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Research Paper

A GENERALIZATION OF THE REGULARITY OF COVERING MAPS

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Abstract. We introduce H-regular covering spaces, a natural generalization of classical regular covers where normality in the whole fundamental group is replaced by invariance under conjugation by an arbitrary subgroup $H \leq \pi_1(X, x_0)$. We give three equivalent characterizations of H-regularity and describe the deck transformation group via a normalizer quotient. When $H = \pi_1(X, x_0)$, H-regularity reduces to the usual normality condition $p_*\pi_1(\widetilde{X}, \widetilde{x}) \trianglelefteq \pi_1(X, x_0)$; hence H-regular coverings are exactly ordinary regular coverings. In this case the H-orbit on each fiber is the whole fiber and the deck transformation group specializes to the classical quotient $\mathrm{Cov}(\widetilde{X}/X) \cong \pi_1(X, x_0)/p_*\pi_1(\widetilde{X}, \widetilde{x})$.

1. Introduction

1.1. **Background and Motivation.** Covering theory plays a key role in algebraic topology by revealing fundamental properties of topological spaces and laying the groundwork for deeper understanding of mathematical structures and their applications in other fields. In algebraic topology, the profound relationship between covering maps and the fundamental group

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has been extensively studied (see [5, 6]). Covering maps have simplified the establishment of connections with covering spaces, thereby enhancing the understanding of the underlying structure of the original space. Their core function in preserving local homeomorphism and facilitating connections between fundamental groups has underscored their critical role in the study of algebraic topology and various homotopy types (see [5]). Covering maps are essential for distilling complex topological spaces, providing a more traceable approach to analyzing their properties by covering spaces. They support the lifting of paths and homotopies, which are crucial for clarifying the connections between fundamental groups and covering spaces. Furthermore, exploring covering maps provides insights into local properties and plays a significant role in the classification of spaces, thereby enhancing our understanding of their overall topological framework (see [1]). For the automorphism group of a covering space, we refer the reader also to [7]. Recent works have extended covering theory to broader contexts; for example, the theory of semicovering maps (see [2, 3]) relaxes the local homeomorphism condition to accommodate spaces that are not semilocally simply connected.

1.2. **Generalization of Regular Covering Maps.** In this work, we consider a generalization of regular covering maps called H-regular coverings, which replace the normality condition in $\pi_1(X)$ by invariance under conjugation by a fixed subgroup H.

1.3. **Motivation.** Classical regular coverings require that the associated subgroup of $\pi_1(X)$ be normal, limiting applicability to those spaces admitting Galois (normal) covers. Many natural examples—such as covers corresponding to non-normal subgroups—thus lie outside this theory. Our work addresses this gap by presenting H-regular covering maps, a generalization that extends the theory to any subgroup. This progress modestly enhances covering space theory by including covers that were previously not covered by the classical regular covering theory. Related concepts such as small loop transfer spaces with respect to subgroups have been investigated in [4].

1.4. **Objectives.** This research introduces H-regular covering maps, thereby broadening the understanding of covering spaces relative to subgroups of the fundamental group. It elucidates the intrinsic characteristics of H-regular covering maps and establishes a theoretical framework that extends classical covering theory.

1.5. **Structure of the Article.**

- **Section 2** recalls the essential principles of covering space theory, comprising definitions and standard lemmas on group actions and the fundamental group.
- **Section 3** defines H-regularity, provides nontrivial examples, and proves the equivalent characterisations and the normalizer description of the deck transformation group.

2. Definitions and notations

In this section we recall a few standard facts from covering space theory and group actions that will be needed later. All the material below can be found in [5, Chapter 10] (or [1]).

Definition 2.1. A continuous map $p : \widetilde{X} \to X$ is a *covering map* if every point of X has an open neighborhood U that is evenly covered, i.e. $p^{-1}(U)$ is a disjoint union of open subsets of $\widetilde{X}$ each mapped homeomorphically onto U by p. The pair $(\widetilde{X}, p)$ is then called a *covering space* of X. Throughout the paper we assume X is connected and locally path-connected.

The induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}) \to \pi_1(X, x)$ yields the well-known correspondence between connected covering spaces and subgroups of the fundamental group. A covering space $(\widetilde{X}, p)$ is called *regular* if $p_*\pi_1(\widetilde{X}, \widetilde{x})$ is a normal subgroup of $\pi_1(X, x)$ for every $x \in X$.

We shall use the following facts about the action of the fundamental group on the fiber.

Lemma 2.2 (fiber subgroups are conjugate)**.** *Let $(\widetilde{X}, p)$ be a covering space of X, $x_0 \in X$, and let $Y = p^{-1}(x_0)$.*

(1) *If $\widetilde{x}_0, \widetilde{x}_1 \in Y$, then $p_*\pi_1(\widetilde{X}, \widetilde{x}_0)$ and $p_*\pi_1(\widetilde{X}, \widetilde{x}_1)$ are conjugate subgroups of $\pi_1(X, x_0)$.*
(2) *Conversely, if a subgroup $S \leq \pi_1(X, x_0)$ is conjugate to $p_*\pi_1(\widetilde{X}, \widetilde{x}_0)$ for some $\widetilde{x}_0 \in Y$, then there exists $\widetilde{x}_1 \in Y$ with $S = p_*\pi_1(\widetilde{X}, \widetilde{x}_1)$.*

(See [5, Lemma 10.7].)

Now let a group G act (on the right) on a set Y. For $y \in Y$, the orbit is $o(y) = \{y \cdot g : g \in G\}$ and the stabilizer is $G_y = \{g \in G : y \cdot g = y\}$. The action is *transitive* if $o(y) = Y$ for every $y \in Y$.

