mathcal{L}$. If H is a filter of $\mathcal{L}$ with $H \wedge T(x) = \mathcal{L}$, then by (3), either $H = \mathcal{L}$ or there is a maximal filter M of $\mathcal{L}$ with $H \subseteq M$ and $x \notin M$. If $x \in J(\mathcal{L})$, then the latter cannot occur; thus $x \in J(\mathcal{L})$ forces $T(x) \ll \mathcal{L}$, and we have equality. □

Proposition 4.4. *The following hold in a lattice $\mathcal{L}$:*

(1) If F is a filter of $\mathcal{L}$, then $F \ll \mathcal{L}$ if and only if $F \subseteq J(\mathcal{L})$;

(2) If F is a nontrivial filter and $J(\mathcal{L}) \subsetneqq F$, then F is not small in $\mathcal{L}$.

Proof. (1) If $F \ll \mathcal{L}$, then $F \subseteq J(\mathcal{L})$ by Lemma 4.3(4). To prove the other side, assume to the contrary, that F is not small in $\mathcal{L}$. Then there exists a nontrivial filter G of $\mathcal{L}$ such that $F \wedge G = \mathcal{L}$. By [15, Lemma 2.1], $G \subseteq M$ for some maximal filter M of $\mathcal{L}$. Since $F \subseteq J(\mathcal{L}) \subseteq M$, we conclude that $\mathcal{L} = F \wedge G \subseteq G \wedge M = M$ which implies that $M = \mathcal{L}$ which is a contradiction. Therefore, $F \ll \mathcal{L}$.

(2) This is a direct consequence of (1). □

The converse of Proposition 4.4(2) is not true, as the following example shows:

Example 4.5. Let $\mathcal{L} = \{0, a, b, c, d, 1\}$ be a lattice with $0 \leq a \leq c \leq 1$, $0 \leq b \leq d \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$, $a \wedge b = 0$, $c \wedge d = b$ and $c \vee d = 1$. An inspection demonstrate that $J(\mathcal{L}) = \{1, c\}$. However, $N = \{1, d\}$ is a non-small filter of $\mathcal{L}$ such that $J(\mathcal{L}) = \{1, c\} \not\subset N$.

Proposition 4.6. *The following hold in* $\mathbb{MG}(\mathcal{L})$*:*

(1) If $M \in \mathrm{Max}(\mathcal{L})$ *and* $F \subsetneqq J(\mathcal{L})$*, then* $M \backsim F$*;*

(2) If $M \notin \mathrm{Max}(\mathcal{L})$ *and* $N \subsetneqq M$*, then* M *is not adjacent to* N*.*

Proof. (1) Since $F \subsetneqq J(\mathcal{L}) \subseteq M$, we conclude that $F \wedge M = M$ and so $M \backsim F$.

(2) If $N \subsetneqq M$, then $N \wedge M = M$. Since $M \notin \mathrm{Max}(\mathcal{L})$, we infer that M is not adjacent to N. $\square$

The converse of Proposition 4.6 is not true, as the following example prove:

Example 4.7. Let $\mathcal{L}$ be as in Example 4.5. Then $N = \{1, d\}$ and $M = \{1, a, c\}$ are not adjacent, however $M \notin \mathrm{Max}(\mathcal{L})$ and $N \not\subset M$. This implies that the converse of Proposition 4.6(2) is not true. Moreover, $F = \{1, d, b, c\} \in \mathrm{Max}(\mathcal{L})$ and it is adjacent to $\{1, d\}$, however $\{1, d\} \not\subset J(\mathcal{L})$. It disprove the converse of Proposition 4.6(1).

The following theorem shows that when the meet-maximal filter graph is a complete bipartite graph.

