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Research Paper

SOME ASPECTS OF $*-$SYMMETRIC MODULES

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Abstract. In this paper, the notion of $*-$symmetric modules is introduced as an extension of $*-$symmetric rings to modules. Specifically, modules whose endomorphism rings satisfy the $*-$symmetric condition are investigated. Furthermore, we demonstrate that every $*-$symmetric module is a symmetric module, exhibits reversible properties, and satisfies the Insertion of Factors Property. In particular, fundamental results concerning $*-$symmetric and symmetric rings are extended to the context of $*-$symmetric modules.

1. Introduction

Throughout the discussion, let R denote an associative ring with unity, and all modules are unitary. The set of idempotent elements of the endomorphism ring of a module M is denoted as $I(M)$, and the set of all regular elements of the endomorphism ring of a module M is denoted as $RG(M)$. For an $R-$module M, let $X = End(M)$, which denotes the endomorphism ring of M. The notions of right annihilator and left annihilator are introduced as follows:

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The right annihilator of a module M with respect to a nonempty subset I of endomorphisms is defined as $r_M(I) = \{m \in M \mid Im = 0\}$.

The left annihilator, on the other hand, is defined as $l_X(I) = \{\phi \in X \mid \phi I = 0\}$.

Furthermore, specific full and upper triangular matrices over a module M are denoted as $Mat_n(M)$ and $U_n(M)$, respectively. In addition, a particular subset of upper triangular matrices with equal diagonal entries is denoted by $D_n(M) = \{(a_{ij}) \in U_n(M) \mid \text{all diagonal entries are equal}\}$.

Within a ring R, an ideal I is called symmetric if, for any elements $r, s, t \in R$, the condition $rst \in I$ implies $rts \in I$. A ring R itself is characterized as symmetric when its zero ideal satisfies this condition. J. Lambek first introduced this concept in 1971, developing it from his investigations into module representation using sheaves of factor modules [12]. Since its initial formulation, extensive research has been conducted by numerous scholars on various related concepts and properties pertaining to symmetric rings.

A ring R is called a $*-$symmetric ring if for any $a, b, c \in R$, $abc = 0$ implies $acb^* = 0$. This notion of a $*-$symmetric ring was introduced by Wang et al. in the study of $*-$symmetric rings [16]. Motivated by this work, we try to introduce the concept of a $*-$symmetric module. In addition, we establish the relations among $*-$symmetric modules, symmetric modules, IFP modules, reversible modules, and reduced modules. We further discuss some properties of $*-$symmetric modules.

A ring R is called a reversible ring if the product $ab = 0$, for each $a, b \in R$, necessitates the reverse product $ba = 0$. This notion of a reversible ring was first introduced by P. M. Cohn as early as 1999 in his study of Reversible Rings [5]. Since then, many authors have studied various concepts related to the reversible ring.

In a 1999 study, Anderson and Comillo investigated rings whose zero products commute, categorizing them using the designations ZC_3 (symmetric rings) and ZC_2 (reversible rings). They determined that a semigroup with no non-zero nilpotent elements satisfies both the conditions ZC_3 and ZC_2, subsequently examining rings adhering to the ZC_3 criterion. Notably, commutative rings are inherently symmetric, while non-reduced commutative rings, such as $\mathbb{Z}_{n^l}$ for $n, l \geq 2$, exhibit symmetry as demonstrated by Anderson and Comillo. According to Anderson et al.[[3], Theorem 1.1], a ring is a symmetric ring if and only if $r_1 r_2 r_3 ... r_n = 0$ implies $r_{\sigma(1)} r_{\sigma(2)} r_{\sigma(3)} ... r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, 3, ..., n\}$, where $r_i \in R$ and n is any positive integer. This result was also independently verified by Anderson-Camillo in their aforementioned study.[3]

In 2004, Lee and Zhou generalized the concept of reduced rings to the realm of module theory, introducing reduced modules as a theoretical analogue [13]. A module M over a ring R is categorised as reduced if, given any m in M and a in R, the condition $ma^2 = 0$ necessarily yields $mRa = 0$. Later, in 2024, Kimuli-Ssevviiri [10] extended this concept to endo-reduced modules by defining a module M as endo-reduced if its ring of endomorphisms, denoted as $End(M)$, forms a reduced ring.

