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Research Paper

SOME GENERAL FORM OF SMALL SUBMODULES

PORAMATE SANGCHAN, NATTHAPHONG CHITMUNG AND SAMRUAM BAUPRADIST*

Abstract. In this paper, we introduce and study the concept of S-F-small submodules, where S is a multiplicatively closed subset of a ring R, and F is a proper submodule of a right R-module M. This notion generalizes both S-small submodules and F-small submodules. Moreover, we establish conditions under which a right R-module M is an S-F-small submodule. Finally, we present the relationships among S-essential submodules, S-small submodules, S-D-essential submodules, and S-F-small submodules under special conditions, where D is a nonzero submodule of a right R-module M.

1. Introduction

Throughout this paper, R is an associative ring with identity, and all right R-modules are assumed to be unitary. We denote by $\operatorname{Ann}_R(M)$ the right annihilator of M, that is, $\operatorname{Ann}_R(M) = \{ r \in R \mid Mr = 0 \}$, and $U(R)$ as the set of all units of R. In module theory, submodules of a right R-module are extensively studied, and an important example is the class

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*Corresponding author

of essential submodules. A nonzero submodule N of a right R-module M is called an *essential submodule* (or *large submodule*) if the following condition is satisfied. For every submodule H of M, the condition $N \cap H = 0$ implies $H = 0$. Generalizations of essential submodules have also been investigated extensively, see e.g. [2, 4, 5].

In 2022, S. Rajaee [6] introduced and studied the notion of S-essential submodules. Let S be a multiplicatively closed subset of a ring R, and let M be a right R-module. A submodule N of M is called an *S-essential submodule* (or *S-large submodule*) of M, denoted by $N \ll_e^S M$, if for every submodule L of M, the condition $N \cap L = 0$ implies that there exists $s \in S$ such that $Ls = 0$. In [6], S. Rajaee showed that if M is an S-co-m module satisfying the S-DAC and $N \ll M$, then $N \ll_e^S M$ if and only if there exists $I \ll R_R$ such that $s(0 :_M I) \ll N \ll (0 :_M I)$ for some $s \in S$. Later, in 2023 [8], we further studied this topic and established conditions under which a right R-module has proper S-essential submodules.

The dual notion of an essential submodule is a small submodule (or superfluous submodule), which is well-known and plays an important role in module theory, particularly in the study of projective modules. Let M be a right R-module. A submodule K of M is called *small* if for every submodule L of M, the equality $K + L = M$ implies $L = M$. From this definition, it follows that 0 is always a small submodule of M. Furthermore, if M is small in itself, then $M = 0$; that is, M is not a small submodule of a nonzero right R-module M. There are many ways to generalize the concept of small submodules (e.g., [1, 2, 3, 6]).

In 2015, M. D. Cisse and D. Sow [4] introduced the concept of an F-small submodule, which generalizes small submodules where F is a proper submodule of M. Let M be a right R-module and F a proper submodule of M. A submodule K of M is called *F-small* if for every submodule L of M with $F \ll M$, the equality $K + L = M$ implies $L = M$. In the case $F = 0$, the notions of an F-small submodule and a small submodule coincide. Moreover, M cannot be an F-small submodule for every proper submodule F of M.

Another way to generalize small submodules is by using a multiplicatively closed subset of a ring R. In 2021, S. Rajaee [6] introduced the concept of S-small submodules. In that work, every ring R was assumed to be commutative with $1 \neq 0$. A multiplicatively closed subset (m.c.s.) of R is a nonempty subset S such that $0 \notin S$, $1 \in S$, and $ss' \in S$ for all $s, s' \in S$. A submodule N of a right R-module M is called an *S-small submodule* (or *S-superfluous submodule*) of M if for every submodule U of M and $s \in S$, the condition $Ms \ll N + U$ implies that there exists $t \in S$ such that $Mt \ll U$, and denoted by $N \ll_s^S M$. Moreover, M can be an S-small submodule of some right R-module M and multiplicatively closed subset S of R. The conditions under which M is an S-small submodule of itself, and the relationship between 0 being an S-essential submodule of M and M being an S-small submodule of M, can be determined (see [6]).

In this paper, we study some properties of S-small submodules. Inspired by the work of M. D. Cisse and D. Sow [4] on F-small submodules, we extend the concept of S-small submodules to that of S-F-small submodules. This extension is analogous to the relationship between small submodules and F-small submodules. Throughout our study of S-F-small submodules, we assume that R is commutative. We call a submodule K of M an S-F-*small submodule*, denoted by $K \ll^{S}_{s,F} M$, if for each submodule L of M containing F and for each $s \in S$, the condition $Ms \ll K+L$ implies that there exists $t \in S$ such that $Mt \ll L$. We provide examples of right R-modules M, multiplicatively closed subsets $S \subseteq R$, and proper submodules $F \subset M$ for which $M \ll^{S}_{s,F} M$. We also establish conditions under which M is an S-F-small submodule and study some of its properties. In the final part of this research, we conclude by presenting the relationships among an S-small submodule, an S-essential submodule, an S-D-essential submodule, and an S-F-small submodule in certain special cases.

2. S-small submodules

In 2021, S. Rajaee [6] introduced the concept of S-small submodules, where S is a multiplicatively closed subset of a ring R, but did not provide an example. Therefore, in Section 2, we present several examples of S-small submodules.

Definition 2.1. [6] Let M be a right R-module and S a multiplicatively closed subset of a ring R. A submodule N of M is an S-*small* (S-*superfluous*) submodule, denoted by $N \ll^{S}_{s} M$, if for each submodule U of M and $s \in S$, $Ms \ll N + U$ implies that there exists $t \in S$ such that $Mt \ll U$.

Example 2.2. Let M be a right R-module and S a multiplicatively closed subset of a ring R. Then 0 is an S-small submodule of M.