Lemma 2.3. *Let a group G act transitively on a set Y and let $x, y \in Y$. Then $G_x = G_y$ iff there exists $\phi \in \mathrm{Aut}(Y)$ with $\phi(x) = y$, where $\mathrm{Aut}(Y)$ denotes the group of G-equivariant bijections $Y \to Y$.*

Lemma 2.4. *If G acts on Y and $y \in Y$, then $|o(y)| = [G : G_y]$. In particular, if G acts transitively then $|Y| = [G : G_y]$.*

Lemma 2.5. *Let G act transitively on a set Y and let $y_0 \in Y$. Then $\mathrm{Aut}(Y) \cong N_G(G_0)/G_0$, where G_0 denotes the stabilizer of y_0.*

3. Main results

In this section, we define H-regular coverings and study their properties. We analyze how subgroup structures of $\pi_1(X)$ influence covering spaces. In particular, we extend covering space theory by formulating H-regular coverings, where regularity is evaluated relative to a subgroup H of $\pi_1(X)$. By generalizing regularity to arbitrary subgroups, the paper expands the types of covering spaces that can be studied and deepens our understanding of how algebraic symmetries relate to topology.

For every subgroup H of $\pi_1(X, x_0)$, H is not a subgroup of $\pi_1(X, x)$ for any point $x \neq x_0$. To present a similar fact, we can consider subgroups corresponding to H in $\pi_1(X, x)$ by the isomorphism $\psi_\alpha : \pi_1(X, x_0) \to \pi_1(X, x)$ for every path α from x_0 to x. Here, we denote $[\alpha]^{-1}H[\alpha] = \{[\alpha]^{-1}h[\alpha] | h \in H\}$ by $\alpha^{-1}H\alpha$.

Definition 3.1. Let $H \leq \pi_1(X, x_0)$ be a subgroup. A covering space $(\widetilde{X}, p)$ of X is called *H-regular* if for every point $x \in X$ and $\widetilde{x} \in p^{-1}(x)$, the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ of $\pi_1(X, x)$ is invariant under conjugation by elements of H (transported to the same basepoint). Equivalently, for every point $x \in X$, for any lift $\widetilde{x} \in p^{-1}(x)$, and for any path α_x in X from x_0 to x, the following holds: if $h \in \alpha_x^{-1}H\alpha_x$, then

$$h^{-1}p_*(\pi_1(\widetilde{X}, \widetilde{x}))h = p_*(\pi_1(\widetilde{X}, \widetilde{x})).$$

In other words, conjugating the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ by any element of the transported subgroup of H yields the same subgroup.

Remark 3.2. Every covering space $(\widetilde{X}, p)$ of X is $\{e\}$-regular, where $\{e\}$ is the trivial subgroup of $\pi_1(X, x_0)$. Indeed, for any $x \in X$, any lift $\widetilde{x} \in p^{-1}(x)$, and any path α_x from x_0 to x, the transported subgroup $\alpha_x^{-1}\{e\}\alpha_x$ is again the trivial subgroup $\{e\}$. Since conjugation by the identity element leaves every subgroup unchanged, the invariance condition

$$h^{-1}p_*\pi_1(\widetilde{X}, \widetilde{x})h = p_*\pi_1(\widetilde{X}, \widetilde{x}),$$

holds vacuously for the only element $h = e$. Thus every covering is $\{e\}$-regular.

Example 3.3. Let $X = S^1 \vee S^1$ with basepoint x_0. Its fundamental group is the free product

$$\pi_1(X, x_0) = \langle a, b \rangle \cong \mathbb{Z} * \mathbb{Z},$$

where a and b are the homotopy classes of the two circles. Recall that elements of $\mathbb{Z} * \mathbb{Z}$ are reduced words in a and b, and two such words represent the same element if and only if they are identical after cancellation of adjacent inverse pairs.

Let $K = \langle b \rangle \leq \pi_1(X, x_0)$ be the infinite cyclic subgroup generated by b. We first show that K is not a normal subgroup. Consider the element $a \in \pi_1(X, x_0)$. Then

$$a^{-1}ba,$$

is a reduced word of length 3 in the free group $\mathbb{Z} * \mathbb{Z}$. It cannot be written as a power of b, because any power b^n is a reduced word involving only the letter b (or its inverse). Hence $a^{-1}ba \notin K$, so K is not normal in $\pi_1(X, x_0)$.

Since $S^1 \vee S^1$ is semilocally simply connected, the classification of covering spaces (see [1, Proposition 1.36]) provides a covering map

$$p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0),$$

such that

$$p_* \pi_1(\widetilde{X}, \widetilde{x}_0) = K = \langle b \rangle.$$

Because K is not normal, this covering is not regular in the classical sense.

We now verify that p is H-regular for $H = K$. Let $x \in X$, let $\widetilde{x} \in p^{-1}(x)$, and choose a path α_x from x_0 to x whose lift starting at $\widetilde{x}_0$ ends at $\widetilde{x}$. Then the subgroup associated to $\widetilde{x}$ is

$$p_* \pi_1(\widetilde{X}, \widetilde{x}) = \alpha_x^{-1} K \alpha_x = \alpha_x^{-1} \langle b \rangle \alpha_x = \langle \alpha_x^{-1} b \alpha_x \rangle.$$

The transported subgroup of $H = K$ is

$$\alpha_x^{-1} H \alpha_x = \alpha_x^{-1} \langle b \rangle \alpha_x = \langle \alpha_x^{-1} b \alpha_x \rangle.$$

Thus the transported H coincides with $p_* \pi_1(\widetilde{X}, \widetilde{x})$. Since this subgroup is infinite cyclic, it is abelian, and hence every element of it commutes with every element of $p_* \pi_1(\widetilde{X}, \widetilde{x})$. Consequently, for any $h \in \alpha_x^{-1} H \alpha_x$,

$$h^{-1}\, p_* \pi_1(\widetilde{X}, \widetilde{x})\, h = p_* \pi_1(\widetilde{X}, \widetilde{x}).$$

Therefore the invariance condition of Definition 3.1 holds, and p is H-regular for $H = K$. This gives an example of an H-regular covering that is not a regular covering.

