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Research Paper

THE GENERALIZED ADJACENCY SPECTRA OF TRIPLE VERTEX AND TRIPLE EDGE JOIN OF GRAPHS

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Abstract. Consider three graphs G_1, G_2, and H with orders n_1, n_2, n_3, and sizes m_1, m_2, m_3, respectively, where the sets of vertices for all three graphs are disjoint. Let $S(G_1, H)$ be the graph obtained from G_1 and H in the following way:

(1) Delete all the edges of G_1 and consider m_1 disjoint copies of H.

(2) Join each vertex of the i^{th} copy of H to the end vertices of the i^{th} edge of G_1.

Using this construction, the graph $G_1(\vee_H)G_2$ is defined to be the graph obtained from $S(G_1, H)$ by joining every vertex of G_1 to every vertex of G_2 (see [19]). In a similar manner, we define $G_1(\Lambda_H)G_2$ to be the graph formed by connecting every vertex in each of the m_1 inserted copies of H in $S(G_1, H)$ to all vertices of G_2. We refer to $G_1(\vee_H)G_2$ and $G_1(\Lambda_H)G_2$ as the *triple vertex join* and *triple edge join* of G_1 and G_2 with respect to H, respectively. In this article, we determine the generalized adjacency spectra (also known as the A_α-spectra) of $G_1(\vee_H)G_2$ when G_1 is regular, and of $G_1(\Lambda_H)G_2$ when both G_1 and H are regular. Furthermore, we present an application demonstrating the construction of infinitely many A_α-cospectral graphs.

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1. Introduction

Throughout this paper, all graphs are assumed to be finite, simple, and undirected. Let G be a graph with vertex set $V(G) = \{v_i \mid i = 1, 2, \ldots, n\}$ and edge set $E(G) = \{e_j \mid j = 1, 2, \ldots, m\}$. The *adjacency matrix* of G, denoted by $A(G) = [a_{ij}]$, is a symmetric matrix of order n, where rows and columns are indexed by the vertices of G. The entries of $A(G)$ are defined as

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G; \\ 0, & \text{otherwise.} \end{cases}$$

Let $d_G(v)$ denote the degree of a vertex v in a graph G, and let $Deg(G)$ be the $n \times n$ diagonal matrix whose i-th diagonal entry is $d_G(v_i)$. The *Laplacian matrix* of G is defined as $L(G) = Deg(G) - A(G)$, and the *signless Laplacian matrix* is defined as $Q(G) = Deg(G) + A(G)$. Extensive studies have been conducted on both the adjacency matrix and the signless Laplacian matrix, revealing numerous spectral similarities. Motivated by these parallels, Nikiforov [18] introduced a unifying matrix known as the *generalized adjacency matrix* or A_α*-matrix*, defined for $\alpha \in [0, 1]$ as

$$A_\alpha(G) = \alpha\, Deg(G) + (1 - \alpha)\, A(G).$$

This definition interpolates between the adjacency matrix and the degree matrix: specifically, $A_0(G) = A(G)$, $A_1(G) = Deg(G)$, and $A_{1/2}(G) = \frac{1}{2}Q(G)$.

Let P be a real matrix of order n. Denote by P^T and $\det(P)$ the transpose and determinant of P, respectively. The *characteristic polynomial* of P is defined by

$$\Psi(P; x) = \det(xI_n - P),$$

where I_n is the $n \times n$ identity matrix. The eigenvalues of P are precisely the roots of $\Psi(P; x)$. For a symmetric matrix P, it is well known that all eigenvalues are real, and that their algebraic multiplicities coincide with their geometric multiplicities.

The multiset of eigenvalues of P is referred to as its *spectrum*, denoted by $\sigma(P) = \{\lambda_1^{p_1}, \lambda_2^{p_2}, \cdots, \lambda_k^{p_k}\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of P, and $p_1, p_2, \ldots, p_k$ are their respective algebraic multiplicities.

Let $\lambda_1(A(G)) \leq \lambda_2(A(G)) \leq \cdots \leq \lambda_n(A(G))$ and $\lambda_1(A_\alpha(G)) \leq \lambda_2(A_\alpha(G)) \leq \cdots \leq \lambda_n(A_\alpha(G))$ denote the eigenvalues of the adjacency matrix $A(G)$ and the A_α-matrix of a graph G, respectively. Throughout this paper, we use this ordering convention for the eigenvalues of any matrix. The spectra of the matrices $A(G)$, $L(G)$, $Q(G)$, and $A_\alpha(G)$ are referred to as the *adjacency spectrum*, *Laplacian spectrum*, *signless Laplacian spectrum*, and A_α*-spectrum* of G, respectively. Two graphs are said to be *adjacency cospectral* (respectively, A_α*-cospectral*) if they have identical adjacency spectra (respectively, identical A_α-spectra). A graph is said to be an A_α*-DS graph* if it is uniquely determined by its A_α-spectrum, i.e., there does not exist any graph non-isomorphic to it that is A_α-cospectral with it. It is known that every graph with at most 8 vertices is an A_α-DS graph, while the smallest order for which a pair of non-isomorphic A_α-cospectral graphs exists is 9. For further results concerning A_α-cospectral graphs, we refer the reader to [13, 14].

Graph operations provide systematic ways of constructing larger and more complex graphs from simpler ones while preserving or modifying important structural and spectral characteristics. A variety of such operations have been studied extensively in the literature. In particular, the spectra and characteristic polynomials associated with graph operations such as the Cartesian product, disjoint union, corona, edge corona, neighborhood corona, Kronecker product, join, subdivision-vertex join, subdivision-edge join, R-vertex join, and R-edge join have been analyzed in detail; see, for example, [2, 5–11, 16, 17, 22]. Furthermore, the A_α-spectra corresponding to some of these graph operations have been investigated in [3, 12, 13, 20]. Such graph constructions have found wide applications in communication and interconnection networks, where large and complex network topologies are formed by systematically combining smaller subnetworks. They are also fundamental in chemical graph theory, where graph operations are used to model molecular structures and to investigate their physicochemical properties through spectral descriptors. A brief account of these applications can be found in [4]. Motivated by these applications, we consider two new graph operations, namely the *triple vertex join* and the *triple edge join*. These operations provide a unified and flexible framework for constructing new families of graphs with rich spectral properties, thereby enriching the study of graph spectra and offering potential applications in network topology, communication networks, and mathematical chemistry.

The following definition can be found in [19].

Definition 1.1. [19] Let G_1, G_2 and H be three graphs on disjoint sets of vertices and G_1 has m_1 edges. Let $S(G_1, H)$ be the graph obtained from G_1 and H in the following way:

(1) Delete all the edges of G_1 and consider m_1 disjoint copies of H.
(2) Join each vertex of the i^{th} copy of H to the end vertices of the i^{th} edge of G_1.

Then $G_1(\vee_H)G_2$ is the graph obtained from $S(G_1, H)$ by joining each vertex of G_1 with each vertex of G_2.

Let G_1, G_2, and H be graphs of orders n_1, n_2, and n_3, and sizes m_1, m_2, and m_3, respectively. Then the graph $G_1(\vee_H)G_2$ has order $n_1 + n_2 + m_1n_3$ and size $2m_1n_3 + m_1m_3 + m_2 + n_1n_2$. Define $G_1(\Lambda_H)G_2$ as the graph formed from $S(G_1, H)$ by joining each vertex in all m_1 inserted copies of H to every vertex of G_2. We refer to $G_1(\vee_H)G_2$ and $G_1(\Lambda_H)G_2$ as the *triple vertex join* and *triple edge join* of graphs G_1 and G_2 with respect to H, respectively. In [19], Paul derived the adjacency, Laplacian, and signless Laplacian spectra of $G_1(\vee_H)G_2$ in terms of the spectra of G_1, G_2, and H, under the assumption that G_1 is regular. In [24], the authors derived the $\{1\}$-inverse of the Laplacian matrix associated with $G_1(\vee_H)G_2$. Utilizing the relationship between resistance distances and $\{1\}$-inverses of Laplacian matrices, they obtained explicit expressions for the resistance distances in $G_1(\vee_H)G_2$.

The remainder of this paper is organized as follows. In Section 2, we present the necessary preliminaries and auxiliary results. In Section 3, we extend the results of [19] by determining the A_α-spectrum of $G_1(\vee_H)G_2$ when G_1 is regular, and that of $G_1(\Lambda_H)G_2$ when both G_1 and H are regular. As an application, we construct infinitely many pairs of A_α-cospectral graphs. Finally, in Section 4, we provide two examples to illustrate the theoretical results.

2. Preliminaries

Let G be a graph with n vertices and m edges. The *incidence matrix* of G, denoted by $R(G)$, is the $n \times m$ matrix whose rows and columns are indexed by the vertex set and edge set of G, respectively. The (i,j)-th entry of $R(G)$ is defined as

$$R(G)_{ij} = \begin{cases} 1, & \text{if the } i\text{-th vertex is incident to the } j\text{-th edge;} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\ell(G)$ denote the *line graph* of G, where each vertex of $\ell(G)$ corresponds to an edge of G, and two vertices in $\ell(G)$ are adjacent if and only if the corresponding edges in G share a common vertex. It is well known (see [7]) that

$$R(G)^T R(G) = A(\ell(G)) + 2I_m, \tag{1}$$

and, if G is r-regular, then

$$R(G)R(G)^T = A(G) + rI_n. \tag{2}$$

Let $\mathbf{0}_n$ and $\mathbf{1}_n$ denote the column vectors of size n consisting of all zeros and all ones, respectively, and let $\mathbf{0}_{m\times n}$ and $J_{m\times n}$ denote the $m \times n$ zero matrix and all-one matrix, respectively.

The *P-coronal* of an $n \times n$ matrix P, denoted by $\Gamma_P(x)$, is defined (in [6]) as

$$\Gamma_P(x) = \mathbf{1}_n^T (xI_n - P)^{-1} \mathbf{1}_n. \tag{3}$$

If every row sum of P is equal to a constant t, then it follows (see [6]) that

$$\Gamma_P(x) = \frac{n}{x-t}. \tag{4}$$

The *Kronecker product* of the matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is denoted by $A \otimes B$ (following the notation adopted in [1]), and defined as the block matrix of order $mp \times nq$, given by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

The Kronecker product is an associative operation. Moreover, if the products AC and BD are defined, then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

In particular, if A and B are nonsingular matrices, then their Kronecker product is also nonsingular, and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

If A and B are square matrices of orders n and p, respectively, then the determinant of their Kronecker product satisfies

$$\det(A \otimes B) = (\det A)^p (\det B)^n.$$

Furthermore, if $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ are the eigenvalues of A, and $\lambda_1(B), \lambda_2(B), \ldots, \lambda_p(B)$ are the eigenvalues of B, then the eigenvalues of $A \otimes B$ are given by the set $\{\lambda_i(A)\lambda_j(B) \mid 1 \leq i \leq n,\ 1 \leq j \leq p\}$.

Lemma 2.1. ([23]) *Let P_1, P_2, P_3, and P_4 be matrices of sizes $x \times x$, $x \times y$, $y \times x$, and $y \times y$, respectively. Then*

$$\begin{aligned} \det \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} &= \det(P_4) \cdot \det\left(P_1 - P_2 P_4^{-1} P_3\right), \quad \textit{when } P_4 \textit{ is invertible;} \\ &= \det(P_1) \cdot \det\left(P_4 - P_3 P_1^{-1} P_2\right), \quad \textit{when } P_1 \textit{ is invertible.} \end{aligned}$$

The matrices $P_1 - P_2P_4^{-1}P_3$ and $P_4 - P_3P_1^{-1}P_2$ are called the Schur complements *of P_4 and P_1, respectively.*

Lemma 2.2 ([15])**.** *Let A be an $n \times n$ real matrix and $c \in \mathbb{R}$. Then,*

$$\det(xI_n - A - cJ_{n\times n}) = (1 - c\,\Gamma_A(x)) \cdot \det(xI_n - A),$$

where $\Gamma_A(x)$ is the A-coronal.