Theorem 4.8. *If* $J(\mathcal{L}) \neq \{1\}$*, then the following assertions are equivalent:*

(1) Every vertex of $\mathbb{MG}(\mathcal{L})$ *is either maximal or small;*

(2) $\mathbb{MG}(\mathcal{L})$ *is a complete bipartite graph.*

Proof. (1) $\Rightarrow$ (2) Let $V_1 = \mathrm{Max}(\mathcal{L})$ and V_2 be the set of all small filters of $\mathcal{L}$. If $M, M' \in V_1$, then M and M' are not adjacent by Proposition 3.3. If $F, G \in V_2$, then $F \wedge G \ll \mathcal{L}$ by Lemma 4.3(2) and so $F \wedge G \subsetneqq J(\mathcal{L})$ by Proposition 4.4(1) which implies that any two vertices of V_2 are not adjacent. Moreover, every vertex in V_1 is adjacent to each vertex V_2 by Proposition 4.6. Therefore, $\mathbb{MG}(\mathcal{L})$ is a complete bipartite graph.

(2) $\Rightarrow$ (1) Suppose that V_1 and V_2 are parts of $\mathbb{MG}(\mathcal{L})$ and let M be a maximal filter of $\mathcal{L}$. Without loss of generality, let $M \in V_1$. If $M \neq M' \in \mathrm{Max}(\mathcal{L})$ with $M' \notin V_1$, then $M' \in V_2$ implies that $M \wedge M'$ is maximal which is impossible. Thus $\mathrm{Max}(\mathcal{L}) \subseteq V_1$. If $N \in V_1$ with $N \notin \mathrm{Max}(\mathcal{L})$, then there exists a maximal filter N' of $\mathcal{L}$ (so $N' \in V_1$) such that $N \backsim N'$ by Lemma 3.2, a contradiction. Therefore, $V_1 = \mathrm{Max}(\mathcal{L})$. If $F \subseteq J(\mathcal{L})$ for some vertex F, then F is small in $\mathcal{L}$ by Proposition 4.4(1) and so

$$\mathcal{M} = \{F \in \mathcal{V}(\mathbb{MG}(\mathcal{L}) : F \subseteq J(\mathcal{L})\},$$

is the set of all small filters of $\mathcal{L}$. An easy inspection will show that $V_2 = \mathcal{M}$, i.e. the result holds. $\square$

Example 4.9. Let $\mathcal{L} = \{0, 1, a_1, a_2, ..., a_n\}$ be a lattice (chain) with $0 < a_1 < a_2 < ... < a_n < 1$. Then $M = \mathcal{L} \setminus \{0\}$ is the unique maximal filter of $\mathcal{L}$ and another proper filters are small. It is clear that $\mathbb{MG}(\mathcal{L})$ is a complete bipartite graph (a star graph).

5. Clique number, domination number and planarity of $\mathbb{MG}(\mathcal{L})$

We continue this section with the investigation of the stability of meet-maximal filter graphs in various lattice-theoretic constructions. In the following results we show that domination numbers are really of interest in indecomposable lattices.

Theorem 5.1. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice such that $\mathrm{Max}(\mathcal{L}_i) = \{M_i\}$ for $i \in \{1, 2\}$. Then $\gamma(\mathbb{MG}(\mathcal{L})) = 2$.*

Proof. By [11, Proposition 3.17], $\mathrm{Max}(\mathcal{L}) = \{M_1', M_2'\}$, where we have that $M_1' = M_1 \times \mathcal{L}_2$ and $M_2' = \mathcal{L}_1 \times M_2$. If $F = F_1 \times F_2$ is a nontrivial filter of $\mathcal{L}$, then either $F \subseteq M_1'$ or $F \subseteq M_2'$ by [15, Lemma 2.1]. Without loss of generality, let $F \subseteq M_1'$ (so $F_1 \subseteq M_1$). Since $F \wedge_c M_1' = (F_1 \wedge M_1) \times (F_2 \wedge \mathcal{L}_2) = M_1 \times \mathcal{L}_2$, we infer that any vertex of the graph $\mathbb{MG}(\mathcal{L})$ is adjacent to at least one of the elements of the set $\{M_1', M_2'\}$. This shows that $\gamma(\mathbb{MG}(\mathcal{L})) = 2$. $\square$

Following example shows that the converse of Theorem 5.1 is not true.