The next example provides a nontrivial geometric illustration of an H-regular covering that is not regular.

Example 3.4. Let X be a closed orientable surface of genus 2, i.e. the connected sum of two tori $T^2 \# T^2$ (see [1, Example 0.8]). Choose a basepoint $x_0 \in X$. Let $[a_1], [b_1], [a_2], [b_2]$ be the homotopy classes of the standard longitudinal and meridian loops on each torus; these generate $\pi_1(X, x_0)$. By van Kampen's theorem,

$$\pi_1(X, x_0) = \big\langle\, [a_1], [b_1], [a_2], [b_2] \;\big|\; [a_1, b_1]\, [a_2, b_2] = 1 \,\big\rangle.$$

(see [1, Example 1.28]).

The surface X is an Eilenberg–MacLane space $K(G, 1)$ with $G = \pi_1(X)$. Indeed, a closed orientable surface of genus $g \geq 2$ admits a complete hyperbolic metric, so its universal cover is the hyperbolic plane $\mathbb{H}^2$, which is contractible. Consequently, all higher homotopy groups vanish: $\pi_n(X) = 0$ for $n \geq 2$. A covering space of a $K(G, 1)$ is again a $K(H, 1)$ for the corresponding subgroup H (see [1, Example 1B.2]).

Let

$$H = \langle [a_1] \rangle \leq \pi_1(X, x_0),$$

be the infinite cyclic subgroup generated by the class of a non-separating simple closed curve. We first show that H is not normal in $\pi_1(X, x_0)$. Consider the element $[b_1][a_1][b_1]^{-1}$. We claim that this element does not belong to H. To see this, recall that the subgroup generated by $[a_1]$ and $[b_1]$ is free of rank 2. This is a standard fact about surface groups: cutting the surface along a non-separating simple closed curve representing $[a_1]$ yields a surface of genus 1 with two boundary components, whose fundamental group is free of rank 2; the inclusion of this subsurface into X induces an injection on fundamental groups. Hence $\langle [a_1], [b_1] \rangle \cong F_2$, the free group on two generators.

In the free group $F_2 = \langle [a_1], [b_1] \rangle$, the element $[b_1][a_1][b_1]^{-1}$ is a reduced word of length 3. If it were equal to a power $[a_1]^k$ for some $k \in \mathbb{Z}$, then since $[a_1]^k$ is a reduced word of length $|k|$, we would need $k = 0$ or $k = \pm 1$. The case $k = 0$ is impossible because $[b_1][a_1][b_1]^{-1} \neq 1$ in the free group. The cases $k = \pm 1$ would imply that $[a_1]$ and $[b_1]$ commute, which is false in F_2. Therefore $[b_1][a_1][b_1]^{-1} \notin H$, so H is not a normal subgroup of $\pi_1(X, x_0)$.

Since X is semilocally simply connected, the classification of covering spaces (see [1, Proposition 1.36]) provides a covering map

$$p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0),$$

such that

$$p_* \pi_1(\widetilde{X}, \widetilde{x}_0) = H = \langle [a_1] \rangle.$$

Because H is not normal, this covering is not regular in the classical sense.

We now verify that p is H-regular in the sense of Definition 3.1. Let $x \in X$, let $\widetilde{x} \in p^{-1}(x)$, and choose a path α_x from x_0 to x whose lift starting at $\widetilde{x}_0$ ends at $\widetilde{x}$. The isomorphism

$$\psi_{\alpha_x} : \pi_1(X, x_0) \to \pi_1(X, x),$$

sends H to the transported subgroup

$$H_{\alpha_x} := \alpha_x^{-1} H \alpha_x = \langle \alpha_x^{-1} [a_1] \alpha_x \rangle.$$

By Lemma 2.2, the subgroup associated to $\widetilde{x}$ is conjugate to H:

$$p_* \pi_1(\widetilde{X}, \widetilde{x}) = \alpha_x^{-1} H \alpha_x = H_{\alpha_x}.$$

Thus the transported subgroup H_{α_x} coincides with $p_* \pi_1(\widetilde{X}, \widetilde{x})$. Since H is infinite cyclic, it is abelian; consequently, its conjugate H_{α_x} is also abelian. Therefore, for every $h \in H_{\alpha_x}$,

$$h^{-1}\, p_* \pi_1(\widetilde{X}, \widetilde{x})\, h = p_* \pi_1(\widetilde{X}, \widetilde{x}),$$

because h commutes with every element of $p_*\pi_1(\widetilde{X},\widetilde{x}) = H_{\alpha_x}$. Hence the invariance condition of Definition 3.1 holds, and p is H-regular.

Finally, we determine the group of deck transformations of this covering. For a covering $p : \widetilde{X} \to X$ with $p_*\pi_1(\widetilde{X},\widetilde{x}_0) = H$, the group of deck transformations is isomorphic to $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X,x_0)$ (see [5, Corollary 10.28]). In the surface group of genus 2, the normalizer of the cyclic subgroup generated by a primitive element such as $[a_1]$ is the centralizer of $[a_1]$, which is the infinite cyclic subgroup H itself. Indeed, the centralizer of a primitive element in a torsion-free surface group is infinite cyclic generated by that element, and the normalizer coincides with the centralizer for primitive elements. Hence $N(H) = H$, and therefore

$$\mathrm{Cov}(\widetilde{X}/X) \cong N(H)/H = \{1\}.$$

Thus the covering has trivial deck transformation group. This is consistent with the fact that the covering is not regular: for a regular covering, the deck group acts transitively on each fiber, whereas here the fiber is not a single $\pi_1(X,x_0)$-orbit.