Lemma 2.3 ([18])**.** *The A_α-spectrum of the complete bipartite graph $K_{r,s}$ is given by*

$$\begin{aligned} \sigma(A_\alpha(K_{r,s})) \quad = \quad & \left\{ \frac{\alpha(r+s) + \sqrt{\alpha^2(r+s)^2 + 4rs(1-2\alpha)}}{2}^{1}, \alpha r^{s-1}, \alpha s^{r-1}, \right. \\ & \left. \frac{\alpha(r+s) - \sqrt{\alpha^2(r+s)^2 + 4rs(1-2\alpha)}}{2}^{1} \right\}. \end{aligned}$$

Lemma 2.4 ([20]). *For the complete bipartite graph $K_{r,s}$, the A_α-coronal is given by*

$$\Gamma_{A_\alpha(K_{r,s})}(x) = \frac{(r+s)x - \alpha(r+s)^2 + 2rs}{x^2 - \alpha(r+s)x + (2\alpha-1)rs}.$$

Lemma 2.5 ([7]). *Let G be an r-regular graph with n vertices, and let $\ell(G)$ denote its line graph. Suppose that the characteristic polynomials of $A(G)$ and $A(\ell(G))$ are given by $\Psi(A(G);x)$ and $\Psi(A(\ell(G));x)$, respectively. Then the following relation holds*

$$\Psi(A(\ell(G));x) = (x+2)^{m-n}\,\Psi(A(G);x-r+2).$$

3. Main result

3.1. **A_α-spectrum of $G_1(\vee_H)G_2$, when G_1 is regular.** In this section, we determine the A_α-spectrum of the triple vertex join graph $G_1(\vee_H)G_2$ formed from two graphs G_1 and G_2. The following theorem provides the A_α-characteristic polynomial of $G_1(\vee_H)G_2$ when G_1 is a regular graph.

Theorem 3.1. *Let G_1, G_2, and H be three graphs of orders n_1, n_2, and n_3, respectively. Suppose that G_1 is r_1-regular with m_1 edges. Then the A_α-characteristic polynomial of the triple vertex join graph $G_1(\vee_H)G_2$ is given by*

$$\begin{aligned}\Psi(A_\alpha(G_1(\vee_H)G_2);x) = {} & \Psi(A_\alpha(G_2);x-\alpha n_1)\cdot\Psi(A_\alpha(H);x-2\alpha)^{m_1}\\ & \times\big[(x-\alpha r_1 n_3-\alpha n_2) - 2r_1(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)\\ & -n_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)\big]\times\prod_{i=1}^{n_1-1}\big[(x-\alpha r_1 n_3-\alpha n_2)\\ & -(1-\alpha)\{(1-2\alpha)r_1+\lambda_i(A_\alpha(G_1))\}\Gamma_{A_\alpha(H)}(x-2\alpha)\big].\end{aligned}$$

Proof. Let R_1 be the incidence matrix of G_1. Suppose the vertex sets of G_1, the m_1 copies of H, and G_2 are given by

$$\{v_1,\ldots,v_{n_1}\},\quad \{u_{11},\ldots,u_{1n_3},\ldots,u_{m_11},\ldots,u_{m_1n_3}\},\quad\text{and}\quad\{w_1,\ldots,w_{n_2}\},$$

respectively. Then the adjacency matrix of $G_1(\vee_H)G_2$ is

$$A(G_1(\vee_H)G_2) = \left[\begin{array}{c|c|c} \mathbf{0}_{n_1\times n_1} & R_1\otimes\mathbf{1}_{n_3}^T & J_{n_1\times n_2}\\ \hline R_1^T\otimes\mathbf{1}_{n_3} & I_{m_1}\otimes A(H) & \mathbf{0}_{m_1n_3\times n_2}\\ \hline J_{n_2\times n_1} & \mathbf{0}_{n_2\times m_1n_3} & A(G_2)\end{array}\right].$$

From the construction of $G_1(\vee_H)G_2$, the degrees of its vertices are computed as follows

$$\begin{aligned} d_{G_1(\vee_H)G_2}(v_i) &= r_1n_3+n_2, \quad \text{for } i=1,2,\ldots,n_1;\\ d_{G_1(\vee_H)G_2}(u_{jk}) &= d_H(u_{jk})+2, \quad \text{for } j=1,2,\ldots,m_1,\ k=1,2,\ldots,n_3;\\ d_{G_1(\vee_H)G_2}(w_\ell) &= d_{G_2}(w_\ell)+n_1, \quad \text{for } \ell=1,2,\ldots,n_2. \end{aligned}$$

Thus, the diagonal degree matrix of $G_1(\vee_H)G_2$, of order $(n_1+m_1n_3+n_2)\times(n_1+m_1n_3+n_2)$, is given by

$$Deg(G_1(\vee_H)G_2)=\left[\begin{array}{c|c|c} (r_1n_3+n_2)I_{n_1} & \mathbf{0}_{n_1\times m_1n_3} & \mathbf{0}_{n_1\times n_2}\\ \hline \mathbf{0}_{m_1n_3\times n_1} & I_{m_1}\otimes(Deg(H)+2I_{n_3}) & \mathbf{0}_{m_1n_3\times n_2}\\ \hline \mathbf{0}_{n_2\times n_1} & \mathbf{0}_{n_2\times m_1n_3} & Deg(G_2)+n_1I_{n_2} \end{array}\right],$$

and

$$A_\alpha(G_1(\vee_H)G_2)=\left[\begin{array}{c|c|c} \alpha(r_1n_3+n_2)I_{n_1} & (1-\alpha)R_1\otimes\mathbf{1}_{n_3}^T & (1-\alpha)J_{n_1\times n_2}\\ \hline (1-\alpha)R_1^T\otimes\mathbf{1}_{n_3} & \begin{array}{c}\alpha[I_{m_1}\otimes(Deg(H)+2I_{n_3})]\\ +(1-\alpha)[I_{m_1}\otimes A(H)]\end{array} & \mathbf{0}_{m_1n_3\times n_2}\\ \hline (1-\alpha)J_{n_2\times n_1} & \mathbf{0}_{n_2\times m_1n_3} & A_\alpha(G_2)+\alpha n_1I_{n_2} \end{array}\right].$$

Therefore, the characteristic polynomial of $A_\alpha(G_1(\vee_H)G_2)$ is given by

$$\begin{aligned} \Psi(A_\alpha(G_1(\vee_H)G_2);x) &= \det[xI_{n_1+m_1n_3+n_2}-A_\alpha(G_1(\vee_H)G_2)]\\ &= \det\left[\begin{array}{c|c|c} (x-\alpha r_1n_3-\alpha n_2)I_{n_1} & -(1-\alpha)R_1\otimes\mathbf{1}_{n_3}^T & -(1-\alpha)J_{n_1\times n_2}\\ \hline -(1-\alpha)R_1^T\otimes\mathbf{1}_{n_3} & \begin{array}{c}I_{m_1}\otimes[(x-2\alpha)I_{n_3}\\ -A_\alpha(H)]\end{array} & \mathbf{0}_{m_1n_3\times n_2}\\ \hline -(1-\alpha)J_{n_2\times n_1} & \mathbf{0}_{n_2\times m_1n_3} & \begin{array}{c}(x-\alpha n_1)I_{n_2}\\ -A_\alpha(G_2)\end{array} \end{array}\right]\\ &= \det[(x-\alpha n_1)I_{n_2}-A_\alpha(G_2)]\times\det S \quad \text{[by Lemma 2.1]}, \end{aligned} \tag{5}$$

where,

$$\begin{aligned} S \;=\; &\left[\begin{array}{c|c} (x-\alpha r_1n_3-\alpha n_2)I_{n_1} & -(1-\alpha)R_1\otimes\mathbf{1}_{n_3}^T\\ \hline -(1-\alpha)R_1^T\otimes\mathbf{1}_{n_3} & I_{m_1}\otimes[(x-2\alpha)I_{n_3}-A_\alpha(H)] \end{array}\right]\\ &-\left[\begin{array}{c} -(1-\alpha)J_{n_1\times n_2}\\ \hline \mathbf{0}_{m_1n_3\times n_2} \end{array}\right][(x-\alpha n_1)I_{n_2}-A_\alpha(G_2)]^{-1}\left[\begin{array}{c|c} -(1-\alpha)J_{n_2\times n_1} & \mathbf{0}_{n_2\times m_1n_3} \end{array}\right], \end{aligned}$$

is the Schur complement of $[(x-\alpha n_1)I_{n_2}-A_\alpha(G_2)]$.

Now,

$$\begin{aligned}
\det S &= \det\left(\left[\begin{array}{c|c} (x-\alpha r_1 n_3 - \alpha n_2) I_{n_1} & -(1-\alpha) R_1 \otimes \mathbf{1}_{n_3}^T \\ \hline -(1-\alpha) R_1^T \otimes \mathbf{1}_{n_3} & I_{m_1} \otimes [(x-2\alpha) I_{n_3} - A_\alpha(H)] \end{array}\right] \right. \\
&\quad \left. -(1-\alpha)^2 \left[\begin{array}{c|c} \Gamma_{A_\alpha(G_2)}(x-\alpha n_1) J_{n_1\times n_1} & \mathbf{0}_{n_1\times m_1 n_3} \\ \hline \mathbf{0}_{m_1 n_3 \times n_1} & \mathbf{0}_{m_1 n_3 \times m_1 n_3} \end{array}\right] \right) \\
&= \det \left[\begin{array}{c|c} \begin{array}{c}(x-\alpha r_1 n_3 - \alpha n_2) I_{n_1} \\ -(1-\alpha)^2 \Gamma_{A_\alpha(G_2)}(x-\alpha n_1) J_{n_1\times n_1}\end{array} & -(1-\alpha) R_1 \otimes \mathbf{1}_{n_3}^T \\ \hline -(1-\alpha) R_1^T \otimes \mathbf{1}_{n_3} & I_{m_1} \otimes [(x-2\alpha) I_{n_3} - A_\alpha(H)] \end{array}\right] \\
&= \times \det[I_{m_1} \otimes \{(x-2\alpha) I_{n_3} - A_\alpha(H)\}] \\
&\quad \times \det[\{(x-\alpha r_1 n_3 - \alpha n_2) I_{n_1} - (1-\alpha)^2 \Gamma_{A_\alpha(G_2)}(x-\alpha n_1) J_{n_1\times n_1}\} \\
&\quad -(1-\alpha)^2 (R_1 \otimes \mathbf{1}_{n_3}^T)\{I_{m_1} \otimes [(x-2\alpha) I_{n_3} - A_\alpha(H)]\}^{-1} (R_1^T \otimes \mathbf{1}_{n_3}) \\
&= \det[I_{m_1} \otimes \{(x-2\alpha) I_{n_3} - A_\alpha(H)\}] \cdot \det[\{(x-\alpha r_1 n_3 - \alpha n_2) I_{n_1} \\
&\quad -(1-\alpha)^2 \Gamma_{A_\alpha(G_2)}(x-\alpha n_1) J_{n_1\times n_1}\} - (1-\alpha)^2 \Gamma_{A_\alpha(H)}(x-2\alpha) R_1 R_1^T] \\
&= \det[(x-2\alpha) I_{n_3} - A_\alpha(H)]^{m_1} \cdot \det[(x-\alpha r_1 n_3 - \alpha n_2) I_{n_1} \\
&\quad -(1-\alpha)^2 \Gamma_{A_\alpha(H)}(x-2\alpha) R_1 R_1^T] \\
&\quad \times [1 - (1-\alpha)^2 \Gamma_{A_\alpha(G_2)}(x-\alpha n_1) \Gamma_{(1-\alpha)^2 \Gamma_{A_\alpha(H)}(x-2\alpha) R_1 R_1^T}(x-\alpha r_1 n_3 - \alpha n_2)], \\
&\quad \text{from Lemma 2.2.}
\end{aligned} \tag{6}$$

Now $2r_1(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)$ is the sum of each row of the matrix $(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)R_1R_1^T$.