Example 2.3. Consider $\mathbb{Z}_6$ as a right $\mathbb{Z}_6$-module and $S = \{\overline{1}, \overline{2}, \overline{4}\}$. It is clear that S is a multiplicatively closed subset of $\mathbb{Z}_6$. Then $\langle\overline{2}\rangle$ is an S-small submodule of $\mathbb{Z}_6$.

Proof. Let L be a submodule of $\mathbb{Z}_6$ and $s \in S$ such that $\mathbb{Z}_6 s \ll \langle\overline{2}\rangle + L$. In this case, one may take $t = \overline{2}$. Then $\mathbb{Z}_6 t \ll L$. Hence $\langle\overline{2}\rangle$ is an S-small submodule of $\mathbb{Z}_6$. □

Example 2.4. Consider $\mathbb{Z}_{12}$ as a right $\mathbb{Z}_{12}$-module and $S = \{\overline{1}, \overline{3}, \overline{9}\}$. It is clear that S is a multiplicatively closed subset of $\mathbb{Z}_{12}$. Then $\langle\overline{4}\rangle$ is an S-small submodule of $\mathbb{Z}_{12}$.

Proof. Let L be a submodule of $\mathbb{Z}_{12}$ and $s \in S$ such that $\mathbb{Z}_{12} s \ll \langle\overline{4}\rangle + L$. In this case, one may take $t = \overline{3}$. Then $\mathbb{Z}_{12} t \ll L$. Hence $\langle\overline{4}\rangle$ is an S-small submodule of $\mathbb{Z}_{12}$. □

Proposition 2.5. *Let M be a right R-module and S a multiplicatively closed subset of a ring R. If $S \subseteq U(R)$, then every S-small submodule of M is a small submodule of M.*

Proof. Let K be an S-small submodule of M and L a submodule of M such that $M = K + L$. Since $M1 \ll K + L$ and $K \ll_s^S M$, there exists $t \in S$ such that $Mt \ll L$. But $t \in S \subseteq U(R)$, so we have $L \ll M = Mt \ll L$, and hence $M = L$. Therefore K is a small submodule of M. □

Proposition 2.6. *Let M be a right R-module and $S \subseteq U(R)$ a multiplicatively closed subset of a ring R. If M is a uniserial module, then every proper submodule of M is an S-small submodule of M.*

Proof. Let H be a proper submodule of M. Since every proper submodule H of a uniserial module M is a small submodule and every S-small submodule of M is a small submodule of M when $S \subseteq U(R)$, it follows that H is an S-small submodule of M. □

In the case that $S = \{1\}$, every S-small submodule is a small submodule; that is, an S-small submodule is a generalization of a small submodule.

Let S be a multiplicatively closed subset of R. We now give a characterization of an S-small submodule. We introduce the notion of an S-small epimorphism and then use it to characterize an S-small submodule.

Definition 2.7. Let M, N be right R-modules and S a multiplicatively closed subset of a ring R. An epimorphism $f : M \to N$ is said to be an S-small epimorphism if $\mathrm{Ker}\,(f)$ is an S-small submodule of M.

Proposition 2.8. *Let M, N be right R-modules and S a multiplicatively closed subset of a ring R. If $S \subseteq U(R)$, then every S-small epimorphism from M to N is a small epimorphism from M to N.*

Proof. This follows from Proposition 2.5. □

Proposition 2.9. *Let A, B, and C be right R-modules and S a multiplicatively closed subset of a ring R. If $\alpha : A \to B$ and $\beta : B \to C$ are S-small epimorphisms such that α is a monomorphism, then $\beta \circ \alpha : A \to C$ is also an S-small epimorphism.*

Proof. Let $\alpha : A \to B$ and $\beta : B \to C$ be S-small epimorphisms, $s_1 \in S$, and $U \ll A$ such that $As_1 \ll \mathrm{Ker}\,(\beta\circ\alpha)+U$. Then $\alpha(A)s_1 \ll \alpha(\mathrm{Ker}\,(\beta\circ\alpha))+\alpha(U)$. Since $\mathrm{Ker}\,(\beta\circ\alpha) = \alpha^{-1}(\mathrm{Ker}\,(\beta))$ and α is an epimorphism, $Bs_1 \ll \mathrm{Ker}\,(\beta) + \alpha(U)$. Since $\mathrm{Ker}\,(\beta) \ll_s^S B$, there exists $s_2 \in S$ such that $Bs_2 \ll \alpha(U)$. Then $As_2 = \alpha^{-1}(B)s_2 \ll \alpha^{-1}(Bs_2) \ll U \ll \mathrm{Ker}\,(\alpha) + U$. Since

$\mathrm{Ker}\,(\alpha) \ll_s^S A$, there exists $s_3 \in S$ such that $As_3 \ll U$. Therefore $\beta \circ \alpha$ is an S-small epimorphism. □

Theorem 2.10. *Let M, N be right R-modules, S a multiplicatively closed subset of a ring R, and f an epimorphism from M to N. If $f : M \to N$ is an S-small epimorphism, then for each right R-module L and R-homomorphism $h : L \to M$ such that $f \circ h$ is an epimorphism, there exists $t \in S$ such that $Mt \ll Im\,(h)$.*

Proof. Let L be a right R-module and $h : L \to M$ an R-homomorphism such that $f \circ h : L \to N$ is an epimorphism. Choose $s \in S$. We will show that $Ms \ll \mathrm{Im}\,(h) + \mathrm{Ker}\,(f)$. Let $ms \in Ms$ where $m \in M$. Thus $f(m) \in N$. Since $f \circ h$ is an epimorphism, there exists $l \in L$ such that $f(m) = (f \circ h)(l)$. Then $f(ms - h(ls)) = f(ms) - (f \circ h)(ls) = 0$. Since $ms = h(ls) + [ms - h(ls)]$ where $h(ls) \in \mathrm{Im}\,(h)$ and $ms - h(ls) \in \mathrm{Ker}\,(f)$, it follows that $Ms \ll \mathrm{Im}\,(h) + \mathrm{Ker}\,(f)$. Since f is an S-small epimorphism, there exists $t \in S$ such that $Mt \ll \mathrm{Im}\,(h)$. □