This example shows that H-regularity is a strictly weaker condition than classical regularity: the covering is H-regular for the non-normal subgroup $H = \langle [a_1] \rangle$, yet it is not a regular covering and its deck transformation group is trivial.

Notation: Let $H \leq \pi_1(X,x_0)$ be a subgroup. For a fixed covering $p : \widetilde{X} \to X$, basepoint $x_0 \in X$, point $x \in X$, lift $\widetilde{x} \in p^{-1}(x)$, and a path α_x from x_0 to x, we define

$$G_{\alpha_x,\widetilde{x}} := \langle \alpha_x^{-1} H \alpha_x,\ p_*\pi_1(\widetilde{X},\widetilde{x}) \rangle.$$

The subgroup $K_H(p,x_0,x,\widetilde{x}) := \bigcap_{\alpha_x} G_{\alpha_x,\widetilde{x}}$ (intersection over all paths α_x from x_0 to x) will be denoted simply by K_H when the context is clear.

We also define an equivalence relation on the fiber $p^{-1}(x_0)$: $\widetilde{x} \sim_H \widetilde{y}$ if there exists $h \in H$ such that the lift of h at $\widetilde{x}$ ends at $\widetilde{y}$. The equivalence class of $\widetilde{x}_0$ is denoted $p^{-1}_{H,\widetilde{x}_0}(x_0)$. For a general point x, with a path α_x from x_0 to x, we define $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$ analogously using $\alpha_x^{-1} H \alpha_x$. For fixed $x \in X$, a path α_x from x_0 to x, and a lift $\widetilde{x} \in p^{-1}(x)$, put

$$H_{\alpha_x} := \alpha_x^{-1} H \alpha_x \leq \pi_1(X,x).$$

The H-compatible subset of the fiber is the orbit

$$p^{-1}_{H,\alpha_x,\widetilde{x}}(x) = \{\, \widetilde{x} \cdot h \mid h \in H_{\alpha_x} \,\} \subseteq p^{-1}(x),$$

where $\cdot$ denotes the usual right action of $\pi_1(X,x)$ on $p^{-1}(x)$ by path lifting. We define

$$\mathrm{Cov}_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X) = \big\{ f \in \mathrm{Cov}(\widetilde{X}/X) \;\big|\; f\big(p^{-1}_{H,\alpha_x,\widetilde{x}}(x)\big) = p^{-1}_{H,\alpha_x,\widetilde{x}}(x) \big\}.$$

Equivalently, since every deck transformation commutes with the $\pi_1(X,x)$-action, this is the set of $f \in \mathrm{Cov}(\widetilde{X}/X)$ with

$$f(\widetilde{x}) \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x).$$

Thus $\mathrm{Cov}_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$ is the stabilizer in $\mathrm{Cov}(\widetilde{X}/X)$ of the H_{α_x}-orbit of $\widetilde{x}$.

Lemma 3.5. *Let $(\widetilde{X},p)$ be an H-regular covering of X, and let $x \in X$, $\widetilde{x} \in p^{-1}(x)$, and let α_x be a path from x_0 to x. Set*

$$G_{\alpha_x} = \langle\, \alpha_x^{-1} H \alpha_x,\ p_*\pi_1(\widetilde{X},\widetilde{x})\,\rangle.$$

Then $p_*\pi_1(\widetilde{X},\widetilde{x}) \trianglelefteq G_{\alpha_x}$.

Proof. Let $(\widetilde{X},p)$ be an H-regular covering of X with $H \leq \pi_1(X,x_0)$. Fix a point $x \in X$, a lift $\widetilde{x} \in p^{-1}(x)$, and a path α_x from x_0 to x. For brevity, denote

$$N = p_*\pi_1(\widetilde{X},\widetilde{x}) \leq \pi_1(X,x), \qquad G = \langle\, \alpha_x^{-1} H \alpha_x,\ N\,\rangle \leq \pi_1(X,x).$$

We must show that $N \trianglelefteq G$, i.e., for every $g \in G$, $g^{-1}Ng = N$.

By Definition 3.1, the H-regularity of p means that for every $h \in \alpha_x^{-1} H \alpha_x$,

$$h^{-1}Nh = N. \tag{1}$$

Thus the subset $\alpha_x^{-1} H \alpha_x$ is contained in the normalizer

$$\mathcal{N} = \{g \in \pi_1(X,x) \mid g^{-1}Ng = N\}.$$

Moreover, N itself is trivially contained in $\mathcal{N}$ because N is a subgroup: for any $n \in N$, $n^{-1}Nn = N$. Hence

$$\alpha_x^{-1} H \alpha_x \,\cup\, N \,\subseteq\, \mathcal{N}.$$

The normalizer $\mathcal{N}$ is a subgroup of $\pi_1(X,x)$; indeed, if $g_1, g_2 \in \mathcal{N}$, then $(g_1g_2)^{-1}N(g_1g_2) = g_2^{-1}(g_1^{-1}Ng_1)g_2 = g_2^{-1}Ng_2 = N$, and $g_1^{-1} \in \mathcal{N}$ as well. Since G is the smallest subgroup containing $\alpha_x^{-1} H \alpha_x$ and N, the inclusion above forces

$$G \subseteq \mathcal{N}.$$

Therefore every $g \in G$ satisfies $g^{-1}Ng = N$, which is precisely the statement $N \trianglelefteq G$.