Thus from equation (4), we get

$$\Gamma_{(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)R_1R_1^T}(x-\alpha r_1 n_3 - \alpha n_2) = \frac{n_1}{(x-\alpha r_1 n_3 - \alpha n_2) - 2r_1(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)}.$$

By equation (2), $r_1 + \lambda_i(A(G_1))$ is the eigenvalue of the matrix $R_1R_1^T$, where $\lambda_i(A(G_1))$ is the eigenvalue of $A(G_1)$ for $i = 1, 2, \ldots, n_1$. Thus, equation (6) becomes

$$\begin{aligned}
\det S &= \Psi(A_\alpha(H); x-2\alpha)^{m_1} \cdot \Bigg[1 - (1-\alpha)^2 \Gamma_{A_\alpha(G_2)}(x-\alpha n_1) \\
&\quad \times \frac{n_1}{(x-\alpha r_1 n_3 - \alpha n_2) - 2r_1(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)}\Bigg] \\
&\quad \times \det\left[(x-\alpha r_1 n_3 - \alpha n_2) I_{n_1} - (1-\alpha)^2 \Gamma_{A_\alpha(H)}(x-2\alpha) R_1 R_1^T\right]
\end{aligned}$$

$$= \Psi(A_\alpha(H); x-2\alpha)^{m_1}$$
$$\times \left[\frac{(x-\alpha r_1 n_3-\alpha n_2)-2r_1(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)-n_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)}{(x-\alpha r_1 n_3-\alpha n_2)-2r_1(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)}\right]$$
$$\times \prod_{i=1}^{n_1}[(x-\alpha r_1 n_3-\alpha n_2)-\{r_1+\lambda_i(A(G_1))\}(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)]$$

$$(7) = \Psi(A_\alpha(H); x-2\alpha)^{m_1}[(x-\alpha r_1 n_3-\alpha n_2)-2r_1(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)$$
$$-n_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)] \times \prod_{i=1}^{n_1-1}\big[(x-\alpha r_1 n_3-\alpha n_2)$$
$$-\{r_1+\lambda_i(A(G_1))\}(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)\big].$$

Since $A_\alpha(G_1)=\alpha r_1 I_{n_1}+(1-\alpha)A(G_1)$, we have $\lambda_i(A_\alpha(G_1))=\alpha r_1+(1-\alpha)\lambda_i(A(G_1))$ is the eigenvalue of $A_\alpha(G_1)$ for $i=1,2,\dots,n_1$. Therefore,

$$\begin{aligned}\det S &= \Psi(A_\alpha(H); x-2\alpha)^{m_1}[(x-\alpha r_1 n_3-\alpha n_2)-2r_1(1-\alpha)^2\Gamma_{A_\alpha(H)}(x-2\alpha)\\ &\quad -n_1(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha n_1)]\times\prod_{i=1}^{n_1-1}[(x-\alpha r_1 n_3-\alpha n_2)-(1-\alpha)\\ &\quad \times\{(1-2\alpha)r_1+\lambda_i(A_\alpha(G_1))\}\Gamma_{A_\alpha(H)}(x-2\alpha)].\end{aligned}$$

Using this result in equation (5), we get the required characteristic equation. □

Corollary 3.2. *Let G_i be a graph of order n_i and size m_i with regularity r_i, for $i=1,2$, and H be a graph of order n_3 and size m_3 with regularity r_3.*

I. *For $r_1=1$, the eigenvalues of $A_\alpha(G_1(\vee_H)G_2)$ are given by*
   (a) $\alpha(n_2+n_3)$, *with multiplicity 1;*
   (b) $2\alpha+\lambda_i(A_\alpha(G_2))$, *with multiplicity 1, for $i=1,2,\dots,n_2-1$;*
   (c) $2\alpha+\lambda_i(A_\alpha(H))$, *with multiplicity 1, for $i=1,2,\dots,n_3-1$;*
   (d) *three zeroes of the equation* $(x-\alpha n_3-\alpha n_2)(x-2\alpha-r_3)(x-2\alpha-r_2)-2(1-\alpha)^2\{n_3(x-2\alpha-r_2)+n_2(x-2\alpha-r_3)\}=0$.

II. *For $r_1\geq 2$, the eigenvalues of $A_\alpha(G_1(\vee_H)G_2)$ are given by*
   (a) $\alpha n_1+\lambda_i(A_\alpha(G_2))$, *with multiplicity 1, for $i=1,2,...,n_2-1$;*
   (b) $2\alpha+r_3$, *with multiplicity m_1-n_1;*
   (c) $2\alpha+\lambda_i(A_\alpha(H))$, *with multiplicity m_1, for $i=1,2,\dots,n_3-1$;*
   (d) *three zeroes of the equation* $(x-2\alpha-r_3)(x-\alpha n_1-r_2)(x-\alpha r_1 n_3-\alpha n_2)-2r_1(1-\alpha)^2 n_3(x-\alpha n_1-r_2)-(1-\alpha)^2 n_1 n_2(x-2\alpha-r_3)=0$;
   (e) *two zeroes of the equation $G_i(x)=0$, $i=1,2,\dots,n_1-1$, where* $G_i(x)=(x-2\alpha-r_3)(x-\alpha n_3 r_1-\alpha n_2)-(1-\alpha)n_3[(1-2\alpha)r_1+\lambda_i(A_\alpha(G_1))]$.

Proof. Since G_2, H are r_2-regular and r_3-regular graphs on n_2, n_3 vertices, respectively, by equation (4) we have, $\Gamma_{A_\alpha(H)}(x-2\alpha) = \frac{n_3}{x-2\alpha-r_3}$ and $\Gamma_{A_\alpha(G_2)}(x-\alpha n_1) = \frac{n_2}{x-\alpha n_1-r_2}$.

I. For $r_1 = 1$, G_1 is a path with two vertices i.e. P_2. Then $n_1 = 2$, $m_1 = 1$, and the eigenvalues of $A(G_1)$ are 1 and -1. Thus by theorem (3.1) we have,

$$\begin{aligned}\Psi(A_\alpha(G_1(\vee_H)G_2);x) = {} & \Psi(A_\alpha(G_2);x-2\alpha)\Psi(A_\alpha(H);x-2\alpha)[(x-\alpha n_3-\alpha n_2) \\ & -2(1-\alpha)^2\frac{n_3}{x-2\alpha-r_3} - 2(1-\alpha)^2\frac{n_2}{x-2\alpha-r_2}](x-\alpha n_3-\alpha n_2) \\ = {} & \prod_{i=1}^{n_2-1}(x-2\alpha-\lambda_i(A_\alpha(G_2)))\prod_{i=1}^{n_3-1}(x-2\alpha-\lambda_i(A_\alpha(H)))(x-\alpha n_3-\alpha n_2) \\ & \times[(x-2\alpha-r_3)(x-2\alpha-r_2)(x-\alpha n_3-\alpha n_2) \\ & -2(1-\alpha)^2\{n_3(x-2\alpha-r_2)+n_2(x-2\alpha-r_3)\}].\end{aligned}$$

II. If $r_1 \geq 2$, then $m_1 \geq n_1$, and by Theorem 3.1 we get,

$$\begin{aligned}\Psi(A_\alpha(G_1(\vee_H)G_2);x) = {} & (x-\alpha n_1-r_2)\prod_{i=1}^{n_2-1}(x-\alpha n_1-\lambda_i(A_\alpha(G_2)))(x-2\alpha-r_3)^{m_1} \\ & \times\prod_{i=1}^{n_3-1}(x-2\alpha-\lambda_i(A_\alpha(H)))^{m_1}\left[(x-\alpha r_1 n_3-\alpha n_2) - \frac{2r_1(1-\alpha)^2 n_3}{x-2\alpha-r_3}\right. \\ & \left.-\frac{(1-\alpha)^2 n_1 n_2}{x-\alpha n_1-r_2}\right]\prod_{i=1}^{n_1-1}\left[(x-\alpha n_3 r_1-\alpha n_2) - \frac{(1-\alpha)n_3}{x-2\alpha-r_3}\right. \\ & \left.\times\{(1-2\alpha)r_1+\lambda_i(A_\alpha(G_1))\}\right] \\ = {} & (x-2\alpha-r_3)^{m_1-n_1}\prod_{i=1}^{n_2-1}(x-\alpha n_1-\lambda_i(A_\alpha(G_2))) \\ & \times\prod_{i=1}^{n_3-1}(x-2\alpha-\lambda_i(A_\alpha(H)))^{m_1}[(x-2\alpha-r_3)(x-\alpha n_1-r_2) \\ & \times(x-\alpha r_1 n_3-\alpha n_2) - 2r_1(1-\alpha)^2 n_3(x-\alpha n_1-r_2) \\ & \times-(1-\alpha)^2 n_1 n_2(x-2\alpha-r_3)]\prod_{i=1}^{n_1-1}[(x-2\alpha-r_3)(x-\alpha n_3 r_1-\alpha n_2) \\ & -(1-\alpha)n_3\{(1-2\alpha)r_1+\lambda_i(A_\alpha(G_1))\}].\end{aligned}$$

Hence, the result follows. □

In a similar manner, the following corollaries can also be proved using Lemma 2.3 and Lemma 2.4.

Corollary 3.3. *Let G_1 be a graph of order n_1 and size m_1 with regularity r_1. Let $G_2 = K_{a,b}$ and $H = K_{s,t}$ where $a, b, s, t \geq 1$.*

I. *For $r_1 = 1$, the eigenvalues of $A_\alpha(G_1(\vee_H)G_2)$ are given by*

(a) $\alpha(2 + a)$ *and* $\alpha(2 + b)$, *with multiplicities $(b - 1)$ and $(a - 1)$, respectively;*

(b) $\alpha(2 + s)$ *and* $\alpha(2 + t)$, *with multiplicities $(t - 1)$ and $(s - 1)$, respectively;*

(c) $\alpha(a + b + s + t)$, *with multiplicity* 1;

(d) *five zeroes of the equation*

$$(x - \alpha(s+t) - \alpha(a+b))(x - 2\alpha - \tfrac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x - 2\alpha - \tfrac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$$
$$(x-2\alpha-\tfrac{\alpha(a+b)+\sqrt{\alpha^2(a+b)^2+4ab(1-2\alpha)}}{2})(x-2\alpha-\tfrac{\alpha(a+b)-\sqrt{\alpha^2(a+b)^2+4ab(1-2\alpha)}}{2})-2(1-\alpha)^2\{(s+t)(x-2\alpha)-\alpha(s+t)^2+2st\}(x-2\alpha-\tfrac{\alpha(a+b)+\sqrt{\alpha^2(a+b)^2+4ab(1-2\alpha)}}{2})(x-2\alpha-\tfrac{\alpha(a+b)-\sqrt{\alpha^2(a+b)^2+4ab(1-2\alpha)}}{2})$$
$$-2(1-\alpha)^2\{(a+b)(x-2\alpha)-\alpha(a+b)^2+2ab\}(x-2\alpha-\tfrac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-2\alpha-\tfrac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2}) = 0.$$

II. *For $r_1 \geq 2$, the eigenvalues of $A_\alpha(G_1(\vee_H)G_2)$ are given by*

(a) $\alpha(n_1 + a)$ *and* $\alpha(n_1 + b)$, *with multiplicities $(b - 1)$ and $(a - 1)$, respectively;*

(b) $\alpha(2 + s)$ *and* $\alpha(2 + t)$, *with multiplicities $m_1(t - 1)$ and $m_1(s - 1)$, respectively;*