Theorem 2.11. *Let M be a right R-module, S a multiplicatively closed subset of a ring R, and K a submodule of M. Then the following statements are equivalent.*

(1) *K is an S-small submodule of M.*

(2) *The projection map $\pi_K : M \to M/K$ is an S-small epimorphism.*

(3) *For each right R-module N, $h \in Hom_R(N, M)$, and $s \in S$, if $Ms \ll Im\,(h) + K$, then there exists $t \in S$ such that $Mt \ll Im\,(h)$.*

Proof. (1) $\Longleftrightarrow$ (2): It is obvious.

(1) $\Longrightarrow$ (3): Since $\mathrm{Im}\,(h) \ll M$ and (1) holds, we have (3).

(3) $\Longrightarrow$ (1): Let L be a submodule of M and $s \in S$ such that $Ms \ll K + L$, and let i_L be the inclusion map from L to M. Then $\mathrm{Im}\,(i_L) = L$, hence $Ms \ll K + \mathrm{Im}\,(i_L)$. By (3), there exists $t \in S$ such that $Mt \ll \mathrm{Im}\,(i_L) = L$. □

3. Some properties of S-small submodules

In general, a right R-module M is not a small submodule of itself. However, in the S-version, we can exhibit a nonzero right R-module M and a multiplicatively closed subset S of R such that M is an S-small submodule of itself.

Example 3.1. Consider $\mathbb{Z}_m$ as a right $\mathbb{Z}$-module, where $m = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_n^{\alpha_n}$, with p_i prime numbers and α_i positive integers for all $i = 1, 2, \ldots, n$. Let $p = \min\{p_i \mid i = 1, 2, \ldots, n\}$ and $S = \{1, p, 2p, 3p, \ldots\}$. Clearly, S is a multiplicatively closed subset of $\mathbb{Z}$. Then $\mathbb{Z}_m$ is an S-small submodule of itself.

Proof. Since $\mathbb{Z}_m m = 0$ and $m \in S$, it follows that $\mathbb{Z}_m$ is an S-small submodule of itself. □

Next, we find some conditions under which a right R-module M is an S-small submodule of itself.

Theorem 3.2. *Let M be a right R-module and S a multiplicatively closed subset of R. Then the following statements are equivalent:*

(1) $M \ll_s^S M$.

(2) *For each submodule L of M, $L \ll_s^S M$.*

(3) *For each nonzero submodule L of M, $L \ll_s^S M$.*

(4) *For each submodule L of M, there exists $s \in S$ such that $Ms \ll L$.*

(5) $Ann_R(M) \cap S \neq \emptyset$.

Proof. (1) $\Longrightarrow$ (2): Assume that $M \ll_s^S M$. Let $L \ll M$, K be submodules of M, and $s \in S$ such that $Ms \ll L + K$. Since $M1 \ll M + K$ and $M \ll_s^S M$, there exists $t \in S$ such that $Mt \ll K$. Hence $L \ll_s^S M$.

(2) $\Longrightarrow$ (3): It is obvious.

(3) $\Longrightarrow$ (4): Case 1: $M = 0$. It is obvious.

Case 2: $M \neq 0$. Let $L \ll M$. Since $M \ll_s^S M$ and $M1 \ll M + L$, there exists $s \in S$ such that $Ms \ll L$.

(4) $\Longrightarrow$ (1): Let $L \ll M$ and $s \in S$ such that $Ms \ll M + L = M$. By assumption, there exists $t \in S$ such that $Mt \ll L$.

(5) $\Longrightarrow$ (1): Let $a \in \mathrm{Ann}_R(M) \cap S$. Then $Ma = 0$. Let $L \ll M$ and $s \in S$ such that $Ms \ll M + L = M$. Thus $Ma = 0 \ll L$.

(1) $\Longrightarrow$ (5): Since $M1 \ll M + 0 = M$ and $M \ll_s^S M$, there exists $t \in S$ such that $Mt \ll 0$, so $Mt = 0$. Hence $t \in \mathrm{Ann}_R(M) \cap S$. □

Corollary 3.3. *Let M and N be right R-modules, S a multiplicatively closed subset of R, and $M \cong N$. Then $M \ll_s^S M$ if and only if $N \ll_s^S N$.*

Proof. ($\Longrightarrow$) Suppose that $M \ll_s^S M$. Since $M \cong N$, there exists an isomorphism $f : M \to N$. Let L be a submodule of N. There exists a submodule $L' \subseteq M$ such that $f(L') = L$. Since $M \ll_s^S M$ and by Theorem 3.2, there exists $s \in S$ such that $Ms \ll L'$. Thus $Ns = f(M)s = f(Ms) \ll f(L') = L$. By Theorem 3.2, we have $N \ll_s^S N$.