Example 5.2. Let $X = \{a, b, c\}$. Then $\mathcal{L} = \{T : T \subseteq X\}$ forms a distributive lattice under set inclusion greatest element X and least element $\emptyset$; (note that if $A, B \in \mathcal{L}$, then $A \wedge B = A \cap B$ and $A \vee B = A \cup B$). Now, the set of all filters of $\mathcal{L}$ is $\{F_1, F_2, F_3, F_4, F_5, F_6\}$, where

$$F_1 = \{X, \{a, b\}\},\ F_2 = \{X, \{a, c\}\},\ F_3 = \{X, \{b, c\}\},$$

$$F_4 = \{X, \{a, b\}, \{a, c\}, \{a\}\},\ F_5 = \{X, \{a, b\}, \{b, c\}, \{b\}\},$$

$$F_6 = \{X, \{a, c\}, \{b, c\}, \{c\}\}.$$

By drawing $\mathbb{MG}(\mathcal{L})$ (see Figure 1), it can be easily checked that $\gamma(\mathbb{MG}(\mathcal{L})) = 2$, $\mathrm{Max}(\mathcal{L}) = \{F_4, F_5, F_6\}$ and $\mathcal{L} \neq \mathcal{L}_1 \times \mathcal{L}_2$, for some lattices $\mathcal{L}_1$, $\mathcal{L}_2$. Hence the converse of Theorem 5.1 is not true.

Theorem 5.3. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ $(n \geq 3)$ be a decomposable lattice such that $\mathrm{Max}(\mathcal{L}_i) = \{M_i\}$ for $i \in \{1, 2, \cdots, n\}$. Then $\gamma(\mathbb{MG}(\mathcal{L})) \leq n$.*

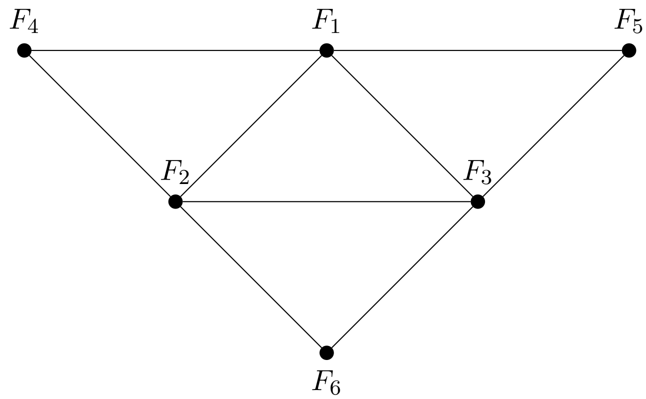

FIGURE 1. $\mathbb{MG}(\mathcal{L})$

Proof. By [11, Proposition 3.17], $\mathrm{Max}(\mathcal{L}) = \{M'_1, \cdots, M'_n\}$, where we have that $M'_k = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_{k-1} \times M_k \times \mathcal{L}_{k+1} \times \cdots \times \mathcal{L}_n$ for $k \in \{1, 2, \cdots, n\}$. Then the set $\mathrm{Max}(\mathcal{L})$ dominates all the vertices of the graph $\mathbb{MG}(\mathcal{L})$; hence $\gamma(\mathbb{MG}(\mathcal{L})) \leq n$. □

A simple lattice is a lattice that has no filters besides the $\{1\}$ and itself. The following example shows that in general Theorem 5.3 is not true in the case $\gamma(\mathbb{MG}(\mathcal{L})) = n$.