(c) $2\alpha + \frac{\alpha(s+t)\pm\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2}$, *each with multiplicity $(m_1 - n_1)$;*

(d) *five zeroes of the equation*

$$(x - \alpha r_1(s+t) - \alpha(a+b))(x - 2\alpha - \tfrac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x - 2\alpha - \tfrac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$$
$$(x-\alpha n_1-\tfrac{\alpha(a+b)+\sqrt{\alpha^2(a+b)^2+4ab(1-2\alpha)}}{2})(x-\alpha n_1-\tfrac{\alpha(a+b)-\sqrt{\alpha^2(a+b)^2+4ab(1-2\alpha)}}{2})-2r_1(1-\alpha)^2\{(s+t)(x-2\alpha)-\alpha(s+t)^2+2st\}(x-\alpha n_1-\tfrac{\alpha(a+b)+\sqrt{\alpha^2(a+b)^2+4ab(1-2\alpha)}}{2})(x-\alpha n_1-\tfrac{\alpha(a+b)-\sqrt{\alpha^2(a+b)^2+4ab(1-2\alpha)}}{2}) - n_1(1-\alpha)^2\{(a+b)(x-\alpha n_1)-\alpha(a+b)^2+2ab\}(x-2\alpha-\tfrac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-2\alpha-\tfrac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2}) = 0;$$

(e) *three zeroes of the equation*

$$(x - \alpha r_1 n_3 - \alpha n_2)(x - 2\alpha - \tfrac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x - 2\alpha - \tfrac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2}) -$$
$$(1-\alpha)\{(1-2\alpha)r_1 + \lambda_i(A_\alpha(G_1))\}\{(s+t)(x-2\alpha)-\alpha(s+t)^2+2st\} = 0,$$ *for each* $i = 1, 2, \ldots, n_1 - 1$.

Corollary 3.4. *Let G_i be a graph of order n_i and size m_i with regularity r_i and $H = K_{s,t}$, where $s, t \geq 1$ and $i = 1, 2$.*

I. *For $r_1 = 1$, the eigenvalues of $A_\alpha(G_1(\vee_H)G_2)$ are given by*

(a) $2\alpha + \lambda_i(A_\alpha(G_2))$, *with multiplicity 1, for $i = 1, 2, \ldots, n_2 - 1$;*

(b) $\alpha(2 + s)$ *and* $\alpha(2 + t)$, *with multiplicities $(t - 1)$ and $(s - 1)$, respectively;*

(c) $\alpha[(s+t)+n_2]$, *with multiplicity* 1;

(d) *four zeroes of the equation*

$$(x-\alpha(s+t)-\alpha n_2)(x-2\alpha-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-2\alpha-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$$
$$(x-2\alpha-r_2)-2(1-\alpha)^2\{(s+t)(x-2\alpha)-\alpha(s+t)^2+2st\}(x-2\alpha-r_2)-2(1-\alpha)^2n_2(x-2\alpha-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-2\alpha-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})=0.$$

II. *For* $r_1\geq 2$, *the eigenvalues of* $A_\alpha(G_1(\vee_H)G_2)$ *are given by*

(a) $\alpha n_1+\lambda_i(A_\alpha(G_2))$, *with multiplicity 1, for* $i=1,2,\ldots,n_2-1$;

(b) $\alpha(2+s)$ *and* $\alpha(2+t)$, *with multiplicities* $(t-1)m_1$ *and* $(s-1)m_1$, *respectively;*

(c) $2\alpha+\frac{\alpha(s+t)\pm\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2}$, *each with multiplicity* (m_1-n_1);

(d) *four zeroes of the equation*

$$(x-\alpha r_1n_3-\alpha n_2)(x-2\alpha-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-2\alpha-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$$
$$(x-\alpha n_1-r_2)-2r_1(1-\alpha)^2\{(s+t)(x-2\alpha)-\alpha(s+t)^2+2st\}(x-\alpha n_1-r_2)-(1-\alpha)^2n_1n_2$$
$$(x-2\alpha-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-2\alpha-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})=0;$$

(e) *three zeroes of the equation*

$$(x-\alpha r_1n_3-\alpha n_2)(x-2\alpha-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-2\alpha-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$$
$$-(1-\alpha)\left[(1-2\alpha)+\lambda_i(A_\alpha(G_1))\right]\{(s+t)(x-2\alpha)-\alpha(s+t)^2+2st\}=0,$$ *for each* $i=1,2,\ldots,n_1-1$.

Corollary 3.5. *Let* G_1 *be a graph of order* n_1 *and size* m_1 *with regularity* r_1. *Let* $G_2=K_{s,t}$ *and* H *be a graph of order* n_3 *with regularity* r_3, *where* $s,t\geq 1$.

I. *For* $r_1=1$, *the eigenvalues of* $A_\alpha(G_1(\vee_H)G_2)$ *are given by*

(a) $\alpha(2+s)$ *and* $\alpha(2+t)$, *with multiplicities* $(t-1)$ *and* $(s-1)$, *respectively;*

(b) $2\alpha+\lambda_i(A_\alpha(H))$, *with multiplicity* 1 *for each* $i=1,2,\ldots,n_3-1$;

(c) $\alpha n_3+\alpha(s+t)$, *with multiplicity* 1;

(d) *four zeroes of the equation*

$$(x-\alpha n_3-\alpha(s+t))(x-2\alpha-r_3)(x-2\alpha-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$$
$$(x-2\alpha-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-2(1-\alpha)^2n_3(x-2\alpha-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-2\alpha-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-$$
$$2(1-\alpha)^2(x-2\alpha-r_3)\{(s+t)(x-2\alpha)-\alpha(s+t)^2+2st\}=0.$$

II. *For* $r_1\geq 2$, *the eigenvalues of* $A_\alpha(G_1(\vee_H)G_2)$ *are given by*

(a) $\alpha(n_1+s)$ *and* $\alpha(n_1+t)$, *with multiplicities* $(t-1)$ *and* $(s-1)$, *respectively;*

(b) $2\alpha+r_3$, *with multiplicity* (m_1-n_1);

(c) $2\alpha+\lambda_i(A_\alpha(H))$, *with multiplicity* m_1, *for* $i=1,2,\ldots,n_3-1$;

(d) *four zeroes of the equation*
$(x-\alpha r_1n_3-\alpha n_2)(x-2\alpha-r_3)(x-\alpha n_1-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$ $(x-\alpha n_1-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-2r_1(1-\alpha)^2n_3(x-\alpha n_1-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-\alpha n_1-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-(1-\alpha)^2n_1(x-2\alpha-r_3)\{(s+t)(x-\alpha n_1)-\alpha(s+t)^2+2st\}=0$;

(e) *two zeroes of the equation*
$(x-\alpha r_1n_3-\alpha n_2)(x-2\alpha-r_3)-(1-\alpha)n_3[(1-2\alpha)r_1+\lambda_i(A_\alpha(G_1))]=0$, *for* $i=1,2,\ldots,n_1-1$.

Corollary 3.6. *Let $\{S_i|i\in\mathbb{N}\}$, $\{T_j|j\in\mathbb{N}\}$ and $\{W_k|k\in\mathbb{N}\}$ be three families of A_α-cospectral graphs such that the graph S_i is r-regular for all i. Also consider $\Gamma_{A_\alpha(T_j)}(x)=f(x)$, for all $j\in\mathbb{N}$ and $\Gamma_{A_\alpha(W_k)}(x)=g(x)$, for all $k\in\mathbb{N}$. Then, $\{S_i(\vee_{T_j})W_k|i,j,k\in\mathbb{N}\}$ and $\{S_i(\vee_{W_k})T_j|i,j,k\in\mathbb{N}\}$ are also two families of A_α-cospectral graphs.*

3.2. **A_α-Spectrum of $G_1(\Lambda_H)G_2$, when G_1 and H both are regular.** In this section, we derive the A_α-spectrum of the graph $G_1(\Lambda_H)G_2$. When both G_1 and H are regular graphs, the A_α-characteristic polynomial of the graph $G_1(\Lambda_H)G_2$ is established in the following theorem.

Theorem 3.7. *Let G_1 be a graph of order n_1 and size m_1 with regularity r_1, H be a graph of order n_3 and size m_3 with regularity r_3, and G_2 be any arbitrary graph with order n_2.*

I. *For $r_1=1$, the A_α-characteristic polynomial of $G_1(\Lambda_H)G_2$ is given by*

$$\begin{aligned}\Psi(A_\alpha(G_1(\Lambda_H)G_2);x) &= \Psi(A_\alpha(G_2);x-\alpha n_3)(x-\alpha n_3)[(x-\alpha(2+n_2))(x-\alpha n_3)\\ &\quad -r_3(x-\alpha n_3)-2n_3(1-\alpha)^2-n_3(1-\alpha)^2(x-\alpha n_3)\\ &\quad \times\Gamma_{A_\alpha(G_2)}(x-\alpha n_3)]\prod_{i=1}^{n_3-1}[x-\alpha(2+n_2)-\lambda_i(A_\alpha(H))].\end{aligned}$$

II. *For $r_1\geq 2$, the A_α-characteristic polynomial of $G_1(\Lambda_H)G_2$ is given by*

$$\begin{aligned}\Psi(A_\alpha(G_1(\Lambda_H)G_2);x) &= \Psi(A_\alpha(G_2);x-\alpha m_1n_3)(x-2\alpha-\alpha n_2-r_3)^{m_1-n_1}\\ &\quad \times[(x-2\alpha-\alpha n_2)(x-\alpha r_1n_3)-r_3(x-\alpha r_1n_3)-2(1-\alpha)^2r_1n_3\\ &\quad -m_1n_3(1-\alpha)^2(x-\alpha r_1n_3)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1n_3)]\\ &\quad \times\prod_{i=m_1-n_1+1}^{m_1-1}[(x-2\alpha-\alpha n_2)(x-\alpha r_1n_3)-r_3(x-\alpha r_1n_3)-n_3(1-\alpha)\\ &\quad \times(\lambda_i(A_\alpha(G_1))+(1-2\alpha)r_1)]\prod_{i=1}^{n_3-1}[x-2\alpha-\alpha n_2-\lambda_i(A_\alpha(H))]^{m_1}.\end{aligned}$$

Proof. Suppose that R_1 denotes the 0-1 vertex-edge incidence matrix of the graph G_1. Let the vertex sets of G_1, the m_1 copies of H, and G_2 be given by

$$\{v_1, \ldots, v_{n_1}\}, \quad \{u_{11}, \ldots, u_{1n_3}, \ldots, u_{m_11}, \ldots, u_{m_1n_3}\}, \quad \text{and} \quad \{w_1, \ldots, w_{n_2}\},$$

respectively. Then, the adjacency matrix of the graph $G_1(\Lambda_H)G_2$ can be expressed as follows

$$A(G_1(\Lambda_H)G_2) = \left[\begin{array}{c|c|c} \mathbf{0}_{n_1} & R_1 \otimes \mathbf{1}_{n_3}^T & \mathbf{0}_{n_1\times n_2} \\ \hline R_1^T \otimes \mathbf{1}_{n_3} & I_{m_1} \otimes A(H) & J_{m_1n_3\times n_2} \\ \hline \mathbf{0}_{n_2\times n_1} & J_{n_2\times m_1n_3} & A(G_2) \end{array}\right].$$