Now consider the subgroup

$$K_H = \bigcap_{\alpha_x} \langle\, \alpha_x^{-1} H \alpha_x,\ p_*\pi_1(\widetilde{X},\widetilde{x})\,\rangle,$$

where the intersection runs over *all* paths α_x from x_0 to x. For each such path we have just proved that $p_*\pi_1(\widetilde{X},\widetilde{x}) \trianglelefteq \langle \alpha_x^{-1} H \alpha_x,\ p_*\pi_1(\widetilde{X},\widetilde{x})\rangle$. Hence $p_*\pi_1(\widetilde{X},\widetilde{x})$ is a normal subgroup of each term in the intersection. If a subgroup N is normal in a family of subgroups $\{G_i\}$, then

N is also normal in the intersection $\bigcap_i G_i$: for any $g \in \bigcap_i G_i$ we have $g \in G_i$ for all i, so $g^{-1}Ng = N$ because the equality holds in each G_i. Consequently,

$$p_*\pi_1(\widetilde{X}, \widetilde{x}) \trianglelefteq K_H,$$

as claimed. $\square$

Definition 3.6 (H-orbits and H-equivariant deck transformations)**.** Let $p : \widetilde{X} \to X$ be a covering map, $x_0 \in X$, and $H \leq \pi_1(X, x_0)$. The group $\pi_1(X, x_0)$ acts on the fiber $p^{-1}(x_0)$ via lifting of loops: for $\widetilde{x} \in p^{-1}(x_0)$ and $[g] \in \pi_1(X, x_0)$, let $\widetilde{x} \cdot [g]$ denote the endpoint of the unique lift of g starting at $\widetilde{x}$.

(1) **Orbit at the basepoint.** For $\widetilde{x}, \widetilde{y} \in p^{-1}(x_0)$, write $\widetilde{x} \sim_H \widetilde{y}$ if there exists $h \in H$ such that $\widetilde{x} \cdot h = \widetilde{y}$. Because H is a subgroup, $\sim_H$ is an equivalence relation on $p^{-1}(x_0)$. The equivalence class of a point $\widetilde{x}_0 \in p^{-1}(x_0)$ is denoted

$$p^{-1}_{H,\widetilde{x}_0}(x_0) = \{\widetilde{x} \in p^{-1}(x_0) \mid \widetilde{x} \sim_H \widetilde{x}_0\}.$$

In other words, $p^{-1}_{H,\widetilde{x}_0}(x_0)$ is precisely the orbit of $\widetilde{x}_0$ under the restricted action of $H \leq \pi_1(X, x_0)$.

(2) **Deck transformations preserving the basepoint orbit.** A deck transformation $f \in \mathrm{Cov}(\widetilde{X}/X)$ is a homeomorphism $f : \widetilde{X} \to \widetilde{X}$ with $p \circ f = p$. Define

$$\mathrm{Cov}_{H,\widetilde{x}_0}(\widetilde{X}/X) = \{\, f \in \mathrm{Cov}(\widetilde{X}/X) \mid f\big(p^{-1}_{H,\widetilde{x}_0}(x_0)\big) = p^{-1}_{H,\widetilde{x}_0}(x_0) \,\}.$$

(3) **Transport to an arbitrary point.** Let $x \in X$ and choose a path α_x from x_0 to x. The isomorphism $\psi_{\alpha_x} : \pi_1(X, x_0) \to \pi_1(X, x)$ sends H to the subgroup $H_{\alpha_x} := \alpha_x^{-1} H \alpha_x \leq \pi_1(X, x)$. The fiber $p^{-1}(x)$ is a right $\pi_1(X, x)$-set by the analogous lifting construction. For a chosen lift $\widetilde{x} \in p^{-1}(x)$, we define

$$p^{-1}_{H,\alpha_x,\widetilde{x}}(x) = \{\, \widetilde{w} \in p^{-1}(x) \mid \widetilde{w} \sim_{H_{\alpha_x}} \widetilde{x} \,\},$$

i.e. the orbit of $\widetilde{x}$ under the action of the transported subgroup H_{α_x}. The associated set of deck transformations that preserve this orbit is

$$\mathrm{Cov}_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X) = \{\, f \in \mathrm{Cov}(\widetilde{X}/X) \mid f\big(p^{-1}_{H,\alpha_x,\widetilde{x}}(x)\big) = p^{-1}_{H,\alpha_x,\widetilde{x}}(x) \,\}.$$

(4) **Intersection over all paths.** Finally, we define

$$\mathrm{Cov}_{H,x_0,\widetilde{x}}(\widetilde{X}/X) = \bigcap_{\alpha_x} \mathrm{Cov}_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X),$$

where the intersection runs over *all* paths α_x from x_0 to x. Thus a deck transformation belongs to $\mathrm{Cov}_{H,x_0,\widetilde{x}}(\widetilde{X}/X)$ iff it preserves every orbit $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$, regardless of the path used to transport H from x_0 to x.

Remark 3.7. If $H = \pi_1(X, x_0)$, then $p^{-1}_{H,\alpha_x,\widetilde{x}}(x) = p^{-1}(x_0)$ and $Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X) = Cov(\widetilde{X}/X)$.

By Remark 3.7, the subsequent lemma is an extension of [5, Theorem 10.9].

Lemma 3.8. *Let $(\widetilde{X}, p)$ be a covering space of X, let $x_0, x \in X$, and let $H \leq \pi_1(X, x_0)$. For any path α_x from x_0 to x and any $\widetilde{x} \in p^{-1}(x)$, define*

$$G_{\alpha_x,\widetilde{x}} = \langle \alpha_x^{-1} H \alpha_x,\ p_*\pi_1(\widetilde{X}, \widetilde{x}) \rangle.$$

Then the following hold.

(1) *$G_{\alpha_x,\widetilde{x}}$ acts transitively (on the right) on the set $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$.*
(2) *For every $\widetilde{w} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$, the stabilizer of $\widetilde{w}$ under this action is $p_*\pi_1(\widetilde{X}, \widetilde{w})$.*
(3) $\left|p^{-1}_{H,\alpha_x,\widetilde{x}}(x)\right| = \left[G_{\alpha_x,\widetilde{x}} : p_*\pi_1(\widetilde{X}, \widetilde{x})\right]$.