($\Longleftarrow$) The proof follows by the same argument as in the proof of ($\Longrightarrow$). □

Theorem 3.4. *Let M be a right R-module and S a multiplicatively closed subset of R. Then the following statements are equivalent:*

(1) $0 \ll_e^S M$.
(2) *For each submodule L of M, $L \ll_e^S M$.*
(3) *For each proper submodule L of M, $L \ll_e^S M$.*
(4) *For each submodule L of M with complement L', $L' \ll_e^S M$.*
(5) *For each submodule L of M, there exists $s \in S$ such that $Ls = 0$.*
(6) $Ann_R(M) \cap S \neq \emptyset$.
(7) *There exists $s \in S$ such that $Ls = 0$ for all submodules L of M.*
(8) $M \ll_s^S M$.
(9) *For each submodule L of M, $L \ll_s^S M$.*
(10) *For each nonzero submodule L of M, $L \ll_s^S M$.*
(11) *For each submodule L of M, there exists $s \in S$ such that $Ms \ll L$.*

Proof. This follows from Theorem 3.2 and [[7], Theorem 2.14]. □

Corollary 3.5. *Let M and N be right R-modules, S a multiplicatively closed subset of R, and $M \cong N$. Then $0 \ll_e^S M$ if and only if $0 \ll_e^S N$.*

Proof. ($\Longrightarrow$) Assume that $0 \ll_e^S M$. By Theorem 3.4, there exists $s \in S$ such that $Ms = 0$. Since $M \cong N$, there exists an isomorphism $f : M \to N$. Then $Ns = f(M)s = f(Ms) = f(0) = 0$. Hence, by Theorem 3.4, $0 \ll_e^S N$.

($\Longleftarrow$) The proof follows by the same argument as in the proof of ($\Longrightarrow$). □

Corollary 3.6. *Let M and N be right R-modules, S a multiplicatively closed subset of R, and $M \cong N$. Then the following statements are equivalent:*

(1) $0 \ll_e^S M$.
(2) $0 \ll_e^S N$.
(3) $N \ll_s^S N$.
(4) $M \ll_s^S M$.

Proof. This result follows from Corollary 3.3, Theorem 3.4, and Corollary 3.5. □

Proposition 3.7. *Let M be a right R-module, S a multiplicatively closed subset of R, and for each $n \in \mathbb{N}$, let A_i be a submodule of M for each $i \in \{1, 2, \ldots, n\}$. Then $A_i \ll_s^S M$ for all $i \in \{1, 2, \ldots, n\}$ if and only if $\sum_{i=1}^{n} A_i \ll_s^S M$.*

Proof. ($\Longrightarrow$) Suppose that $A_i \ll_s^S M$ for all $i \in \{1, 2, \ldots, n\}$. We proceed by induction on n. For $n = 1$, the statement is true by assumption. Suppose that $\sum_{i=1}^{n-1} A_i \ll_s^S M$. Let L be a submodule of M and $s_1 \in S$ such that $Ms_1 \ll (\sum_{i=1}^{n} A_i) + L$. Since $Ms_1 \ll (\sum_{i=1}^{n} A_i) + L =$

$(\sum_{i=1}^{n-1} A_i) + A_n + L$ and $\sum_{i=1}^{n-1} A_i \ll_s^S M$, there exists $s_2 \in S$ such that $Ms_2 \ll_s^S A_n + L$. Since $A_n \ll_s^S M$, there exists $s_3 \in S$ such that $Ms_3 \ll L$. Therefore $\sum_{i=1}^{n} A_i \ll_s^S M$.

($\Longleftarrow$) Assume that $\sum_{j=1}^{n} A_j \ll_s^S M$. Let $1 \leq i \leq n$, $E \ll M$, and $s \in S$ such that $Ms \ll A_i + E$. Then $Ms \ll A_i + E \ll \sum_{j=1}^{n} A_j + E$. Since $\sum_{j=1}^{n} A_j \ll_s^S M$, there exists $t \in S$ such that $Mt \ll E$. Thus $A_i \ll_s^S M$. Hence, we have $A_i \ll_s^S M$ for all $i \in \{1, 2, \ldots, n\}$. □

Proposition 3.8. *Let M, N be right R-modules and A a submodule of M. If $A \ll_s^S M$, then $\varphi(A) \ll_s^S \varphi(M)$ for all $\varphi \in Hom_R(M, N)$.*

Proof. Suppose that $A \ll_s^S M$. Let $\varphi \in \mathrm{Hom}_R(M, N)$, $L \ll N$, and $s_1 \in S$ such that $\varphi(M)s_1 \ll \varphi(A) + L$. Let $m \in M$. Since $\varphi(ms_1) = \varphi(m)s_1 \in \varphi(M)s_1 \ll \varphi(A) + L$, there exist $a \in A$ and $l \in L$ such that $\varphi(ms_1) = \varphi(a) + l$. Then $ms_1 - a \in \varphi^{-1}(L)$, so $ms_1 \in A + \varphi^{-1}(L)$. Thus $Ms_1 \ll A + \varphi^{-1}(L)$. Since $A \ll_s^S M$, there exists $s_2 \in S$ such that $Ms_2 \ll \varphi^{-1}(L)$, and hence $\varphi(M)s_2 \ll L$. Therefore $\varphi(A) \ll_s^S \varphi(M)$. □

Corollary 3.9. *Let M_i be right R-modules for all $i \in \lambda$, $K_i \ll M_i$ for all $i \in \lambda$, and S a multiplicatively closed subset of R. Let $M = \bigoplus_{i\in\lambda} M_i$ and $K = \bigoplus_{i\in\lambda} K_i$. If $K \ll_s^S M$, then $K_i \ll_s^S M_i$ for all $i \in \lambda$.*

Proof. Assume that $K \ll_s^S M$, and let $\pi_i : M \to M_i$ be the projection maps. By Proposition 3.8, $K_i = \pi_i(K) \ll_s^S \pi_i(M) = M_i$ for all $i \in \lambda$. Hence $K_i \ll_s^S M_i$ for all i. □

4. S-F-small submodules

In this section, we define an S-F-small submodule, where F is a proper submodule of a right R-module M and S is a multiplicatively closed subset of R. An S-F-small submodule generalizes both S-small submodules and F-small submodules.

Definition 4.1. Let M be a right R-module, S a multiplicatively closed subset of R, and F a proper submodule of M. A submodule K of M is said to be an S-F-*small* submodule, denoted by $K \ll_{s,F}^S M$, if for each submodule L of M such that F is a submodule of L and $s \in S$ with $Ms \ll K + L$ implies that there exists $t \in S$ such that $Mt \ll L$.