Since the degrees of vertices of $G_1(\Lambda_H)G_2$ are as follows

$$\begin{aligned} d_{G_1(\Lambda_H)G_2}(v_i) &= r_1n_3, \text{for } i = 1, 2, \ldots, n_1; \\ d_{G_1(\Lambda_H)G_2}(u_{jk}) &= d_H(u_{jk}) + 2 + n_2, \text{for } j = 1, 2, \ldots, m_1, \text{ and } k = 1, 2, \ldots, n_3; \\ d_{G_1(\Lambda_H)G_2}(w_l) &= d_{G_2}(w_l) + m_1n_3, \text{for } l = 1, 2, \ldots, n_2, \end{aligned}$$

we have

$$Deg(G_1(\Lambda_H)G_2) = \left[\begin{array}{c|c|c} r_1n_3I_{n_1} & \mathbf{0}_{n_1\times m_1n_3} & \mathbf{0}_{n_1\times n_2} \\ \hline \mathbf{0}_{m_1n_3\times n_1} & I_{m_1} \otimes [(2+n_2)I_{n_3} + Deg(H)] & \mathbf{0}_{m_1n_3\times n_2} \\ \hline \mathbf{0}_{n_2\times n_1} & \mathbf{0}_{n_2\times m_1n_3} & Deg(G_2) + m_1n_3I_{n_2} \end{array}\right],$$

and

$$A_\alpha(G_1(\Lambda_H)G_2) = \left[\begin{array}{c|c|c} \alpha r_1n_3I_{n_1} & (1-\alpha)R_1 \otimes \mathbf{1}_{n_3}^T & \mathbf{0}_{n_1\times n_2} \\ \hline (1-\alpha)R_1^T \otimes \mathbf{1}_{n_3} & \begin{array}{c} \alpha[I_{m_1} \otimes ((2+n_2)I_{n_3} + Deg(H))] \\ +(1-\alpha)[I_{m_1} \otimes A(H)] \end{array} & (1-\alpha)J_{m_1n_3\times n_2} \\ \hline \mathbf{0}_{n_2\times n_1} & (1-\alpha)J_{n_2\times m_1n_3} & \begin{array}{c} A_\alpha(G_2) \\ +\alpha m_1n_3I_{n_2} \end{array} \end{array}\right].$$

Therefore, the characteristic polynomial of $A_\alpha(G_1(\Lambda_H)G_2)$ is given by

$$\begin{aligned} &\Psi(A_\alpha(G_1(\Lambda_H)G_2); x) \\ &\quad = \det[xI_{n_1+m_1n_3+n_2} - A_\alpha(G_1(\Lambda_H)G_2)] \\ &\quad = \det\left[\begin{array}{c|c|c} (x-\alpha r_1n_3)I_{n_1} & -(1-\alpha)R_1 \otimes \mathbf{1}_{n_3}^T & \mathbf{0}_{n_1\times n_2} \\ \hline -(1-\alpha)R_1^T \otimes \mathbf{1}_{n_3} & \begin{array}{c} I_{m_1}\otimes \\ [(x-\alpha(2+n_2))I_{n_3} - A_\alpha(H)] \end{array} & -(1-\alpha)J_{m_1n_3\times n_2} \\ \hline \mathbf{0}_{n_2\times n_1} & -(1-\alpha)J_{n_2\times m_1n_3} & \begin{array}{c} (x-\alpha m_1n_3)I_{n_2} \\ -A_\alpha(G_2) \end{array} \end{array}\right] \\ &\quad = \det[(x-\alpha m_1n_3)I_{n_2} - A_\alpha(G_2)] \times \det S' \quad [\text{by Lemma 2.1}], \end{aligned}$$

where

$$
\begin{aligned}
S' &= \left[\begin{array}{c|c} (x-\alpha r_1 n_3)I_{n_1} & -(1-\alpha)R_1\otimes \mathbf{1}_{n_3}^T \\ \hline -(1-\alpha)R_1^T\otimes \mathbf{1}_{n_3} & I_{m_1}\otimes[(x-\alpha(2+n_2))I_{n_3}-A_\alpha(H)] \end{array}\right] \\
&\quad -\left[\begin{array}{c} \mathbf{0}_{n_1\times n_2} \\ \hline -(1-\alpha)J_{m_1n_3\times n_2} \end{array}\right][(x-\alpha m_1 n_3)I_{n_2}-A_\alpha(G_2)]^{-1} \\
&\quad \times \left[\begin{array}{c|c} \mathbf{0}_{n_2\times n_1} & -(1-\alpha)J_{n_2\times m_1n_3} \end{array}\right],
\end{aligned}
$$

is the Schur complement of $[(x-\alpha m_1 n_3)I_{n_2}-A_\alpha(G_2)]$. Now,

$$
\begin{aligned}
\det S' &= \det\Big(\left[\begin{array}{c|c} (x-\alpha r_1 n_3)I_{n_1} & -(1-\alpha)R_1\otimes \mathbf{1}_{n_3}^T \\ \hline -(1-\alpha)R_1^T\otimes \mathbf{1}_{n_3} & I_{m_1}\otimes[(x-\alpha(2+n_2))I_{n_3}-A_\alpha(H)] \end{array}\right] \\
&\quad -(1-\alpha)^2\left[\begin{array}{c|c} \mathbf{0}_{n_1\times n_1} & \mathbf{0}_{n_1\times m_1n_3} \\ \hline \mathbf{0}_{m_1n_3\times n_1} & \Gamma_{A_\alpha(G_2)}(x-\alpha m_1n_3)J_{m_1n_3\times m_1n_3} \end{array}\right]\Big) \\
&= \det\left[\begin{array}{c|c} (x-\alpha r_1 n_3)I_{n_1} & -(1-\alpha)R_1\otimes \mathbf{1}_{n_3}^T \\ \hline -(1-\alpha)R_1^T\otimes \mathbf{1}_{n_3} & \begin{array}{c} I_{m_1}\otimes[(x-\alpha(2+n_2))I_{n_3}-A_\alpha(H)] \\ -(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1n_3)J_{m_1n_3\times m_1n_3} \end{array} \end{array}\right] \\
&= \det[(x-\alpha r_1n_3)I_{n_1}]\times\det[I_{m_1}\otimes((x-\alpha(2+n_2))I_{n_3}-A_\alpha(H)) \\
&\quad -(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1n_3)J_{m_1n_3\times m_1n_3} \\
&\quad -(1-\alpha)^2(R_1^T\otimes\mathbf{1}_{n_3})[(x-\alpha r_1n_3)I_{n_1}]^{-1}(R_1\otimes\mathbf{1}_{n_3}^T)] \\
&= (x-\alpha r_1n_3)^{n_1}\times\det[(x-\alpha(2+n_2))I_{m_1n_3}-(I_{m_1}\otimes A_\alpha(H)) \\
&\quad -(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1n_3)J_{m_1n_3\times m_1n_3}-\frac{(1-\alpha)^2}{(x-\alpha r_1n_3)}(R_1^TR_1\otimes J_{n_3\times n_3})] \\
&= (x-\alpha r_1n_3)^{n_1}\cdot[1-(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1n_3) \tag{8}\\
&\quad \times\Gamma_{(I_{m_1}\otimes A_\alpha(H)+\frac{(1-\alpha)^2}{(x-\alpha r_1n_3)}(R_1^TR_1\otimes J_{n_3\times n_3}))}(x-\alpha(2+n_2))] \\
&\quad \times\det[(x-\alpha(2+n_2))I_{m_1n_3}-(I_{m_1}\otimes A_\alpha(H)) \\
&\quad -\frac{(1-\alpha)^2}{(x-\alpha r_1n_3)}(R_1^TR_1\otimes J_{n_3\times n_3})], \quad \text{from Lemma 2.2.}
\end{aligned}
$$

The line graph of G_1 is known to be $(2r_1-2)$-regular if G_1 is r_1-regular. As a result, $I_{m_1}\otimes A_\alpha(H)$ is r_3-regular and $R_1^TR_1$ is $2r_1$-regular. Hence $I_{m_1}\otimes A_\alpha(H)+\frac{(1-\alpha)^2}{(x-\alpha r_1n_3)}(R_1^TR_1\otimes J_{n_3\times n_3})$ is $r_3+\frac{(1-\alpha)^2}{(x-\alpha r_1n_3)}(2r_1n_3)$-regular.

Thus, equation (8) becomes

$$\det S'=(x-\alpha r_1 n_3)^{n_1}\left[1-(1-\alpha)^2\Gamma_{A_\alpha(G_2)}(x-\alpha m_1 n_3)\frac{m_1 n_3}{x-\alpha(2+n_2)-r_3-\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}(2r_1 n_3)}\right]$$
$$\times\ \det\Big[(x-\alpha(2+n_2))I_{m_1 n_3}-[I_{m_1}\otimes A_\alpha(H)]$$
$$-\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}((A(\ell(G_1))+2I_{m_1})\otimes J_{n_3\times n_3})\Big]$$
$$=(x-\alpha r_1 n_3)^{n_1}$$
$$\times\left[1-\frac{m_1 n_3(1-\alpha)^2(x-\alpha r_1 n_3)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1 n_3)}{(x-\alpha(2+n_2))(x-\alpha r_1 n_3)-r_3(x-\alpha r_1 n_3)-2(1-\alpha)^2 r_1 n_3}\right]$$
$$\times\prod_{i=1}^{m_1 n_3}\left[x-\alpha(2+n_2)-\lambda_i(I_{m_1}\otimes A_\alpha(H)+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}((A(\ell(G_1))+2I_{m_1})\otimes J_{n_3\times n_3}))\right]. \tag{9}$$

I. Now for $r_1=1,\ G_1=P_2,\ m_1=1,\ n_1=2$, and in this case $\ell(G_1)$ is an isolated point. Therefore eigenvalue of $A(\ell(G_1))$ is 0. Thus we have,

$$\Psi(A_\alpha(G_1(\Lambda_H)G_2);x)=\Psi(A_\alpha(G_2);x-\alpha n_3)(x-\alpha n_3)^2$$
$$\times\left[1-\frac{n_3(1-\alpha)^2(x-\alpha n_3)\Gamma_{A_\alpha(G_2)}(x-\alpha n_3)}{(x-\alpha(2+n_2))(x-\alpha n_3)-r_3(x-\alpha n_3)-2(1-\alpha)^2 n_3}\right]$$
$$\times\prod_{i=1}^{n_3}\left[x-\alpha(2+n_2)-\lambda_i(A_\alpha(H))-\frac{(1-\alpha)^2}{(x-\alpha n_3)}(2J_{n_3\times n_3})\right]$$
$$=\Psi(A_\alpha(G_2);x-\alpha n_3)(x-\alpha n_3)^2$$
$$\times\left[1-\frac{n_3(1-\alpha)^2(x-\alpha n_3)\Gamma_{A_\alpha(G_2)}(x-\alpha n_3)}{(x-\alpha(2+n_2))(x-\alpha n_3)-r_3(x-\alpha n_3)-2(1-\alpha)^2 n_3}\right]$$
$$\times\left[\frac{(x-\alpha(2+n_2))(x-\alpha n_3)-r_3(x-\alpha n_3)-2(1-\alpha)^2 n_3}{(x-\alpha n_3)}\right]$$
$$\times\prod_{i=1}^{n_3-1}[x-\alpha(2+n_2)-\lambda_i(A_\alpha(H))]$$
$$=\ \Psi(A_\alpha(G_2);x-\alpha n_3)(x-\alpha n_3)[(x-\alpha(2+n_2))(x-\alpha n_3) \tag{10}$$
$$-r_3(x-\alpha n_3)-2n_3(1-\alpha)^2-n_3(1-\alpha)^2(x-\alpha n_3)\Gamma_{A_\alpha(G_2)}(x-\alpha n_3)]$$
$$\times\prod_{i=1}^{n_3-1}[x-\alpha(2+n_2)-\lambda_i(A_\alpha(H))].$$