Example 5.4. Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$ be a decomposable lattice, where we have that $\mathcal{L}_i$ is a simple lattice for $i \in \{1, 2, 3\}$. Then we have $\mathcal{V}(\mathbb{MG}(\mathcal{L})) = \{\mathcal{L}_1 \times \{1\} \times \{1\}, \mathcal{L}_1 \times \mathcal{L}_2 \times \{1\}, \{1\} \times \mathcal{L}_2 \times \{1\}, \{1\} \times \mathcal{L}_2 \times \mathcal{L}_3, \mathcal{L}_1 \times \{1\} \times \mathcal{L}_3, \{1\} \times \{1\} \times \mathcal{L}_3\}$. Then the set $S = \{\mathcal{L}_1 \times \mathcal{L}_2 \times \{1\}, \{1\} \times \mathcal{L}_2 \times \mathcal{L}_3\}$ dominates all the vertices of the graph $\mathbb{MG}(\mathcal{L})$; hence $\gamma(\mathbb{MG}(\mathcal{L})) = 2 \neq 3$.

Proposition 5.5. *Let $A, B \notin \mathrm{Max}(\mathcal{L})$ such that they are adjacent. Then there is a unique maximal filter M of $\mathcal{L}$ such that $M \in N(A) \cap N(B)$.*

Proof. Since $A \wedge B$ is a maximal filter, we infer that $A \wedge B$ is adjacent to both A and B which gives $A \wedge B \in N(A)$ and $A \wedge B \in N(B)$. Let $M \in \mathrm{Max}(\mathcal{L})$ such that $M \in N(A) \cap N(B)$. It is enough to show that $M = A \wedge B$. On the contrary, assume that $M \neq A \wedge B$. First, we prove that $M \in N(A) \cap N(B)$ if and only if $M \in N(A \wedge B)$. If $M \backsim A \wedge B$, then $M \wedge (A \wedge B)$ is a maximal filter of $\mathcal{L}$ which gives $M \wedge A \neq \{1\}$ and $M \wedge B \neq \{1\}$. Since $M \subseteq A \wedge M$ and $M \subseteq M \wedge B$, we conclude that $A \wedge M = M = B \wedge M$ and so $M \backsim A$ and $M \backsim B$. Conversely, assume that $A \backsim M$ and $B \backsim M$. Then $A \wedge M$ and $M \wedge B$ are maximal filters gives $A \wedge M = M = M \wedge B$ which implies that $(A \wedge B) \wedge M = M$; so $M \backsim A \wedge B$. Since $M \wedge (A \wedge B)$ is a maximal filter, we infer that $M \wedge (A \wedge B) \neq \{1\}$ which implies that $A \wedge B \subseteq M$. Therefore, $M = A \wedge B$, as $A \wedge B$ is a maximal filter, a contradiction. Thus $M = A \wedge B$. □

Theorem 5.6. *Let $\mathcal{C}$ be a clique in $\mathbb{MG}(\mathcal{L})$. Then $\mathcal{C}$ is contained in the subgraph induced by $\{F \in \mathcal{V}(\mathbb{MG}(\mathcal{L})) : F \subseteq M\}$ for some maximal filter M of $\mathcal{L}$.*

Proof. By Proposition 3.3, clique $\mathcal{C}$ has at most one maximal filter. If $\mathcal{C} \cap \mathrm{Max}(\mathcal{L}) = \{M\}$, then $X \subseteq M$ for every $X \in \mathcal{C}$ and so $\mathcal{C}$ is contained in the subgraph induced by $\{F \in \mathcal{V}(\mathbb{MG}(\mathcal{L})) : F \subseteq M\}$. So we may assume that $\mathcal{C} \cap \mathrm{Max}(\mathcal{L}) = \emptyset$. The adjacency of every two vertices of $\mathcal{C}$ and Proposition 5.5 shows that there exists a unique maximal filter M' of $\mathcal{L}$ for which $\mathcal{C}$ is a subgraph of the graph induced by $\{F \in \mathcal{V}(\mathbb{MG}(\mathcal{L})) : F \subseteq M'\}$. □

The following theorem provides some condition under which $\mathbb{MG}(\mathcal{L})$ is a split graph.