Proposition 4.2. *Let M be a right R-module. Then the zero submodule is S-F-small for every multiplicatively closed subset S and every proper submodule F.*

Lemma 4.3. *Let M be a right R-module, F a proper submodule of M, S a multiplicatively closed subset of R. Let G, K be proper submodules of M such that $G \ll K$. If $N \ll_{s,G}^S M$, then $N \ll_{s,K}^S M$.*

Proof. Assume that $N \ll_{s,G}^{S} M$. Let $K \ll L \ll M$ and $s \in S$ such that $Ms \ll N + L$. Since $G \ll K \ll L \ll M$ and $N \ll_{s,G}^{S} M$, there exists $t \in S$ such that $Mt \ll L$. Hence, $N \ll_{s,K}^{S} M$. □

Proposition 4.4. *Let M be a right R-module, S a multiplicatively closed subset of R, and $K \ll M$. The following statements are equivalent:*

(1) *For each proper submodule F of M, $K \ll_{s,F}^{S} M$.*
(2) *$K \ll_{s}^{S} M$.*
(3) *$K \ll_{s,0}^{S} M$.*

Proof. (2)$\Longleftrightarrow$(3): It is obvious.

(1)$\Longrightarrow$(2): Choose $F = 0$. Then we have $K \ll_{s,0}^{S} M$. By (3)$\Longrightarrow$(2), we are done.

(2)$\Longrightarrow$(1): By (2)$\Longrightarrow$(3), we have $K \ll_{s,0}^{S} M$. By Lemma 4.3, for each proper submodule F of M, $K \ll_{s,F}^{S} M$. □

From Proposition 4.4, by setting $F = 0$, the properties of an S-small submodule follow from those of an S-F-small submodule, showing that S-F-small submodules generalize S-small submodules. Similarly, when $S = \{1\}$, S-F-small submodules reduce to F-small submodules, further confirming the generalization. In particular, choosing $F = 0$ and $S = \{1\}$ recovers the notion of a small submodule.

Example 4.5. Consider $\mathbb{Z}_6$ as a right $\mathbb{Z}_6$-module. Let $F = \langle\overline{3}\rangle$ and $S = \{\overline{1}, \overline{3}\}$. It is clear that S is multiplicatively closed in $\mathbb{Z}_6$. Then $\langle\overline{3}\rangle$ is an S-$\langle\overline{3}\rangle$-small submodule of $\mathbb{Z}_6$.

Proof. Let L be a submodule of $\mathbb{Z}_6$ such that $F \ll L$, and $s \in S$ with $\mathbb{Z}_6 s \ll \langle\overline{3}\rangle + L$. In this case, one may take $t = \overline{3}$. Then $\mathbb{Z}_6 t \ll L$. □

Example 4.6. Consider $\mathbb{Z}_6$ as a right $\mathbb{Z}_6$-module. Let $F = \langle\overline{3}\rangle$ and $S = \{\overline{1}, \overline{2}, \overline{4}\}$. It is clear that S is multiplicatively closed in $\mathbb{Z}_6$. Then $\langle\overline{2}\rangle$ is an S-$\langle\overline{2}\rangle$-small submodule of $\mathbb{Z}_6$.

Proof. Let L be a submodule of $\mathbb{Z}_6$ such that $F \ll L$, and $s \in S$ with $\mathbb{Z}_6 s \ll \langle\overline{2}\rangle + L$. In this case, one may take $t = \overline{2}$. Then $\mathbb{Z}_6 t \ll L$. □

Proposition 4.7. *Let M be a right R-module, S a multiplicatively closed subset of R, and F a proper submodule of M. If $S \subseteq U(R)$, then every S-F-small submodule of M is an F-small submodule of M.*

Proof. Suppose that $S \subseteq U(R)$. Let K be an S-F-small submodule of M, and let L be a submodule of M such that F is a submodule of L and $s \in S$ with $Ms \ll K + L$. Since K is an S-F-small submodule of M, there exists $t \in S$ such that $Mt \ll L$. Since $t \in S \subseteq U(R)$, we have L$\ll M = Mt \ll L$, and hence $M = L$. Therefore K is an F-small submodule of M. □

Let S be a multiplicatively closed subset of R. We give a characterization of S-F-small submodules. First, we introduce the notion of an S-F-small epimorphism and then use this concept to characterize an S-F-small submodule.

Definition 4.8. Let M, N be right R-modules, F a proper submodule of M, and S a multiplicatively closed subset of R. An epimorphism $f : M \to N$ is said to be an S-F-*small epimorphism* if $\mathrm{Ker}\,(f)$ is an S-F-small submodule of M.

Proposition 4.9. *Let M, N be right R-modules and S a multiplicatively closed subset of R. If $S \subseteq U(R)$, then every S-F-small epimorphism from M to N is an F-small epimorphism from M to N.*

Proof. It follows from Proposition 4.7. □

Theorem 4.10. *Let M, N be right R-modules, F a proper submodule of M, and S a multiplicatively closed subset of R. If an epimorphism $f : M \to N$ is an S-F-small epimorphism, then for each right R-module L and R-homomorphism $h : L \to M$ such that $F \ll Im(h)$ and $f \circ h$ is an epimorphism, there exists $t \in S$ such that $Mt \ll Im(h)$.*