2. Preliminaries

This section introduces fundamental concepts and theoretical frameworks essential for the forthcoming analysis.

A ring R is considered completely semiprime if, and only if, every non-zero element a complies with the condition that $a = 0$ whenever $a^2 = 0$. Consequently, this condition precludes the existence of non-trivial nilpotent elements within R. Such a ring is also termed a reduced ring. A module M over a ring R is considered a semiprime module when its endomorphism ring satisfies the semiprime ring condition.

An $R-$module M is called a reduced module if the condition $ma^2 = 0$, where $a \in R$ and $m \in M$, necessitates $mRa = 0$ [13]. This concept can be considered as a generalization of reduced rings; specifically, a ring is characterized as reduced if it encompasses no non-zero nilpotent elements. An $R-$module M is called a reduced module if its endomorphism ring, $End(M)$, is a reduced ring.

An element $f \in End(M)$ is said to be idempotent if it satisfies $f^2 = f$. Here, the $\mathbb{Z}-$module $\mathbb{Z}_2$ we consider $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then f is an idempotent element.

A ring R is said to satisfy the Insertion of Factors Property (simply an IFP ring) if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. It can be easily checked that the reversible rings are IFP rings. A ring R is usually called an abelian ring if every idempotent element of R is central. IFP rings are abelian rings, but the converse fails in general. [8]

An $R-$module M is called an IFP module (Insertion of Factors Property) if for any element $a \in R$ and any element m in M, whenever $ma = 0$, then $mra = 0$ for every r in R [11]. A module M over a ring R is called an IFP module when its endomorphism ring satisfies the IFP ring condition.

A $2-$degree anti-isomorphism $a \longmapsto a^*$ in a ring R is characterized by the properties, that is, $(a^*)^* = a$; $(a + b)^* = a^* + b^*$; and $(ab)^* = b^*a^*$. A ring R endowed with such an involution $*$ is denoted as a $*-$ring [6]. An $R-$module M is said to possess a $*-$module structure when $End(M)$ satisfies the property of a $*-$ring.

A ring R with an involution $*$ is called a $*-$proper ring if the condition $x^*x = 0$ implies $x = 0$ for all elements x in R [6]. A module M over a ring R equipped with an involution $*$ is called a $*-$proper module if $End(M)$ is a $*-$proper ring.

An element x in a $*-$ring R is called $*-$cancellable if $x^*x = 0$ implies $x = 0$. If each $x \in R$ is $*-$cancellable, then R is called a $*-$cancellable ring[6]. An $R-$module M is called a $*-$cancellable module if $End(M)$ is a $*-$cancellable ring.

A module M over a ring R is called a reversible module if its endomorphism ring $End(M)$ satisfies a specific property, namely that it is a reversible ring. Equivalently, for any $f, g \in End(M)$, $fg = 0$ implies $gf = 0$.

A module M over a ring R is called a symmetric module when the endomorphism ring of M satisfies the criterion of symmetry. This condition is equivalently expressed by the statement: for all endomorphisms f, g, h of $End(M)$, if $fgh = 0$, then the reverse permutation gfh also equals zero. [14]

In a given ring R, if for every element a, there exists an element b, both in R, such that the product aba equals a, then R is called a regular ring (von Neumann). Furthermore, if the endomorphism ring, that is, $End(M)$ is a regular ring, M is called a regular module.

A module M is called an abelian module if it satisfies the condition $mae = mea$ for all elements m in M and idempotents e in R. It is demonstrated in [2] that reduced modules, symmetric modules, semicommutative modules, and Armendariz modules inherently possess this characteristic. An $R-$module M is called an abelian module when its endomorphism ring, that is, $End(M)$ exhibits abelian ring properties.