Theorem 5.7. *If $\mathbb{MG}(\mathcal{L})$ is not empty and $\mathcal{V}(\mathbb{MG}(\mathcal{L})) = \mathrm{Max}(\mathcal{L}) \cup \mathrm{Min}(\mathcal{L})$, then $\mathbb{MG}(\mathcal{L})$ is a split graph.*

Proof. At first, we show that the subgraph induced by the minimal filters of $\mathcal{L}$ is a complete graph. It is enough to show if C and D are minimal filters of $\mathcal{L}$ with $C \neq D$, then $C \backsim D$. We claim that $C \wedge D \neq \mathcal{L}$. On the contrary, let $C \wedge D = \mathcal{L}$. Let $C \subseteq E \subsetneqq \mathcal{L}$ for some maximal filter E of $\mathcal{L}$. As $C \wedge D = \mathcal{L} \subseteq E \wedge D$, we conclude that $E \wedge D = \mathcal{L}$. If $D \subseteq E$, then $E \wedge D = E = \mathcal{L}$, a contradiction. Therefore, $E \cap D = \{1\}$. Let $e \in E$. Then $e \in \mathcal{L} = C \wedge D$. Hence $e = c \wedge d$, for some $c \in C$ and $d \in D$. Therefore $e = e \vee e = e \vee (c \wedge d) = (e \vee c) \wedge (e \vee d)$. As $e \vee d \in E \cap D = \{1\}$, $e \vee d = 1$ and so $e = e \vee c$. This implies that $E \subseteq C$ and so $E = C$. This shows that C is a maximal filter. Similarly, D is a maximal filter. It follows that $\mathcal{V}(\mathbb{MG}(\mathcal{L})) = \mathrm{Max}(\mathcal{L})$ and so $\mathbb{MG}(\mathcal{L})$ is empty by Proposition 3.3 which is impossible. Therefore, $C \wedge D \neq \mathcal{L}$ and so $C \wedge D$ is a vertex of the graph $\mathbb{CG}(\mathcal{L})$ (since $C \wedge D \neq \{1\}$). It is clear that $C \wedge D \notin \mathrm{Min}(\mathcal{L})$; so $C \wedge D \in \mathrm{Max}(\mathcal{L})$, i.e. any two minimal filters are adjacent. Consider the subgraph induced by $\mathrm{Min}(\mathcal{L})$ of $\mathbb{MG}(\mathcal{L})$. Let $C, D \in \mathrm{Min}(\mathcal{L})$ with $C \neq D$. Then $C \wedge D \in \mathrm{Max}(\mathcal{L})$ which implies that the subgraph induced by $\mathrm{Min}(\mathcal{L})$ is complete. Moreover, by Proposition 3.5, the subgraph induced by $\mathrm{Max}(\mathcal{L})$ is empty. Thus, $\mathbb{MG}(\mathcal{L})$ is a split graph. □

Corollary 5.8. *If $\mathbb{MG}(\mathcal{L})$ is not empty and $\mathcal{V}(\mathbb{MG}(\mathcal{L})) = \mathrm{Max}(\mathcal{L}) \cup \mathrm{Min}(\mathcal{L})$, then $|\mathrm{Min}(\mathcal{L})| \leq \omega(\mathbb{MG}(\mathcal{L}))$.*

Proof. Since the subgraph of $\mathbb{MG}(\mathcal{L})$ with the vertex set of $\mathrm{Min}(\mathcal{L})$ is a complete subgraph of $\mathbb{MG}(\mathcal{L})$, we conclude that $|\mathrm{Min}(\mathcal{L})| \leq \omega(\mathbb{MG}(\mathcal{L}))$. □

The following example shows that the equality does not hold necessarily in Corollary 5.8.