Proof. Suppose that $f : M \to N$ is an S-F-small epimorphism. Let L be a right R-module and $h : L \to M$ an R-homomorphism such that $F \ll \mathrm{Im}\,(h)$ and $f \circ h$ is an epimorphism. Choose $s \in S$. Let $ms \in M$ where $m \in M$. Since $f(m) \in N$ and $f \circ h$ is an epimorphism, there exists $l \in L$ such that $(f \circ h)(l) = f(m)$. Then $f(ms - h(ls)) = f(ms) - (f \circ h)(ls) = 0$.But $ms = h(ls) + ms - h(ls)$, and since $h(ls) \in \mathrm{Im}\,(h)$, we have $ms - h(ls) \in \mathrm{Ker}\,(f)$. So $Ms \ll \mathrm{Im}\,(h) + \mathrm{Ker}\,(f)$. Since f is an S-F-small epimorphism, there exists $t \in S$ such that $Mt \ll \mathrm{Im}\,(h)$. □

Theorem 4.11. *Let M be a right R-module, F a proper submodule of M, S a multiplicatively closed subset of R, and K a submodule of M. Then the following statements are equivalent.*

(1) *K is an S-F-small submodule of M.*

(2) *The projection map $\pi_K : M \to M/K$ is an S-F-small epimorphism.*

(3) *For each right R-module N, $h \in Hom_R(N, M)$ such that $F \ll Im(h)$ and $s \in S$, if $Ms \ll Im(h) + K$, then there exists $t \in S$ such that $Mt \ll Im(h)$.*

Proof. (1) $\Longleftrightarrow$ (2): It is obvious.

(1) $\Longrightarrow$ (3): By assumption and since $\mathrm{Im}\,(h) \ll M$, we obtain (3).

(3) $\Longrightarrow$ (1): Let L be a submodule of M such that F is a submodule of L and $s \in S$ with $Ms \ll K + L$. Let i_L be the inclusion map from L to M. Since $\mathrm{Im}\,(i_L) = L$, we have $Ms \ll K + \mathrm{Im}\,(i_L)$. By (3), there exists $t \in S$ such that $Mt \ll \mathrm{Im}\,(i_L) = L$. Hence K is an S-F-small submodule of M. □

5. Some properties of S-F-small submodules

In this section, let M be a right R-module, F a proper submodule of M, and S a multiplicatively closed subset of R. We present some properties of S-F-small submodules of M. It is known that M is not an F-small submodule of any right R-module M. However, it is possible to find a right R-module M, a multiplicatively closed subset S of R, and a proper submodule F of M such that M is an S-F-small submodule of M.

Example 5.1. Consider $\mathbb{Z}_6$ as a right $\mathbb{Z}$-module, $S = \{1, 3, 3^2, 3^3, 3^4, \dots\}$ and $F = \langle\bar{3}\rangle$. It is clear that S is a multiplicatively closed subset of $\mathbb{Z}$. Then $\mathbb{Z}_6 \ll^S_{s,\langle\bar{3}\rangle} \mathbb{Z}_6$.

Proof. Since $\mathbb{Z}_6 \cdot 3 = \langle\bar{3}\rangle$, we have $\mathbb{Z}_6 \cdot 3 \ll L$ for every submodule L of $\mathbb{Z}_6$ containing F. □

We can now find some conditions on a right R-module M such that M is an S-F-small submodule of M.

Theorem 5.2. *Let M be a right R-module, S a multiplicatively closed subset of R, and F a proper submodule of M. Then the following statements are equivalent:*

(1) $M \ll^S_{s,F} M$.

(2) *For each submodule $L \ll M$, $L \ll^S_{s,F} M$.*

(3) *For each nonzero submodule $L \ll M$, $L \ll^S_{s,F} M$.*

(4) *For each submodule L of M such that $F \ll L \ll M$, there exists $s \in S$ such that $Ms \ll L$.*

(5) $\{t \in R : Mt \ll F\} \cap S \neq \emptyset$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $M \ll^S_{s,F} M$. Let L be a submodule of M, and let K be a submodule of M such that F is a submodule of K and $s \in S$ with $Ms \ll L + K$. Since $M1 \ll M + K$, F is a submodule of K, and $M \ll^S_{s,F} M$, there exists $t \in S$ such that $Mt \ll K$. Hence $L \ll^S_{s,F} M$.

(2) $\Longrightarrow$ (3): This is clear.

(3) $\Longrightarrow$ (4): Let L be a submodule of M such that $F \ll L \ll M$.

Case 1. $M \neq 0$. By assumption, we have $M \ll^S_{s,F} M$. Since $M1 \ll M + L$, there exists $s \in S$

such that $Ms \ll L$.

Case 2. $M = 0$. This is trivial.

(4) $\Longrightarrow$ (1): Let L be a submodule of M such that F is a submodule of L and $s \in S$ with $Ms \ll M + L$. By (4), there exists $t \in S$ such that $Mt \ll L$.

(5) $\Longrightarrow$ (1): Suppose that $\{t \in R : Mt \ll F\} \cap S \neq \emptyset$. Let L be a submodule of M such that F is a submodule of L and $s \in S$ with $Ms \ll M + L$. Choose $t \in \{t \in R : Mt \ll F\} \cap S$. Then $Mt \ll F$, and since $F \ll L$, we have $Mt \ll L$. Hence $M \ll^S_{s,F} M$.

(1) $\Longrightarrow$ (5): Suppose that $M \ll^S_{s,F} M$. Since $M1 \ll M + F$ and $M \ll^S_{s,F} M$, there exists $t \in S$ such that $Mt \ll F$. So $\{t \in R : Mt \ll F\} \cap S \neq \emptyset$. □

Corollary 5.3. *Let M, N be right R-modules, F a proper submodule of M, S a multiplicatively closed subset of R, and suppose $M \cong N$ via an isomorphism $f : M \to N$. Then $M \ll^S_{s,F} M$ if and only if $N \ll^S_{s,f(F)} N$.*

Proof. ($\Longrightarrow$) From Theorem 5.2, we have $\{t \in R : Mt \ll F\} \cap S \neq \emptyset$. Since there exists $t \in S$ such that $Mt \ll F$, it follows that $Nt = f(M)t = f(Mt) \ll f(F)$.