Proof. To keep notation compact, write $G = G_{\alpha_x,\widetilde{x}}$ and $F = p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$. Recall that F consists of those points $\widetilde{w} \in p^{-1}(x)$ for which there exists an element $h \in \alpha_x^{-1} H \alpha_x$ such that the lift of h starting at $\widetilde{x}$ ends at $\widetilde{w}$; in the language of the equivalence relation $\sim_{\alpha_x^{-1} H \alpha_x}$, this means $\widetilde{w}$ lies in the class of $\widetilde{x}$.

Definition of the action. For $[g] \in G$ and $\widetilde{w} \in F$, let $\widetilde{g}$ be the unique lift of g with $\widetilde{g}(0) = \widetilde{w}$. We set

$$\widetilde{w}[g] := \widetilde{g}(1).$$

Because the lift of a path depends only on its path-homotopy class (see [5, Corollary 10.6]), the value $\widetilde{w}[g]$ is well defined regardless of which representative of $[g]$ is used. If g is the constant path at x, then its lift is the constant path at $\widetilde{w}$, so $\widetilde{w}[g] = \widetilde{w}$. For two classes $[g_1], [g_2] \in G$, let $\widetilde{g_1}$ lift g_1 from $\widetilde{w}$ and $\widetilde{g_2}$ lift g_2 from $\widetilde{g_1}(1)$. Then $\widetilde{g_1} * \widetilde{g_2}$ is the lift of $g_1 * g_2$ starting at $\widetilde{w}$, and evaluating at the endpoint gives

$$\widetilde{w}[g_1 * g_2] = (\widetilde{g_1} * \widetilde{g_2})(1) = \widetilde{g_2}(1) = (\widetilde{w}[g_1])[g_2].$$

Thus the formula defines a right action of G on the whole fiber $p^{-1}(x)$. It remains to show that the subset F is invariant under this action, i.e. that $\widetilde{w}[g] \in F$ whenever $\widetilde{w} \in F$ and $[g] \in G$.

Every element of G can be written as a product of elements from $\alpha_x^{-1} H \alpha_x$ and from $p_*\pi_1(\widetilde{X}, \widetilde{x})$. If $g \in \alpha_x^{-1} H \alpha_x$, then by definition of F the endpoint $\widetilde{x}[g]$ belongs to F. If $g \in p_*\pi_1(\widetilde{X}, \widetilde{x})$, then g lifts to a loop based at $\widetilde{x}$, so $\widetilde{x}[g] = \widetilde{x} \in F$. By induction on the word length, using the fact that the action of a product is the composition of the actions, one sees that $\widetilde{x}[g] \in F$ for every $[g] \in G$. Now if $\widetilde{w} \in F$, there is $h \in \alpha_x^{-1} H \alpha_x$ with $\widetilde{w} = \widetilde{x}[h]$. Then for any $[g] \in G$,

$$\widetilde{w}[g] = (\widetilde{x}[h])[g] = \widetilde{x}[hg],$$

and $hg \in G$ because G is a subgroup. Hence $\widetilde{w}[g] \in F$. Therefore the action restricts to a right action of G on F.

1. Transitivity. Let $\widetilde{w}, \widetilde{y} \in F$. By definition, there exist $h_w, h_y \in \alpha_x^{-1} H \alpha_x \subseteq G$ such that $\widetilde{w} = \widetilde{x}[h_w]$ and $\widetilde{y} = \widetilde{x}[h_y]$. Consider the element $g = h_w^{-1} * h_y \in \alpha_x^{-1} H \alpha_x \subseteq G$. Then

$$\widetilde{w}[g] = (\widetilde{x}[h_w])[h_w^{-1} * h_y] = \widetilde{x}[h_w * h_w^{-1} * h_y] = \widetilde{x}[h_y] = \widetilde{y}.$$

Thus G acts transitively on F.

2. Stabilizer. Take $\widetilde{w} \in F$ and let $[f] \in G$ be an element that stabilises $\widetilde{w}$, i.e. $\widetilde{w}[f] = \widetilde{w}$. Let $\widetilde{f}$ be the lift of f with $\widetilde{f}(0) = \widetilde{w}$; then $\widetilde{f}(1) = \widetilde{w}$, so $\widetilde{f}$ is a loop at $\widetilde{w}$. Hence $[\widetilde{f}] \in \pi_1(\widetilde{X}, \widetilde{w})$ and $[f] = [p \circ \widetilde{f}] \in p_*\pi_1(\widetilde{X}, \widetilde{w})$.

Conversely, if $[f] \in p_*\pi_1(\widetilde{X}, \widetilde{w})$, write $[f] = [p \circ \widetilde{g}]$ with $[\widetilde{g}] \in \pi_1(\widetilde{X}, \widetilde{w})$. The unique lift of f starting at $\widetilde{w}$ coincides with $\widetilde{g}$, because both project to f and start at the same point. Therefore $\widetilde{f}(1) = \widetilde{g}(1) = \widetilde{w}$, yielding $\widetilde{w}[f] = \widetilde{w}$. Thus the stabiliser $\mathrm{Stab}_G(\widetilde{w})$ is exactly $p_*\pi_1(\widetilde{X}, \widetilde{w})$. (Notice that $p_*\pi_1(\widetilde{X}, \widetilde{w}) \subseteq G$ because any such element can be expressed using a conjugation by a suitable path, but the direct inclusion is also clear from the definition of G and the fact that $p_*\pi_1(\widetilde{X}, \widetilde{w})$ is conjugate to $p_*\pi_1(\widetilde{X}, \widetilde{x})$ by an element of $\alpha_x^{-1} H \alpha_x$.)