A ring R is called a semicommutative ring if it fulfills the following condition: for any elements a and b in R, $ab = 0$ implies $aRb = 0$. Correspondingly, a module M over a ring R is termed semicommutative if for any module element m and any ring element a, $ma = 0$ implies $mRa = 0$. Buhphang and Rege in [4] conducted an investigation of the intrinsic characteristics of semicommutative modules, while Agayev and Harmanci broadened the scope of their analysis to comprehend the properties of semicommutative rings and modules, [1] with particular emphasis on the inherent semicommutativity of subrings within matrix rings.

We conclude this section with the following (Propositions 2.1-2.4) which are easy consequences of well known facts and results (see, for instance: [9], Proposition 1.2; [15], Lemmas 5.1 and 12.2; [2], Lemma 2.2; [7], Corollary 1.5), which will be inherently useful to our study.

Proposition 2.1. *Let R be a reduced ring. Then*

$$D_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in R \right\},$$

is a semicommutative ring.

Proposition 2.2. *The following statements hold for a ring R*

(1) *If R is a reduced ring, then $ab = 0$ implies $ba = 0$ for every $a, b \in R$;*

(2) *If R is a reduced ring, then every idempotent element of R is central in R;*

(3) *R is a reduced ring if and only if $l_R(a^2) = l_R(a)$ if and only if $r_R(a^2) = r_R(a)$ for every $a \in R$.*

Proposition 2.3. (1) *R is an abelian ring if and only if every $R-$ module is an abelian module.*

(2) *R is an abelian ring if and only if R_R is an abelian module.*

Proposition 2.4. *Let R be a reversible ring. If $ab \in I(R)$ for each $a, b \in R$, then $ab = ba$.*

3. Results and Discussion

Our primary findings are delineated in this section.

Definition 3.1. A module M over a ring R is defined as a $*-$symmetric module if the endomorphism ring of M, denoted $End(M)$, is a $*-$symmetric ring, that is, the endomorphism ring satisfies the condition that for any endomorphisms f, g, h of $End(M)$, $fgh = 0$ implies $fhg^* = 0$.

Example 3.2. (1) The $\mathbb{Z}-$module $\mathbb{Z}_3$, consider $M = \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$, which is a reduced module. Let $t = (x, y) \in \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$. Taking $t^* = (y, x)$. Consider $\alpha = (2, 0), \beta = (0, 1), \gamma = (1, 0) \in \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$. Then $\alpha\beta\gamma = (2, 0)(0, 1)(1, 0) = (0, 0) = 0$ while $\alpha\gamma\beta^* = (2, 0)(1, 0)(0, 1)^* = (2, 0)(1, 0)(1, 0) = (2, 0) \neq 0$. Hence, $\mathbb{Z}_3 \bigoplus \mathbb{Z}_3$ is not a $*-$symmetric module.

(2) All domains with some involution $*$ are $*-$symmetric module. Let $\mathbb{Z}[\sqrt{-3}] = \{x + \sqrt{-3}y : x, y \in \mathbb{Z}\}$ is $*-$symmetric with the involution $*$ defined by $(x + \sqrt{-3}y)^* = x - \sqrt{-3}y$.

Theorem 3.3. *Let M be a module with an involution $*$. If M is a $*-$symmetric module, then*

(1) *M is a symmetric module.*

(2) *M is a reversible module.*

Proof. (1): We assume that M is a $*-$symmetric module and $fgh = 0$ for each $f, g, h \in End(M)$. Then $fgh = 0$ implies $fhg^* = 0$ for any $f, g, h \in End(M)$. Thus, $fhg^* = 0 \implies 1(fh)g^* = 0$. Since M is a $*-$symmetric module, we have $1(fh)g^* = 0$ implies $1g^*(fh)^* = 0 \implies g^*(fh)^* = 0$. Now, taking an involution on $g^*(fh)^* = 0$, we have $\{g^*(fh)^*\}^* = 0 \implies \{(fh)^*\}^*(g^*)^* = 0 \implies fhg = 0$. Hence, $fgh = 0$ implies $fhg = 0$ for each $f, g, h \in End(M)$, indicating that M is a symmetric module.