($\Longleftarrow$) The proof follows by the same argument as in the proof of ($\Longrightarrow$). □

Lemma 5.4. *Let M be a right R-module and S a multiplicatively closed subset of R. If $0 \ll^S_e M$, then for each proper submodule F of M, $M \ll^S_{s,F} M$, and for each nonzero submodule D of M, $0 \ll^S_{e,D} M$.*

Proof. Assume that $0 \ll^S_e M$. Let F be a proper submodule of M and D a nonzero submodule of M. By [8, Theorem 2.3], there exists $s \in S$ such that $Ms = 0$. Since $Ms = 0 \ll F$, it follows from Theorem 5.2 that $M \ll^S_{s,F} M$. Moreover, since $Ms = 0$, we have $Ds = 0$. By [7, Theorem 2.3], we can conclude that $0 \ll^S_{e,D} M$. □

Lemma 5.5. *Let M be a right R-module and S a multiplicatively closed subset of R. If there exists a nonzero submodule D of M such that $0 \ll^S_{e,D} M$ and a proper submodule F of M such that $F \ll D$ and $M \ll^S_{s,F} M$, then $0 \ll^S_e M$.*

Proof. Since there exist $t, s \in S$ such that $Mt \ll F$ and $Ds = 0$, and since $F \ll D$, we obtain $Mts \ll Fs \ll Ds = 0$. Thus $Mts = 0$, where $ts \in S$. It follows from [7, Theorem 2.3] that $0 \ll^S_e M$. □

Theorem 5.6. *Let M be a right R-module and S a multiplicatively closed subset of R. If there exists a proper submodule F of M and a nonzero submodule D of M such that $F \ll D \ll M$, then $0 \ll^S_{e,D} M$ and $M \ll^S_{s,F} M$ if and only if $0 \ll^S_e M$.*

Proof. This follows from Lemma 5.4 and Lemma 5.5. □

From Theorem 5.6, we can extend [7, Theorem 2.3] to Theorem 5.7 by using the concepts of S-D-essential submodules and S-F-small submodules.

Theorem 5.7. *Let M be a right R-module and S a multiplicatively closed subset of R. If there exists a proper submodule F of M and a nonzero submodule D of M such that $F \ll D \ll M$, then the following statements are equivalent:*

(1) $0 \ll_e^S M$.
(2) *For each submodule L of M, $L \ll_e^S M$.*
(3) *For each proper submodule L of M, $L \ll_e^S M$.*
(4) *For each submodule L of M with complement L', $L' \ll_e^S M$.*
(5) *For each submodule L of M, there exists $s \in S$ such that $Ls = 0$.*
(6) $Ann_R(M) \cap S \neq \emptyset$.
(7) *There exists $s \in S$ such that for each submodule L of D, $Ls = 0$.*
(8) $M \ll_s^S M$.
(9) *For each submodule L of M, $L \ll_s^S M$.*
(10) *For each nonzero submodule L of M, $L \ll_s^S M$.*
(11) *For each submodule L of M, there exists $s \in S$ such that $Ms \ll L$.*
(12) $0 \ll_{e,D}^S M$ *and* $M \ll_{s,F}^S M$.

Proof. This follows from [7, Theorem 2.3] and Theorem 5.6. □

Proposition 5.8. *Let M be a right R-module, F a proper submodule of M, and S a multiplicatively closed subset of R. Then $K \ll_{s,F}^S M$ for any submodule K of F.*

Proof. Let L be a submodule of M such that $F \ll L \ll M$ and $s \in S$ with $Ms \ll K + L$. Since $K \ll F \ll L$, it follows that $Ms \ll L$. □

Proposition 5.9. *Let M be a right R-module, F a proper submodule of M, S a multiplicatively closed subset of R, and K a submodule of M. Then the following statements are equivalent:*

(1) $K \ll_{s,F}^S M$.
(2) $(K + F)/F \ll_s^S M/F$.

Proof. (1) $\Longrightarrow$ (2): Assume that $K \ll_{s,F}^S M$. Let $L/F \ll M/F$ where $F \ll L \ll M$ and $s \in S$ such that $(M/F)s \ll (K+F)/F+L/F$. Then $(M/F)s \ll (K+F)/F+L/F \ll (K+F+L)/F$. Thus $Ms \ll K + F + L$. Since $K \ll_{s,F}^S M$ and $F \ll F + L$, there exists $t \in S$ such that $Mt \ll F + L$. Hence $(M/F)t \ll (F + L)/F$.

(2) $\Longrightarrow$ (1): Assume that $(K+F)/F \ll_s^S M/F$. Let $F \ll L \ll M$ and $s \in S$ such that $Ms \ll K+L$. Then $(M/F)s \ll (K+F)/F + L/F$. By assumption, there exists $t \in S$ such that $(M/F)t \ll L/F$. Thus $Mt \ll L$. □

Proposition 5.10. *Let M, N be right R-modules, F a proper submodule of M, φ an R-homomorphism from M to N such that $\varphi(F)$ is a proper submodule of N, and A a submodule of M. If $A \ll_{s,F}^S M$, then $\varphi(A) \ll_{s,\varphi(F)}^S \varphi(M)$.*

Proof. The proof is similar to that of Proposition 3.8, using the fact that $F \ll \varphi^{-1}(\varphi(F)) \ll \varphi^{-1}(L) \ll \varphi^{-1}(N) \ll M$, where $\varphi(F) \ll L \ll N$. □

Theorem 5.11. *Let M be a right R-module, F a proper submodule of M, K and N submodules of M such that $K \ll N$ and $K+F$ is a proper submodule of M, and let S be a multiplicatively closed subset of R. Then $N \ll_{s,F}^S M$ if and only if $K \ll_s^S M$ and $N/K \ll_{(F+K)/K}^S M/K$.*

Proof. ($\Longrightarrow$) Suppose that $N \ll_{s,F}^S M$.