II. Let $r_1\geq 2$, $m_1\geq n_1$. Then, by Lemma 2.5, the first largest n_1 eigenvalues of $A(\ell(G_1))$ are $\lambda_i(A(G_1))+r_1-2$ for $i=m_1-n_1+1,m_1-n_1+2,\ldots,m_1$ and -2 repeated for (m_1-n_1) times. Also, the eigenvalues of $J_{n_3\times n_3}$ are n_3 with multiplicity 1 and 0 with multiplicity n_3-1. Therefore, the eigenvalues of $\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}[(A(\ell(G_1))+2I_{m_1})\otimes J_{n_3\times n_3}]$

are $\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}(\lambda_i(A(G_1))+r_1)n_3$ with multiplicity 1, for $i=m_1-n_1+1, m_1-n_1+2,\ldots,m_1$ and 0 with multiplicities $m_1n_3-n_1$. Again, the row sum of $[I_{m_1}\otimes A_\alpha(H)]=B$ (say) is r_3. Thus, the eigenvalues of B are r_3 with multiplicities m_1 and $\lambda_i(A_\alpha(H))$ with multiplicities m_1, for $i=1,2,\ldots,n_3-1$. It can be observed that a set of eigenvectors corresponding to r_3 is $F=\left\{\begin{pmatrix}\mathbf{1}_{n_3}\\ \hline \mathbf{0}_{n_3}\\ \hline \vdots\\ \hline \mathbf{0}_{n_3}\end{pmatrix},\begin{pmatrix}\mathbf{0}_{n_3}\\ \hline \mathbf{1}_{n_3}\\ \hline \mathbf{0}_{n_3}\\ \hline \vdots\\ \hline \mathbf{0}_{n_3}\end{pmatrix},\ldots,\begin{pmatrix}\mathbf{0}_{n_3}\\ \hline \mathbf{0}_{n_3}\\ \hline \vdots\\ \hline \mathbf{1}_{n_3}\end{pmatrix}\right\}$. Let μ_i be a non-zero eigenvalue of $A(\ell(G_1))+2I_{m_1}=C$ (say), and $X_i=\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_{m_1}\end{pmatrix}$ be a corresponding eigenvector. Then

$$\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}\left[C\otimes J_{n_3\times n_3}\right]\begin{pmatrix}x_1\mathbf{1}_{n_3}\\ \hline x_2\mathbf{1}_{n_3}\\ \hline \vdots\\ \hline x_{m_1}\mathbf{1}_{n_3}\end{pmatrix}=\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}n_3\mu_i\begin{pmatrix}x_1\mathbf{1}_{n_3}\\ \hline x_2\mathbf{1}_{n_3}\\ \hline \vdots\\ \hline x_{m_1}\mathbf{1}_{n_3}\end{pmatrix},$$

and in this way n_1 non-zero eigenvalues of $\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}[C\otimes J_{n_3\times n_3}]$ are obtained. We observe that $\begin{pmatrix}x_1\mathbf{1}_{n_3}\\ \hline x_2\mathbf{1}_{n_3}\\ \hline \vdots\\ \hline x_{m_1}\mathbf{1}_{n_3}\end{pmatrix}=Y_i$ (say) is a linear combination of the eigenvectors of the set F. Hence Y_i is an eigenvector of $[I_{m_1}\otimes A_\alpha(H)]$ corresponding to the eigenvalue r_3. This follows that

$$\begin{aligned}BY_i+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}\left[C\otimes J_{n_3\times n_3}\right]Y_i &= r_3Y_i+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}\mu_i n_3 Y_i\\ \Rightarrow\left[B+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}\left[C\otimes J_{n_3\times n_3}\right]\right]Y_i &= (r_3+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}\mu_i n_3)Y_i.\end{aligned}$$

Hence, Y_i is also an eigenvector of $[I_{m_1}\otimes A_\alpha(H)]+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}[(A(\ell(G_1))+2I_{m_1})\otimes J_{n_3\times n_3}]$ corresponding to the eigenvalue $(r_3+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}\mu_i n_3)$. Thus the eigenvalues of $[I_{m_1}\otimes A_\alpha(H)]+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}[(A(\ell(G_1))+2I_{m_1})\otimes J_{n_3\times n_3}]$ are

(a) $r_3+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}(2r_1n_3)$, with multiplicity 1;

(b) $r_3+\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}[\lambda_i(A(G_1))+r_1]n_3$, with multiplicity 1, for $i=m_1-1, m_1-2, \dots, m_1-n_1+1$;

(c) r_3, with multiplicity m_1-n_1; and

(d) $\lambda_i(A_\alpha(H))$, with multiplicity m_1, for $i=1,2,\dots,n_3-1$ (as $I_{m_1}\otimes A_\alpha(H)$ and $\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}[(A(\ell(G_1))+2I_{m_1})\otimes J_{n_3\times n_3}]$ commute with each other).

Therefore, the equation (9) becomes

$$\begin{aligned}\det S' &= (x-\alpha r_1 n_3)^{n_1}\\ &\quad\times\left[1-\frac{m_1 n_3(1-\alpha)^2(x-\alpha r_1 n_3)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1 n_3)}{(x-\alpha(2+n_2))(x-\alpha r_1 n_3)-r_3(x-\alpha r_1 n_3)-2(1-\alpha)^2 r_1 n_3}\right]\\ &\quad\times\left[x-\alpha(2+n_2)-r_3-\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}(2r_1 n_3)\right]\\ &\quad\times\prod_{i=m_1-n_1+1}^{m_1-1}\left[x-\alpha(2+n_2)-r_3-\frac{(1-\alpha)^2}{(x-\alpha r_1 n_3)}(\lambda_i(A(G_1))+r_1)n_3\right]\\ &\quad\times[x-\alpha(2+n_2)-r_3]^{m_1-n_1}\prod_{i=1}^{n_3-1}[x-\alpha(2+n_2)-\lambda_i(A_\alpha(H))]^{m_1}\\ &= [(x-\alpha(2+n_2))(x-\alpha r_1 n_3)-r_3(x-\alpha r_1 n_3)-2(1-\alpha)^2 r_1 n_3\\ &\quad -m_1 n_3(1-\alpha)^2(x-\alpha r_1 n_3)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1 n_3)]\cdot[x-\alpha(2+n_2)-r_3]^{m_1-n_1}\\ &\quad\times\prod_{i=m_1-n_1+1}^{m_1-1}[(x-\alpha(2+n_2))(x-\alpha r_1 n_3)-r_3(x-\alpha r_1 n_3)\\ &\quad -n_3(1-\alpha)^2(\lambda_i(A(G_1))+r_1)]\times\prod_{i=1}^{n_3-1}[x-\alpha(2+n_2)-\lambda_i(A_\alpha(H))]^{m_1}.\end{aligned}$$

Since, G_1 is r_1-regular, $A_\alpha(G_1)=\alpha r_1 I_{n_1}+(1-\alpha)A(G_1)$, and hence $\lambda_i(A(G_1))=\frac{\lambda_i(A_\alpha(G_1))-\alpha r_1}{(1-\alpha)}$ for $i=1,2,\dots,n_1$.

Therefore, we get the required characteristic polynomial as

$$\begin{aligned}(11)\quad \Psi(A_\alpha(G_1(\Lambda_H)G_2);x) &= \Psi(A_\alpha(G_2);x-\alpha m_1 n_3)(x-2\alpha-\alpha n_2-r_3)^{m_1-n_1}\\ &\quad\times[(x-2\alpha-\alpha n_2)(x-\alpha r_1 n_3)-r_3(x-\alpha r_1 n_3)-2r_1 n_3(1-\alpha)^2\\ &\quad -m_1 n_3(1-\alpha)^2(x-\alpha r_1 n_3)\Gamma_{A_\alpha(G_2)}(x-\alpha m_1 n_3)]\\ &\quad\times\prod_{i=m_1-n_1+1}^{m_1-1}[(x-2\alpha-\alpha n_2)(x-\alpha r_1 n_3)-r_3(x-\alpha r_1 n_3)\\ &\quad -n_3(1-\alpha)(\lambda_i(A_\alpha(G_1))+(1-2\alpha)r_1)]\\ &\quad\times\prod_{i=1}^{n_3-1}[x-2\alpha-\alpha n_2-\lambda_i(A_\alpha(H))]^{m_1}.\end{aligned}$$

Hence the results. □

Corollary 3.8. *Let G_i be a graph of order n_i and size m_i with regularity r_i for $i = 1, 2$ and H be a graph of order n_3 with regularity r_3.*

I. *For $r_1 = 1$, the eigenvalues of $A_\alpha(G_1(\Lambda_H)G_2)$ are given by*

  (a) *αn_3, with multiplicity 1;*

  (b) *$\alpha n_3 + \lambda_i(A_\alpha(G_2))$, with multiplicity 1, for $i = 1, 2, \ldots, n_2 - 1$;*

  (c) *$\alpha(2 + n_2) + \lambda_i(A_\alpha(H))$, with multiplicity 1 for $i = 1, 2, \ldots, n_3 - 1$;*

  (d) *three zeroes of the equation $(x-\alpha(2+n_2))(x-\alpha n_3)(x-\alpha n_3-r_2)-r_3(x-\alpha n_3)(x-\alpha n_3 - r_2) - 2n_3(1 - \alpha)^2(x - \alpha n_3 - r_2) - (1 - \alpha)^2(x - \alpha n_3)n_2n_3 = 0$.*

II. *For $r_1 \geq 2$, the eigenvalues of $A_\alpha(G_1(\Lambda_H)G_2)$ are given by*

  (a) *$\alpha(2 + n_2) + r_3$, with multiplicity $m_1 - n_1$;*

  (b) *$\alpha m_1n_3 + \lambda_i(A_\alpha(G_2))$, with multiplicity 1, for $i = 1, 2, \ldots, n_2 - 1$;*

  (c) *$\alpha(2 + n_2) + \lambda_i(A_\alpha(H))$, with multiplicities m_1, for $i = 1, 2, \ldots, n_3 - 1$;*

  (d) *three zeroes of the equation $(x - \alpha(2 + n_2))(x - \alpha r_1n_3)(x - \alpha m_1n_3 - r_2) - r_3(x - \alpha r_1n_3)(x-\alpha m_1n_3-r_2)-2r_1n_3(1-\alpha)^2(x-\alpha m_1n_3-r_2)-(1-\alpha)^2(x-\alpha n_3)n_2n_3m_1 = 0$;*

  (e) *two zeroes of the equation $(x - \alpha(2 + n_2))(x - \alpha r_1n_3) - r_3(x - \alpha r_1n_3) - n_3(1 - \alpha)(\lambda_i(A_\alpha(G_1)) - (1 - 2\alpha)r_1) = 0$, for $i = m_1 - n_1 + 1, m_1 - n_1 + 2, \ldots, m_1 - 1$.*

Proof. I. When $r_1 = 1$, by equation (10) we have

$$\begin{aligned}
\Psi(A_\alpha(G_1(\Lambda_H)G_2); x) &= (x - \alpha n_3 - r_2) \prod_{i=1}^{n_2-1} [x - \alpha n_3 - \lambda_i(A_\alpha(G_2))] \\
&\quad \times (x - \alpha n_3)[(x - \alpha(2 + n_2))(x - \alpha n_3) - r_3(x - \alpha n_3) - 2n_3(1 - \alpha)^2 \\
&\quad - n_3(1 - \alpha)^2(x - \alpha n_3)\frac{n_2}{x - \alpha n_3 - r_2}] \prod_{i=1}^{n_3-1} [x - \alpha(2 + n_2) - \lambda_i(A_\alpha(H))] \\
&= \prod_{i=1}^{n_2-1} [x - \alpha n_3 - \lambda_i(A_\alpha(G_2))](x - \alpha n_3) \\
&\quad \times [(x - \alpha(2 + n_2))(x - \alpha n_3)(x - \alpha n_3 - r_2) \\
&\quad - r_3(x - \alpha n_3)(x - \alpha n_3 - r_2) - 2n_3(1 - \alpha)^2(x - \alpha n_3 - r_2) \\
&\quad - n_2n_3(1 - \alpha)^2(x - \alpha n_3)] \times \prod_{i=1}^{n_3-1} [x - \alpha(2 + n_2) - \lambda_i(A_\alpha(H))].
\end{aligned}$$