(2): Let $fg = 0$ for all $f, g \in End(M)$ and assume that M is a $*-$symmetric module. Then $fg = 0 \implies 1fg = 0$. Thus, by our assumption, we have $1fg = 0 \implies 1gf^* = 0$. Again, using the condition that M is a $*-$symmetric module, we have $1gf^* = 0$ implies

$1f^*g^* = 0 \implies f^*g^* = 0$. Now, taking an involution on $f^*g^* = 0$, we have $\{f^*g^*\}^* = 0 \implies (g^*)^*(f^*)^* = 0 \implies gf = 0$. Therefore, $fg = 0$ implies $gf = 0$ for all $f, g \in End(M)$. Hence, M is a reversible module. □

Corollary 3.4. *Suppose that M is a $*-$symmetric module and $fgh = 0$. Consequently, we obtain the following implications:*
1) $g^*fh = 0$; 2) $h^*fg = 0$; 3) $ghf^* = 0$.

Remark 3.5. In general, symmetric and reversible module need not be a $*-$symmetric module.

In support of Remark 3.5, we consider the following example.

Example 3.6. In a $\mathbb{Z}-$module $\mathbb{Z}_7$, let $M = \mathbb{Z}_7 \bigoplus \mathbb{Z}_7$. We choose $m = (x, y) \in \mathbb{Z}_7 \bigoplus \mathbb{Z}_7$ such that $m^* = (y, x)$. Let $a = (5, 0), b = (0, 3)$, and $c = (2, 0) \in \mathbb{Z}_7 \bigoplus \mathbb{Z}_7$. Then $abc = (5, 0)(0, 3)(2, 0) = (0, 0) = 0$ and $acb = (5, 0)(2, 0)(0, 3) = (0, 0) = 0$, while $acb^* = (5, 0)(2, 0)(0, 3)^* = (5, 0)(2, 0)(3, 0) = (2, 0) \neq 0$. Hence, $\mathbb{Z}_7 \bigoplus \mathbb{Z}_7$ is a symmetric module, but not a $*-$symmetric module.

Corollary 3.7. *Let M be a $*-$proper module. If M is endowed with the structure of a symmetric module, then M inherits the property of the $*-$symmetric module.*

Proof. We assume that M is a symmetric module and let $fgh = 0$ for any $f, g, h \in End(M)$. Then $fgh = 0$ implies $fhg = 0$. Now, $fhg = 0 \implies fhgg^* = 0 \implies (fh)gg^* = 0$. Since M is a symmetric module, we have $(fh)gg^* = 0 \implies (fh)g^*g = 0$. Next, from $fhg^*g = 0$ we get $h^*f^*fhg^*g = 0 \implies (h^*f^*)(fhg^*)g = 0$. Again, using the condition that M is a symmetric module, we have $(h^*f^*)(fhg^*)g = 0 \implies g(h^*f^*)(fhg^*) = 0$. Taking an involution on $g(h^*f^*)(fhg^*) = 0$, we have $(fhg^*)^*(h^*f^*)^*g^* = 0 \implies (fhg^*)^*(fhg^*) = 0$. As M is a $*-$proper module, we have $(fhg^*)^*(fhg^*) = 0 \implies fhg^* = 0$. Thus, $fgh = 0$ implies $fhg^* = 0$ for each $f, g, h \in End(M)$. Hence, M is a $*-$symmetric module. □

Lemma 3.8. *Let M be a $*-$proper module. If M possesses properties inherent to reversible modules, then it necessarily fulfils the requisite criteria of a $*-$symmetric module.*

Proof. Since every symmetric module is a reversible module and from Corollary 3.7, we have M is a $*-$symmetric module. □

Lemma 3.9. [16] *Let M be a module with an involution $*$. Then M is a $*-$symmetric module if and only if $fgh = 0$ implies $fgh^* = 0$ for any $f, g, h \in End(M)$.*

Theorem 3.10. *Let M be a $*-$symmetric module and $f \in End(M)$. If $f \in I(M)$, then $f^2 = ff^* = f^*f$ for all $f^* \in End(M)$.*