3. Cardinality of F. The orbit–stabiliser theorem for a transitive group action yields

$$|F| = [G : \mathrm{Stab}_G(\widetilde{w})],$$

for any $\widetilde{w} \in F$. Choosing $\widetilde{w} = \widetilde{x}$ and using part 2, we obtain

$$|F| = \big[G : p_*\pi_1(\widetilde{X}, \widetilde{x})\big].$$

□

Lemma 3.9. *Let $(\widetilde{X}, p)$ be a covering space of X and $x_0, x \in X$. Let $H \leq \pi_1(X, x_0)$ and α_x is a path from x_0 to x. Then $Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$ act transitively on $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$ if and only if $p_*\pi_1(\widetilde{X}, \widetilde{x}) = p_*\pi_1(\widetilde{X}, \widetilde{y})$ for all $\widetilde{y} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$.*

Proof. Assume that $Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$ acts transitively on $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$; if $\widetilde{y} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$, then there exists $f \in Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$ with $f(\widetilde{x_0}) = \widetilde{y}$. Now $f_*\pi_1(\widetilde{X}, \widetilde{x}) = \pi_1(\widetilde{X}, \widetilde{y})$. Because $p = pf$, it follows that $p_* = p_*f_*$; thus, $p_*\pi_1(\widetilde{X}, \widetilde{x}) = p_*f_*\pi_1(\widetilde{X}, \widetilde{x}) = p_*\pi_1(\widetilde{X}, \widetilde{y})$. Conversely, let $\widetilde{y} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$ so $p_*\pi_1(\widetilde{X}, \widetilde{x}) = p_*\pi_1(\widetilde{X}, \widetilde{y})$. The lifting criterion (see [5, Theorem 10.16]) gives a homeomorphism $f : (\widetilde{X}, \widetilde{x}) \to (\widetilde{X}, \widetilde{y})$ with $pf = p$; thus $f \in Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$ and $f(\widetilde{x_0}) = \widetilde{y}$. Therefore $Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$ acts transitively on $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$. □

Theorem 3.10. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map and $H \leq \pi_1(X, x_0)$. The following are equivalent.*

(1) *p is H-regular.*

(2) *For every $x \in X$, every path α_x from x_0 to x, and every lift $\widetilde{x} \in p^{-1}(x)$, the group* $\mathrm{Cov}_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$ *acts transitively on the fiber subset $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$.*

(3) *For every $x \in X$, every path α_x from x_0 to x, and every lift $\widetilde{x} \in p^{-1}(x)$, we have $p_*\pi_1(\widetilde{X},\widetilde{x}) = p_*\pi_1(\widetilde{X},\widetilde{y})$ for all $\widetilde{y} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$.*

Proof. **(i) $\Leftrightarrow$ (iii).** Suppose p is H-regular and let $x, \alpha_x, \widetilde{x}$ and $\widetilde{y} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$. By definition of the orbit, there exists $h_{\widetilde{y}} \in \alpha_x^{-1}H\alpha_x$ such that the lift of $h_{\widetilde{y}}$ from $\widetilde{x}$ ends at $\widetilde{y}$; hence $[h_{\widetilde{y}}]^{-1}p_*\pi_1(\widetilde{X},\widetilde{x})[h_{\widetilde{y}}] = p_*\pi_1(\widetilde{X},\widetilde{y})$ by Lemma 2.2. H-regularity forces the left-hand side to equal $p_*\pi_1(\widetilde{X},\widetilde{x})$, so $p_*\pi_1(\widetilde{X},\widetilde{x}) = p_*\pi_1(\widetilde{X},\widetilde{y})$. Conversely, if (iii) holds and $h \in \alpha_x^{-1}H\alpha_x$, then taking $\widetilde{y}$ as the endpoint of the lift of h from $\widetilde{x}$, Lemma 2.2 gives $[h]^{-1}p_*\pi_1(\widetilde{X},\widetilde{x})[h] = p_*\pi_1(\widetilde{X},\widetilde{y}) = p_*\pi_1(\widetilde{X},\widetilde{x})$, proving H-regularity.

(ii) $\Leftrightarrow$ (iii). Lemma 3.9 states precisely the equivalence between the transitivity of $\mathrm{Cov}_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$ on the orbit and the equality of the subgroups. □

Theorem 3.11. *Let $(\widetilde{X},p)$ be a covering space of X, where X is a locally path-connected; let $x_0, x \in X$, and let the set $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$ be viewed as $\langle \alpha_x^{-1}H\alpha_x,\ p_*\pi_1(\widetilde{X},\widetilde{x})\rangle$-set, where $H \leq \pi_1(X,x_0)$ and α_x is a path from x_0 to x. Then $f \longmapsto f|_{p^{-1}_{H,\alpha_x,\widetilde{x}}(x)}$ is an isomorphism*

$$Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X) \cong Aut(p^{-1}_{H,\alpha_x,\widetilde{x}}(x)).$$

Proof. Denote $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$ by Y and $G = \langle \alpha_x^{-1}H\alpha_x, p_*\pi_1(\widetilde{X},\widetilde{x})\rangle$. If $f \in Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$, then the restriction $f|_Y : Y \to Y$ is a bijection and a G-map because f commutes with the action (as shown in the proof of Lemma 3.8). The map $f \longmapsto f|_Y$ is a homomorphism.

It is injective: if $f|_Y = \mathrm{id}_Y$, then in particular $f(\widetilde{x}) = \widetilde{x}$, and by the uniqueness of lifts (a deck transformation is determined by its value at a single point, see [5, Theorem 10.16]) we have $f = 1_{\widetilde{X}}$.