II. When $r_1 \geq 2$, by equation (11) we get

$$\begin{aligned}
\Psi(A_\alpha(G_1(\Lambda_H)G_2); x) \quad &= \quad (x - \alpha m_1n_3 - r_2) \prod_{i=1}^{n_2-1} [x - \alpha m_1n_3 - \lambda_i(A_\alpha(G_2))] \\
&\quad \times (x - 2\alpha - \alpha n_2 - r_3)^{m_1-n_1}[(x - 2\alpha - \alpha n_2)(x - \alpha r_1n_3) \\
&\quad - r_3(x - \alpha r_1n_3) - 2(1 - \alpha)^2 r_1n_3 - (1 - \alpha)^2 m_1n_3(x - \alpha r_1n_3)
\end{aligned}$$

$$\times\frac{n_2}{(x-\alpha m_1n_3-r_2)}]\prod_{i=m_1-n_1+1}^{m_1-1}[(x-2\alpha-\alpha n_2)(x-\alpha r_1n_3)-r_3(x-\alpha r_1n_3)$$

$$-n_3(1-\alpha)(\lambda_i(A_\alpha(G_1))+(1-2\alpha)r_1)]\prod_{i=1}^{n_3-1}[x-2\alpha-\alpha n_2-\lambda_i(A_\alpha(H))]^{m_1}$$

$$=\prod_{i=1}^{n_2-1}[x-\alpha m_1n_3-\lambda_i(A_\alpha(G_2))](x-2\alpha-\alpha n_2-r_3)^{m_1-n_1}$$

$$\times[(x-2\alpha-\alpha n_2)(x-\alpha r_1n_3)(x-\alpha m_1n_3-r_2)-r_3(x-\alpha r_1n_3)(x-\alpha m_1n_3-r_2)$$

$$-2r_1n_3(1-\alpha)^2(x-\alpha m_1n_3-r_2)-(1-\alpha)^2m_1n_2n_3(x-\alpha r_1n_3)]$$

$$\times\prod_{i=m_1-n_1+1}^{m_1-1}[(x-2\alpha-\alpha n_2)(x-\alpha r_1n_3)-r_3(x-\alpha r_1n_3)-n_3(1-\alpha)(\lambda_i(A_\alpha(G_1))$$

$$+(1-2\alpha)r_1)]\times\prod_{i=1}^{n_3-1}[x-2\alpha-\alpha n_2-\lambda_i(A_\alpha(H))]^{m_1}.$$

Hence the corollary follows. □

In a similar way, using Lemma 2.2 and Lemma 2.3, we can prove the following

Corollary 3.9. *Let G_1 be a graph of order n_1 and size m_1 with regularity r_1, H be a graph of order n_3 with regularity r_3 and $G_2=K_{s,t}$, where $s,t\geq 1$.*

I. *For $r_1=1$, the eigenvalues of $A_\alpha(G_1(\Lambda_H)G_2)$ are given by*
   (a) $\alpha(n_3+s)$ *and* $\alpha(n_3+t)$, *with multiplicities $(t-1)$ and $(s-1)$, respectively;*
   (b) αn_3, *with multiplicity 1;*
   (c) $\alpha(2+n_2)+\lambda_i(A_\alpha(H))$, *with multiplicity 1, for $i=1,2,\ldots,n_3-1$;*
   (d) *four zeroes of the equation* $(x-\alpha(2+n_2))(x-\alpha n_3)(x-\alpha n_3-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-\alpha n_3-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-r_3(x-\alpha n_3)(x-\alpha n_3-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-\alpha n_3-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-2n_3(1-\alpha)^2(x-\alpha n_3-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})(x-\alpha n_3-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-n_3(1-\alpha)^2(x-\alpha n_3)((s+t)(x-\alpha n_3)-\alpha(s+t)^2+2st)=0.$

II. *For $r_1\geq 2$, the eigenvalues of $A_\alpha(G_1(\Lambda_H)G_2)$ are given by*
   (a) $\alpha(m_1n_3+s)$ *and* $\alpha(m_1n_3+t)$, *with multiplicities $(t-1)$ and $(s-1)$, respectively;*
   (b) $\alpha(2+n_2)+r_3$, *with multiplicities m_1-n_1;*
   (c) *two zeroes of the equation* $(x-2\alpha-\alpha(s+t))(x-\alpha r_1n_3)-r_3(x-\alpha r_1n_3)-n_3(1-\alpha)(\lambda_i(A_\alpha(G_1))+(1-2\alpha)r_1)=0$, *for $i=m_1-n_1+1,m_1-n_1+2,\ldots,m_1-1$;*
   (d) $\alpha(2+(s+t))+\lambda_i(A_\alpha(H))$, *with multiplicity 1, for $i=1,2,\ldots,n_3-1$;*

(e) *four zeroes of the equation*
$(x-\alpha(2+n_2))(x-\alpha r_1n_3)(x-\alpha m_1n_3-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$
$(x-\alpha m_1n_3-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-r_3(x-\alpha r_1n_3)(x-\alpha m_1n_3-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$
$(x-\alpha m_1n_3-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-2r_1n_3(1-\alpha)^2(x-\alpha n_3-\frac{\alpha(s+t)+\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})$
$(x-\alpha m_1n_3-\frac{\alpha(s+t)-\sqrt{\alpha^2(s+t)^2+4st(1-2\alpha)}}{2})-m_1n_3(1-\alpha)^2(x-\alpha r_1n_3)((s+t)(x-\alpha m_1n_3)-\alpha(s+t)^2+2st)=0.$

Corollary 3.10. *Let G_1 be a graph of order n_1 and size m_1 with regularity r_1, H be a graph of order n_3 with regularity r_3 and G_2 be any arbitrary graph of order n_2. For $\alpha=0$, $A_\alpha(G_1(\Lambda_H)G_2)=A(G_1(\Lambda_H)G_2)$. Then,*

I. *for $r_1=1$, the eigenvalues of $A(G_1(\Lambda_H)G_2)$ are given by*
   (a) 0, *with multiplicity 1;*
   (b) $\lambda_i(A(G_2))$, *with multiplicity 1, for $i=1,2,\dots,n_2$;*
   (c) $\lambda_i(A(H))$, *with multiplicity 1, for $i=1,2,\dots,n_3-1$;*
   (d) *two zeroes of the equation* $x^2-r_3x-2n_3-xn_3\Gamma_{(A(G_2))}(x)=0.$

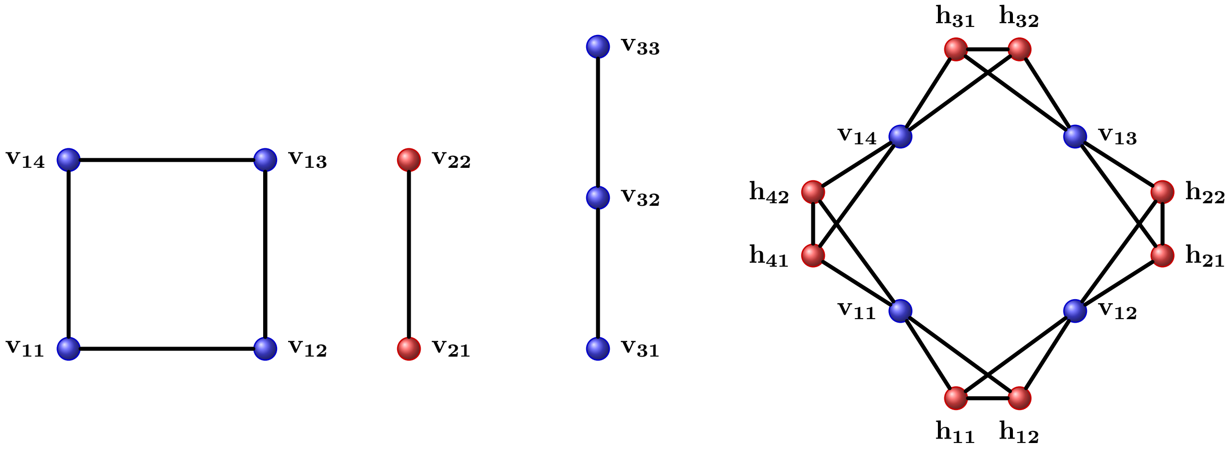
II. *for $r_1\geq 2$, the eigenvalues of $A(G_1(\Lambda_H)G_2)$ are given by*
   (a) r_3, *with multiplicity* (m_1-n_1);
   (b) $\lambda_i(A(G_2))$, *with multiplicity 1, for $i=1,2,\dots,n_2$;*
   (c) $\lambda_i(A(H))$, *with multiplicity m_1, for $i=1,2,\dots,n_3-1$;*
   (d) *two zeroes of the equation $x^2-xr_3-(\lambda_i(A(G_1))+r_1)n_3=0$, for $i=m_1-n_1+1,m_1-n_1+2,\dots,m_1-1$;*
   (e) *two zeroes of the equation* $x^2-r_3x-2r_1n_3-xm_1n_3\Gamma_{A(G_2)}(x)=0.$

Corollary 3.11. *Let $\{G_i|i\in\mathbb{N}\}$ and $\{H_k|k\in\mathbb{N}\}$ be two families of r_1 and r_3 regular A_α-cospectral graphs respectively. Let $\{T_j|j\in\mathbb{N}\}$ be a family of A_α-cospectral graphs such that $\Gamma_{A_\alpha(T_j)}(x)=f(x)$, for all $j\in\mathbb{N}$. Then, all the graphs $G_i(\Lambda_{H_k})T_j$'s are A_α-cospectral, for all $i,j,k\in\mathbb{N}$.*

4. Examples

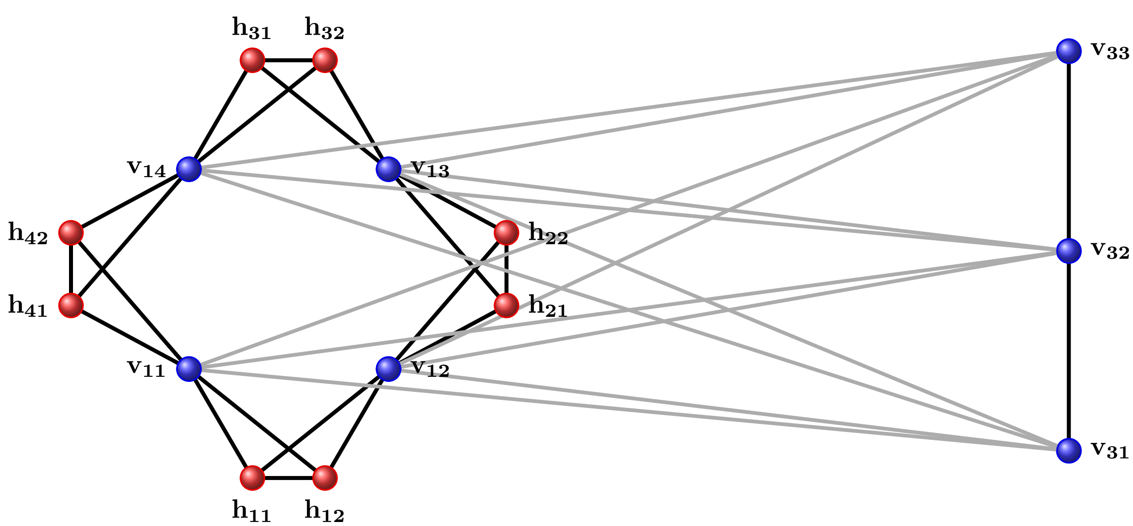
To illustrate the applicability of our theoretical results, we explicitly compute the A_α-characteristic polynomials for two representative cases, namely the graphs $G_1(\vee_H)G_2$ and $G_1(\Lambda_H)G_2$. The methodology employed here can be readily extended to derive analogous expressions for other graph constructions considered in this work.