Proof. Assume $f \in I(M)$, then $f^2 = f \implies f(1-f) = 0 \implies 1f(1-f) = 0$. By our hypothesis, we have $1f(1-f) = 0 \implies 1f(1-f)^* = 0$ by Lemma 3.9. This gives $f = ff^*$ and

$$f^2 = ff = f(ff^*) = fff^* = f^2f^* = ff^*,$$

and in a similar manner $f^2 = f^*f$. Hence, $f^2 = ff^* = f^*f$. □

Lemma 3.11. *For a $*-$symmetric module M, the following conditions are equivalent:*

(1) *M is a $*-$cancellable module;*
(2) *$f^2 \in I(M)$ implies $f \in RG(M)$, for each $f \in End(M)$.*

Proof. (1) $\implies$ (2): We assume that M is a $*-$cancellable module and $f^2 \in I(M)$ for each $f \in End(M)$. Then

$$\begin{aligned}
& f^2 = f^4 \\
\implies\ & f^2(1-f^2) = 0 \\
\implies\ & (f(1-f^2))^2 = 0 \\
\implies\ & 1(f(1-f^2))\{(f(1-f^2))\}^* = 0 \quad \text{(By Lemma 3.9)} \\
\implies\ & (f(1-f^2)) = 0 \\
\implies\ & f = f^3 \\
\implies\ & f = ff^2 \\
\implies\ & f = ff^*f \\
\implies\ & f \in RG(M).
\end{aligned}$$

Hence, $f^2 \in I(M)$ implies $f \in RG(M)$ for all $f \in End(M)$.

(2) $\implies$ (1): Assume condition (2) and $ff^* = 0$, for any $f \in End(M)$. Then $f \in RG(M)$. For all $f^* \in End(M)$, $f = ff^*f = 0$. Thus, $ff^* = 0$ implies $f = 0$. Hence, M is a $*-$cancellable module. □

Theorem 3.12. *Let M be a $*-$cancellable module. Then the following conditions are equivalent:*

(1) *M is a $*-$symmetric module;*
(2) *$fgh \in I(M)$ implies $fhg \in I(M)$, for each $f, g, h \in End(M)$;*
(3) *$fgh \in I(M)$ implies $ghf \in I(M)$, for each $f, g, h \in End(M)$;*

(4) *$fgh \in I(M)$ implies $hfg \in I(M)$, for each $f, g, h \in End(M)$;*

(5) *$fgh \in I(M)$ implies $gfh \in I(M)$, for each $f, g, h \in End(M)$.*

Proof. (1) $\implies$ (2): We assume that M is a $*-$symmetric module and $fgh \in I(M)$ for each $f, g, h \in End(M)$. Then $fgh \in I(M) \implies (fgh)^2 = fgh \implies f(1 - ghf)gh = 0 \implies f\{(1-ghf)g\}h = 0$. Since M is a $*-$symmetric module, we have $f\{(1-ghf)g\}h = 0$ implies $fh\{(1-ghf)g\}^* = 0 \implies 1(fh)\{(1-ghf)g\}^* = 0$. Again, using the condition that M is a $*-$symmetric module, we have $1(fh)\{(1-ghf)g\}^* = 0$ implies $1\{(1-ghf)g\}^*(fh)^* = 0 \implies \{(1-ghf)g\}^*(fh)^* = 0$. Now, taking an involution on $\{(1-ghf)g\}^*(fh)^* = 0$, we have

$$
\begin{aligned}
& fh(1-ghf)g = 0 \\
\implies \quad & fhg - fhghfg = 0 \\
\implies \quad & fhg = fhg(hf)g \\
\implies \quad & fhg = fhgfhg \\
\implies \quad & fhg = (fhg)^2 \\
\implies \quad & fhg \in I(M).
\end{aligned}
$$

Thus, $fgh \in I(M)$ implies $fhg \in I(M)$ for all $f, g, h \in End(M)$.