First, we show that $(F+K)/K$ is a proper submodule of M/K. Let $a \in M \setminus (K+F)$. Then $a+K \in M/K$. Suppose, to the contrary, that $a+K \in (F+K)/K$. Then $a+K = f+k+K = f+K$ for some $f \in F$, $k \in K$, so $a-f \in K$. Then $a = a-f+f \in K+F$. But $a \notin K+F$. So $a+K \in M/K \setminus (F+K)/K$. Therefore, $(F+K)/K$ is a proper submodule of M/K.

Second, we show that $K \ll_{s,F}^S M$. Let $F \ll L \ll M$ and $s \in S$ such that $Ms \ll K+L$. Since $K \ll N$, we have $Ms \ll K+L \ll N+L$, and hence $Ms \ll N+L$. Since $N \ll_{s,F}^S M$ and $Ms \ll N+L$, there exists $t \in S$ such that $Mt \ll L$. Therefore, $K \ll_{s,F}^S M$.

Third, we show that $(M/K)t \ll E/K$. Let $(F+K)/K \ll E/K \ll M/K$, where $K \ll E \ll M$, and $s \in S$ such that $(M/K)s \ll N/K + E/K$. Then $Ms \ll E+N$. Let $f \in F$. Since $(F+K)/K \ll E/K$ and $f \in F$, there exists $e \in E$ such that $f+K = e+K$. Hence $f-e \in K \ll E$, and we conclude that $f = (f-e)+e \in E$. Therefore, $F \ll E$. Since $Ms \ll E+N$, $F \ll E$, and $N \ll_{s,F}^S M$, there exists $t \in S$ such that $Mt \ll E$. Hence, $(M/K)t \ll E/K$.

($\Longleftarrow$) Assume that $K \ll_{s,F}^S M$ and $N/K \ll_{s,(F+K)/K}^S M/K$. Let $F \ll L \ll M$ and $s \in S$ such that $Ms \ll N+L$. Since $F \ll L$, we have $(F+K)/K \ll (L+K)/K$. Also, since $Ms \ll N+L$, it follows that $(M/K)s \ll N/K + (K+L)/K$. Since $N/K \ll_{(F+K)/K}^S M/K$ and $(M/K)s \ll N/K + (K+L)/K$, there exists $t \in S$ such that $(M/K)t \ll (K+L)/K$. Thus, $Mt \ll K+L$. Since $K \ll_{s,F}^S M$ and $F \ll L \ll M$, there exists $l \in S$ such that $Ml \ll L$. Hence, $N \ll_{s,F}^S M$. □

Theorem 5.12. *Let M be a right R-module, S a multiplicatively closed subset of R, and for each $n \in \mathbb{N}$, let A_i be submodules of M for all $i \in \{1, 2, 3, \ldots, n\}$. Then $A_i \ll^S_{s,F} M$ for all $i \in \{1, 2, 3, \ldots, n\}$ if and only if $\sum_{i=1}^n A_i \ll^S_{s,F} M$.*

Proof. ($\Longrightarrow$) Assume that $A_i \ll^S_{s,F} M$ for all $i \in \{1, 2, 3, \ldots, n\}$. We proceed by induction on n. For $n = 1$, this is trivial. Suppose that $\sum_{i=1}^{n-1} A_i \ll^S_{s,F} M$, and let $L \ll M$ with F is a submodule of L and $s \in S$ such that $Ms \ll \sum_{i=1}^n A_i + L$. Since $Ms \ll \sum_{i=1}^n A_i + L = \sum_{i=1}^{n-1} A_i + A_n + L$ and $\sum_{i=1}^{n-1} A_i \ll^S_{s,F} M$, there exists $h \in S$ such that $Mh \ll A_n + L$. Since $A_n \ll^S_{s,F} M$, there exists $t \in S$ such that $Mt \ll L$. Hence, by induction, $\sum_{i=1}^n A_i \ll^S_{s,F} M$.

($\Longleftarrow$) Suppose that $\sum_{i=1}^n A_i \ll^S_{s,F} M$. Let $j \in \{1, 2, \ldots, n\}$ and $L \ll M$ such that F is a submodule of L and $s \in S$ with $Ms \ll A_j + L$. Then $Ms \ll A_j + L \ll \sum_{i=1}^n A_i + L$. Since $\sum_{i=1}^n A_i \ll^S_{s,F} M$, there exists $t \in S$ such that $Mt \ll L$. Hence $A_j \ll^S_{s,F} M$. □

Proposition 5.13. *Let M be a right R-module, S a multiplicatively closed subset of R, K a submodule of M, and F a proper submodule of M such that $F \ll^S_s M$. Then $K \ll^S_{s,F} M$ if and only if $K \ll^S_s M$.*

Proof. ($\Longrightarrow$) Assume that $K \ll^S_{s,F} M$. Let $L \ll M$ and $s \in S$ such that $Ms \ll K + L \ll K + L + F$. Since $K \ll^S_{s,F} M$ and $F \ll L + F \ll M$, there exists $t \in S$ such that $Mt \ll L + F$. As $F \ll^S_s M$, there exists $l \in S$ such that $Ml \ll L$.

($\Longleftarrow$) This follows directly from Proposition 4.4. □

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Poramate Sangchan

Department of Mathematics and Computer Science,

Faculty of Science,

Chulalongkorn University,

Bangkok 10330, Thailand.

poramate345@gmail.com

Natthaphong Chitmung

Department of Mathematics and Computer Science,

Faculty of Science,

Chulalongkorn University,

Bangkok 10330, Thailand.

ying_e-com@hotmail.com

Samruam Baupradist

Department of Mathematics and Computer Science,

Faculty of Science,

Chulalongkorn University,

Bangkok 10330, Thailand.

samruam.b@chula.ac.th