To prove surjectivity, let $\phi \in Aut(Y)$. For any $\widetilde{w} \in Y$, we have $G_{\widetilde{w}} = G_{\phi(\widetilde{w})}$ because ϕ is a G-isomorphism. By Lemma 3.8(2) these stabilisers are $p_*\pi_1(\widetilde{X},\widetilde{w})$ and $p_*\pi_1(\widetilde{X},\phi(\widetilde{w}))$, so they are equal. The lifting criterion [5, Theorem 10.16] then yields a deck transformation $f \in Cov(\widetilde{X}/X)$ with $f(\widetilde{w}) = \phi(\widetilde{w})$. Because Y is a transitive G-set, for any $\widetilde{x_1} \in Y$ there exists $[h] \in G$ with $\widetilde{x_1} = [h]\widetilde{w}$. Hence

$$f(\widetilde{x_1}) = f([h]\widetilde{w}) = [h]f(\widetilde{w}) = [h]\phi(\widetilde{w}) = \phi([h]\widetilde{w}) = \phi(\widetilde{x_1}),$$

so $f|_Y = \phi$. Since f maps Y onto Y, we have $f \in Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X)$, completing the proof. □

The next theorem is the counterpart for the intersection over all paths.

Theorem 3.12. *Let $(\widetilde{X}, p)$ be a covering space of X, where X is locally path connected; let $x_0, x \in X$, and let the set Y_H be viewed as a K_H-set. Then $f \longmapsto f|_{Y_H}$ is an isomorphism*

$$Cov_{H,x_0,\widetilde{x}}(\widetilde{X}/X) \cong Aut(Y_H)$$

Proof. The argument is completely analogous to the previous theorem, using the fact that Y_H is a transitive K_H-set and the deck transformations that preserve Y_H are exactly those in $Cov_{H,x_0,\widetilde{x}}(\widetilde{X}/X)$. The details are omitted. □

Theorem 3.13. *Let $(\widetilde{X}, p)$ be a covering space of X, where X denotes a locally path-connected. If $x_0, x \in X$, $H \leq \pi_1(X, x_0)$, $\widetilde{w} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$ and α_x is a path from x_0 to x, then*

$$Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X) \cong N_{\langle \alpha_x^{-1} H \alpha_x,\ p_*\pi_1(\widetilde{X},\widetilde{x})\rangle}(p_*\pi_1(\widetilde{X},\widetilde{w}))/p_*\pi_1(\widetilde{X},\widetilde{w}).$$

Proof. By Theorem 3.11, $Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X) \cong Aut(p^{-1}_{H,\alpha_x,\widetilde{x}}(x))$ where the fiber $p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$ is viewed as a transitive $\langle \alpha_x^{-1} H \alpha_x, p_*\pi_1(\widetilde{X},\widetilde{x})\rangle$-set (Lemma 3.8 (1)). Stabilizer of $\widetilde{w}$ is $p_*\pi_1(\widetilde{X},\widetilde{w})$, by Lemma 3.8(2). The theorem follows from Lemma 2.5. □

The proof of the following theorem that follows is similar to the previous theorem.

Theorem 3.14. *Let $(\widetilde{X}, p)$ be a covering space of X, where X denotes a locally path-connected. If $x_0, x \in X$ and $\widetilde{w} \in Y$, then*

$$Cov_{H,x_0,\widetilde{x}}(\widetilde{X}/X) \cong N_{K_H}(p_*\pi_1(\widetilde{X},\widetilde{w}))/p_*\pi_1(\widetilde{X},\widetilde{w}).$$

Corollary 3.15. *Let $(\widetilde{X}, p)$ be an H-regular covering space of X, where X is a locally path-connected, $H \leq \pi_1(X, x_0)$ and α_x is a path from x_0 to x. Then, for $x \in X$ and $\widetilde{w} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$,*

$$Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X) \cong \langle \alpha_x^{-1} H \alpha_x,\ p_*\pi_1(\widetilde{X},\widetilde{x})\rangle/p_*\pi_1(\widetilde{X},\widetilde{x}).$$

In the above Corollary, conclude the [5, Corollary 10.28], it is enough to consider $H = \pi_1(X, x)$.

Proof. Because $(\widetilde{X}, p)$ is an H-regular covering space of X, $p_*\pi_1(\widetilde{X},\widetilde{w})$ is a normal subgroup of $\langle \alpha_x^{-1} H \alpha_x, p_*\pi_1(\widetilde{X},\widetilde{x})\rangle$, and so $p_*\pi_1(\widetilde{X},\widetilde{w}) = p_*\pi_1(\widetilde{X},\widetilde{x})$ for all $\widetilde{w} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$. □

In addition, we conclude the following corollary.

Corollary 3.16. *Let $(\widetilde{X}, p)$ be an H-regular covering space of X, where X is locally path connected and $H \leq \pi_1(X, x_0)$. For $x \in X$ and $\widetilde{w} \in Y_H$,*

$$Cov_{H,x_0,\widetilde{x}}(\widetilde{X}/X) \cong K_H/p_*\pi_1(\widetilde{X},\widetilde{x}).$$

Corollary 3.17. *Let $(\widetilde{X}, p)$ be a universal covering space of X, where X is a locally path-connected, $H \leq \pi_1(X, x_0)$ and α_x is a path from x_0 to x. Then, for $x \in X$,*

$$Cov_{H,\alpha_x,\widetilde{x}}(\widetilde{X}/X) \cong \langle \alpha_x^{-1} H \alpha_x \rangle.$$

In the above Corollary, conclude the [5, Corollary 10.29], it is enough to consider $H = \pi_1(X, x)$.

Proof. Because X is simply connected, $\pi_1(\widetilde{X}, \widetilde{w}) = \{1\}$ for every $\widetilde{w} \in p^{-1}_{H,\alpha_x,\widetilde{x}}(x)$, and so $p_*\pi_1(\widetilde{X}, \widetilde{w}) = \{1\}$. □

In addition, we conclude the following corollary.

Corollary 3.18. *Let $(\widetilde{X}, p)$ be a universal covering space of X, where X is a locally path-connected and $H \leq \pi_1(X, x_0)$. Then, for $x \in X$,*

$$Cov_{H,x_0,\widetilde{x}}(\widetilde{X}/X) \cong K_H.$$

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