(1) $\implies$ (3): Consider M to be a $*-$symmetric module and $fgh \in I(M)$ for each $f, g, h \in End(M)$. Then

$$
\begin{aligned}
& fgh \in I(M) \\
\implies \quad & (fgh)^2 = fgh \\
\implies \quad & fg(1-hfg)h = 0.
\end{aligned}
$$

Since M is a $*-$symmetric module, we have $f\{g(1-hfg)\}h = 0$ implies $g(1-hfg)hf^* = 0 \implies 1\{g(1-hfg)h\}f^* = 0$. Again, using the condition that M is a $*-$symmetric module, we have $1\{g(1-hfg)h\}f^* = 0$ implies $1f^*\{g(1-hfg)h\}^* = 0 \implies f^*\{g(1-hfg)h\}^* = 0$. Now, taking an involution on $f^*\{g(1-hfg)h\}^* = 0$, we have

$$
\begin{aligned}
& [f^*\{g(1-hfg)h\}^*]^* = 0 \\
\implies \quad & g(1-hfg)hf = 0 \\
\implies \quad & ghf - ghfghf = 0 \\
\implies \quad & ghf = (ghf)^2 \\
\implies \quad & ghf \in I(M).
\end{aligned}
$$

Hence, $fgh \in I(M)$ implies $ghf \in I(M)$ for all $f, g, h \in I(M)$.

(3) $\implies$ (1): Assume that $fgh \in I(M)$ implies $ghf \in I(M)$ for each $f, g, h \in End(M)$. We first show that M is a symmetric module. Let $fgh = 0$ for each $f, g, h \in End(M)$. Then

$$ghf = ghfghf = 0,$$

by the condition, entailing that M is a symmetric module. As M is a symmetric module, we have $fgh = 0 \implies hfg = 0$. Now,

$$\begin{aligned}
& hfg = 0 \\
\implies \quad & h(fg) = 0 \\
\implies \quad & f^*h(fg) = 0 \\
\implies \quad & h(fg)f^* = 0 \qquad \text{(Since } M \text{ is a symmetric module)} \\
\implies \quad & hf^*(fg) = 0 \\
\implies \quad & h(f^*f)g = 0 \\
\implies \quad & gh(f^*f) = 0 \\
\implies \quad & (f^*f)^*(gh)^* = 0 \qquad \text{(Taking an involution on both sides)} \\
\implies \quad & f^*fh^*g^* = 0 \\
\implies \quad & ghf^*fh^*g^* = 0 \\
\implies \quad & (ghf^*)(ghf^*)^* = 0.
\end{aligned}$$

Since M is a $*-$cancellable module, we have $(ghf^*)(ghf^*)^* = 0$ implies $ghf^* = 0$. Thus, $fgh = 0$ implies $ghf^* = 0$ for each $f, g, h \in End(M)$. Hence, M is a $*-$symmetric module.

(1) $\implies$ (4): We assume that M is a $*-$symmetric module and $fgh \in I(M)$ for each $f, g, h \in End(M)$. Then

$$\begin{aligned}
& fgh \in I(M) \\
\implies \quad & (fgh)^2 = fgh \\
\implies \quad & f(1 - ghf)gh = 0 \\
\implies \quad & f\{(1 - ghf)g\}h = 0.
\end{aligned}$$

Since M is a $*-$symmetric module, we have $f\{(1 - ghf)g\}h = 0$ implies $h^*f(1 - ghf)g = 0 \implies h^*1\{f(1 - ghf)g\} = 0$. Again, using the condition that M is a symmetric module, we have $h^*1\{f(1 - ghf)g\} = 0$ implies $\{f(1 - ghf)g\}^*h^*1 = 0 \implies \{f(1 - ghf)g\}^*h^* = 0$. Now, taking an involution on $\{f(1-ghf)g\}^*h^*=0$, we have

$$\begin{aligned} & h\{f(1-ghf)g\}=0 \\ \Longrightarrow\quad & hfg-hfghfg=0 \\ \Longrightarrow\quad & hfg=(hfg)^2 \\ \Longrightarrow\quad & hfg\in I(M). \end{aligned}$$

Thus, $fgh\in I(M)$ implies $hfg\in I(M)$ for all $f,g,h\in End(M)$.