Example 4.1. Consider the graphs $G_1=K_{2,2}$, $H=K_{1,1}$, and $G_2=K_{1,2}$. By definition, the graph $S(G_1,H)$ is constructed as follows.



$\mathbf{G_1}$ $\mathbf{H}$ $\mathbf{G_2}$ $\mathbf{S(G_1, H)}$

The triple vertex join of G_1 and G_2 with respect to H, denoted by $G_1(\vee_H)G_2$, is constructed as follows.



$\mathbf{G_1(\vee_H)G_2}$

Let $\alpha = \frac{1}{3}$. In this case, we have $n_1 = 4$, $n_2 = 3$, $n_3 = 2$, $m_1 = 4$, $m_2 = 2$, $m_3 = 1$, and $r_1 = 2$, $r_3 = 1$. Accordingly, the A_α-matrix of the graph $G_1(\vee_H)G_2$ is given by

$$A_{\frac{1}{3}}(G_1(\vee_H)G_2) = \left(\begin{array}{cccc|cccccccc|ccc}
\frac{7}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
0 & \frac{7}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
0 & 0 & \frac{7}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
0 & 0 & 0 & \frac{7}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
\hline
\frac{2}{3} & \frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 1 & 0 & 0 & 0 \\
\hline
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{3} & \frac{2}{3} & 0 \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 2 & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{5}{3}
\end{array}\right).$$

We observe that

$$\begin{aligned}
\Psi\big(A_\alpha(G_2); x-\alpha n_1\big) &= \Psi\left(A_{\frac{1}{3}}(G_2); x-\frac{4}{3}\right) = (x-2.79076)(x-1.66667)(x-0.875906),\\
\Psi\big(A_\alpha(H); x-2\alpha\big)^{m_1} &= \Psi\left(A_{\frac{1}{3}}(H); x-\frac{2}{3}\right)^4 = (x-1.66667)^4(x-0.33333)^4,\\
\Gamma_{A_\alpha(G_2)}(x-\alpha n_1) &= \Gamma_{A_{\frac{1}{3}}(G_2)}\left(x-\frac{4}{3}\right) = \frac{3(x-1)}{(x-2.79076)(x-0.875906)},\\
\Gamma_{A_\alpha(H)}(x-2\alpha) &= \Gamma_{A_{\frac{1}{3}}(H)}\left(x-\frac{2}{3}\right) = \frac{6}{3x-5}.\\
\sigma\left(A_{\frac{1}{3}}(G_1)\right) &= \left\{2^1, \left(\frac{2}{3}\right)^2, \left(-\frac{2}{3}\right)^1\right\}
\end{aligned}$$

Substituting these expressions into the polynomial obtained from Theorem 3.1 and simplifying yields

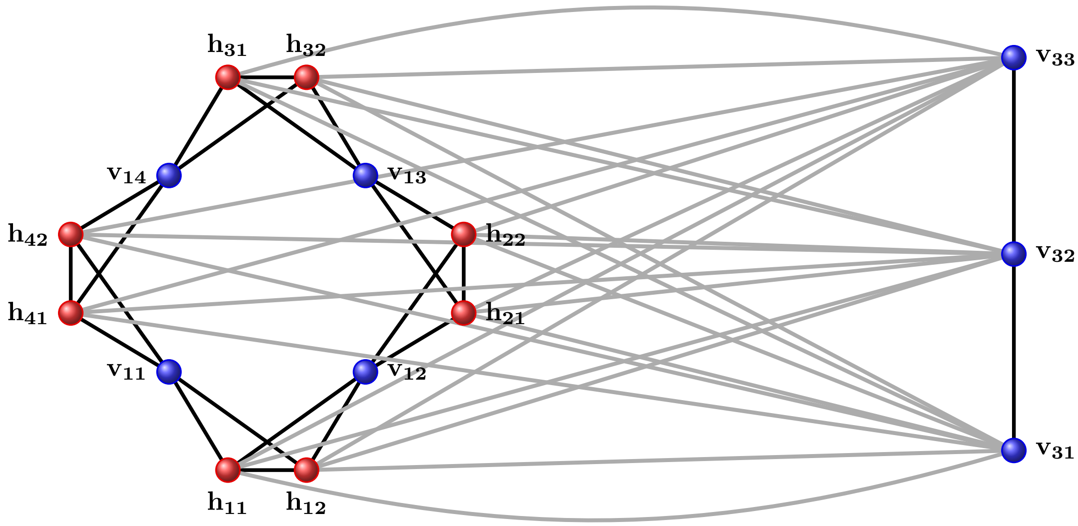
$$\begin{aligned}
P_1(x) &= (x-5.33768)(x-3.37437)^2(x-2.33333)(x-2.15094)(x-1.66667)^2\\
&\quad \times(x-0.932338)(x-0.625631)^2(x-0.333333)^4(x+0.75429).
\end{aligned}$$

Solving $P_1(x)=0$ gives the eigenvalues

$$\left\{5.33768^1, 3.37437^2, 2.33333^1, 2.15094^1, 1.66667^2, 0.932338^1, 0.625631^2, 0.333333^4, -0.75429^1\right\},$$

which constitute the A_α-spectrum of the triple vertex join $G_1(\vee_H)G_2$.

Example 4.2. The triple edge join of G_1 and G_2 with respect to H, denoted by $G_1(\Lambda_H)G_2$, is constructed as follows.



$$\mathbf{G_1(\Lambda_H)G_2}$$

Accordingly, the A_α-matrix of the triple edge join $G_1(\Lambda_H)G_2$ is given by

$$A_\alpha\big(G_1(\Lambda_H)G_2\big) = \left(\begin{array}{cccc|cccccccc|ccc}
\frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{4}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 \\
\hline
\frac{2}{3} & \frac{2}{3} & 0 & 0 & 2 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & \frac{2}{3} & 0 & 0 & \frac{2}{3} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 2 & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} & 2 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 2 & \frac{2}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 2 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 2 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
\hline
0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 3 & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{10}{3} & \frac{2}{3} \\
0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & \frac{2}{3} & 3
\end{array}\right).$$

We have

$$\Psi\big(A_\alpha(G_2); x - \alpha m_1 n_3\big) = \Psi\left(A_{\frac{1}{3}}(G_2); x - \frac{8}{3}\right) = (x - 4.12409)(x - 3)(x - 2.20924),$$

$$\left(x-2\alpha-\alpha n_2-r_3\right)^{m_1-n_1}=1,$$

$$\Gamma_{A_\alpha(G_2)}(x-\alpha m_1 n_3)=\Gamma_{A_{\frac{1}{3}}(G_2)}\left(x-\frac{8}{3}\right)=\frac{3x-7}{(x-4.12409)(x-2.20924)},$$

$$\sigma\left(A_{\frac{1}{3}}(G_1)\right)=\left\{2^1,\left(\frac{2}{3}\right)^2,\left(-\frac{2}{3}\right)^1\right\}, \text{and } \sigma\left(A_{\frac{1}{3}}(H)\right)=\left\{1^1,\left(-\frac{1}{3}\right)^1\right\}.$$

Substituting these into the polynomial expression from Theorem 3.7 (3.7), and simplifying, we obtain

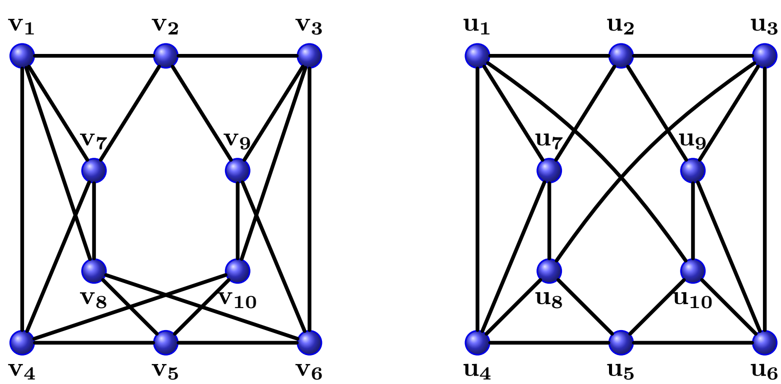
$$\begin{aligned} P_2(x) &= (x-6.96068)(x-3.49071)^2(x-3)(x-2.66667)(x-2.46689)(x-1.91497) \\ &\quad \times(x-1.33333)^5(x-0.509288)^2(x+1.00921). \end{aligned}$$

Solving $P_2(x)=0$ yields the eigenvalues

$$\left\{6.96068^1, 3.49071^2, 3^1, 2.66667^1, 2.46689^1, 1.91497^1, 1.33333^5, 0.509288^2, -1.00921^1\right\},$$

which form the A_α-spectrum of the triple edge join $G_1(\Lambda_H)G_2$.

Consider an adjacency cospectral pair G' and G'', both of which are 4-regular graphs on 10 vertices, first identified by van Dam and Haemers [21].



$\mathbf{G'}$ $\mathbf{G''}$

Since G' and G'' are adjacency cospectral and regular, it follows that they are also A_α-cospectral for every $\alpha\in[0,1]$. In particular, for $\alpha=\frac{1}{3}$, the A_α-spectra of G' and G'' are

$$\sigma\left(A_{\frac{1}{3}}(G')\right)=\sigma\left(A_{\frac{1}{3}}(G'')\right)=\left\{4^1, 2.82405^1, 2.37437^1, 2^1, 0.66667^4, -0.157379^1, -0.374369^1\right\},$$

and their corresponding A_α-coronals are

$$\Gamma_{A_{\frac{1}{3}}(G')}(x) \;=\; \Gamma_{A_{\frac{1}{3}}(G'')}(x) \;=\; \frac{10}{x-4}.$$

Let $S(G_1,H)$ be as in Example 4.1. By constructing $G_1(\vee_H)G'$ and $G_1(\vee_H)G''$ and evaluating their A_α-spectra at $\alpha=\frac{1}{3}$, we obtain

$$\begin{aligned}\sigma\Big(A_{\frac{1}{3}}(G_1(\vee_H)G')\Big)&=\sigma\Big(A_{\frac{1}{3}}(G_1(\vee_H)G'')\Big)\\&=\Big\{9.44619^1,5.1736^2,4.66667^1,4.15738^1,3.7077^1,3.33333^1,2.51929^1,2^4,\\&\quad 1.66667^1,1.17595^1,1.15973^2,0.958965^1,0.33333^4,-0.298815^1\Big\}.\end{aligned}$$

Similarly, forming $G_1(\Lambda_H)G'$ and $G_1(\Lambda_H)G''$ from $S(G_1,H)$ yields

$$\begin{aligned}\sigma\Big(A_{\frac{1}{3}}(G_1(\Lambda_H)G')\Big)&=\sigma\Big(A_{\frac{1}{3}}(G_1(\Lambda_H)G'')\Big)\\&=\Big\{12^1,\ 5.49071^1,\ 5.43358^2,\ 5.04104^1,\ 5^1,\ 4.66667^1,\ 3.66667^4,\ 3.33333^4,\\&\quad 2.50929^1,2.2923^1,\ 2.07233^1,\ 1.33333^1,\ 0.899755^2,\ -1.07233^1\Big\}.\end{aligned}$$

Hence, two new pairs of A_α-cospectral graphs are obtained, namely

$$\{\,G_1(\vee_H)G',\ G_1(\vee_H)G''\,\}\quad\text{and}\quad\{\,G_1(\Lambda_H)G',\ G_1(\Lambda_H)G''\,\}.$$

In view of Corollaries 3.6 and 3.11, this construction can be extended to generate infinitely many new pairs of A_α-cospectral graphs from any given initial pair that satisfies the conditions specified therein.

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