Example 5.9. Let $\mathcal{L} = \{0, a, b, c, 1\}$ be a lattice with the relations $0 \leq c \leq a \leq 1$, $0 \leq c \leq b \leq 1$, $a \vee b = 1$ and $a \wedge b = c$. An inspection will show that the nontrivial filters of $\mathcal{L}$ are $F_1 = \{1, a\}$, $F_2 = \{1, b\}$, $F_3 = \{1, a, b, c\}$, $F_4 = \{1, a, c\}$ and $F_5 = \{1, b, c\}$. Clearly, $F_1 \wedge F_2 = F_1 \wedge F_3 = F_2 \wedge F_3 = F_3 \in \mathrm{Max}(\mathcal{L})$, $\mathrm{Min}(\mathcal{L}) = \{F_1, F_2\}$ and $\mathrm{C} = \{F_1, F_2, F_3\}$ is a clique. Hence $|\mathrm{Min}(\mathcal{L})| = 2 < \omega(\mathbb{MG}(\mathcal{L})) = 3$.

The following theorem provides some condition under which $\mathbb{MG}(\mathcal{L})$ is a planar graph.

Theorem 5.10. *Suppose that $\mathbb{MG}(\mathcal{L})$ is not empty and let $\mathcal{V}$ $(\mathbb{MG}(\mathcal{L})) = \mathrm{Max}(\mathcal{L}) \cup \mathrm{Min}(\mathcal{L})$. If $|\mathrm{Min}(\mathcal{L})| \leq 3$, then $\mathbb{MG}(\mathcal{L})$ is a planar graph.*

Proof. Note that $\mathbb{MG}(\mathcal{L})$ is a split graph by Theorem 5.7. Since $|\mathrm{Min}(\mathcal{L})| \leq 3$, we infer that any subgraph induced by five vertices is not complete; so K_5 is not a subgraph of $\mathbb{MG}(\mathcal{L})$. It suffices to show that $K_{3,3}$ is not a subgraph of $\mathbb{MG}(\mathcal{L})$. On the contrary, assume that $K_{3,3}$ is a subgraph of $\mathbb{MG}(\mathcal{L})$ with partite sets $|V_1| = 3$ and $|V_2| = 3$. Therefore, either $V_1 \subseteq \mathrm{Max}(\mathcal{L})$ or $V_2 \subseteq \mathrm{Max}(\mathcal{L})$. If $V_1 \subseteq \mathrm{Max}(\mathcal{L})$, then $V_2 \subseteq \mathrm{Min}(\mathcal{L})$, a contradiction since the subgraph induced by the minimal filters of $\mathcal{L}$ is a complete graph. Therefore, $\mathbb{MG}(\mathcal{L})$ is a planar graph. □

Example 5.11. Let $\mathcal{L}$ be as in Example 4.5 and F_1, F_2, F_3, F_4, F_5, F_6 its filters. Then $\mathrm{Min}(\mathcal{L}) = \{F_1, F_2, F_3\}$ and $\mathrm{Max}(\mathcal{L}) = \{F_4, F_5, F_6\}$. By Figure 1, $\mathbb{MG}(\mathcal{L})$ is a planar graph.

6. Conclusion and future works

We give some connections between algebraic properties of a lattice and the graph theoretical properties of its meet-maximal filter graph. We prove that if $J(\mathcal{L}) \neq \{1\}$, then the subgraph induced by the non-maximal filters of $\mathcal{L}$ is a connected graph of diameter not bigger than 4. Also, it is shown that every vertex of $\mathbb{MG}(\mathcal{L})$ is either maximal or small iff $\mathbb{MG}(\mathcal{L})$ is complete bipartite. Further, if $\mathbb{MG}(\mathcal{L})$ is not empty and $\mathcal{V}$ $(\mathbb{MG}(\mathcal{L})) = \mathrm{Max}(\mathcal{L}) \cup \mathrm{Min}(\mathcal{L})$, then $\mathbb{MG}(\mathcal{L})$ is a split graph. The research presented in this paper yielded numerous ideas to improve the relation of meet-maximal filter graph $\mathbb{MG}(\mathcal{L})$ and the algebraic structure of $\mathcal{L}$ for instance independence number and metric dimension.

7. Acknowledgments

The authors express their deep gratitude to the referees for their constructive comments, careful reading and fruitful suggestions that improve the quality of the manuscript.

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