(1) $\Longrightarrow$ (5): We assume that M is a $*-$symmetric module and $fgh\in I(M)$ for each $f,g,h\in End(M)$. Then

$$\begin{aligned} & fgh\in I(M) \\ \Longrightarrow\quad & (fgh)^2=fgh \\ \Longrightarrow\quad & f(1-ghf)gh=0 \\ \Longrightarrow\quad & f\{(1-ghf)gh\}=0. \end{aligned}$$

Now, taking an involution on $f\{(1-ghf)gh\}=0$, we have

$$\begin{aligned} \{(1-ghf)gh\}^*f^* &= 0 \\ \Longrightarrow\ (gh)^*(1-ghf)^*f^* &= 0 \\ \Longrightarrow\ h^*g^*(1-ghf)^*f^* &= 0 \\ \Longrightarrow\ h^*\{g^*(1-ghf)^*\}f^* &= 0. \end{aligned}$$

Since M is a $*-$symmetric module, we have $h^*\{g^*(1-ghf)^*\}f^*=0$ implies

$$\begin{aligned} h^*f^*\{g^*(1-ghf)^*\}^* &= 0 \\ \Longrightarrow\ h^*f^*(1-ghf)g &= 0 \\ \Longrightarrow\ 1(fh)^*(1-ghf)g &= 0 \\ \Longrightarrow\ 1(1-ghf)g\{(fh)^*\}^* &= 0 \\ \Longrightarrow\ 1(1-ghf)g(fh) &= 0 \\ \Longrightarrow\ (1-ghf)g(fh) &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} & (1-ghf)gfh = 0 \\ \implies & gfh - ghfgfh = 0 \\ \implies & gfh = g(hf)gfh \\ \implies & gfh = gfhgfh \\ \implies & gfh = (gfh)^2 \\ \implies & gfh \in I(M). \end{aligned}$$

Hence, $fgh \in I(M)$ implies $gfh \in I(M)$ for all $f, g, h \in End(M)$. □

Remark 3.13. Let M be a $*-$symmetric module. If $fgh \in I(M)$ for each $f, g, h \in End(M)$, then the following equalities hold: $fgh = gfh = fhg = ghf = hfg$.

Theorem 3.14. *For a module M, the following conditions are equivalent:*

(1) *M is a $*-$symmetric module;*
(2) *$fgh \in I(M)$ implies $fhg^* = fhg^*(ghf)^*$, for each $f, g, h \in End(M)$;*
(3) *$fgh \in I(M)$ implies $ghf^* = ghf^*(ghf)^*$, for each $f, g, h \in End(M)$;*
(4) *$fgh \in I(M)$ implies $fhg = fhghfg$, for each $f, g, h \in End(M)$.*

Proof. (1) $\implies$ (2) : We assume M to be a $*-$symmetric module and $fgh \in I(M)$ for each $f, g, h \in End(M)$. Then

$$\begin{aligned} fgh &= (fgh)^2 \\ \implies fgh &= fghfgh \\ \implies fg(1-hfg)h &= 0 \\ \implies fh\{g(1-hfg)\} &= 0 \qquad \text{(By Remark 3.13)} \\ \implies fh\{g(1-hfg)\}^* &= 0 \qquad \text{(Since } M \text{ is a } *-\text{symmetric module)} \\ \implies fh(1-hfg)^*g^* &= 0 \\ \implies fhg^* - fh(hfg)^*g^* &= 0 \\ \implies fhg^* &= fhg^*(ghf)^*. \end{aligned}$$

Thus, $fgh \in I(M)$ implies $fhg^* = fhg^*(ghf)^*$ for all $f, g, h \in End(M)$.

(2) $\implies$ (1): Assume $fgh \in I(M)$ implies $fhg^* = fhg^*(ghf)^*$ for each $f, g, h \in End(M)$. We first show that M is a reversible module. Let $mn = 0$ for each $m, n \in End(M)$. Then

$$nm = nm1^* = nm1^*(1mn)^* = 0,$$

by the condition, entailing that M is a reversible module. Next, we assume that $fgh = 0$ for each $f, g, h \in End(M)$. Since M is a reversible module, we have $f(gh) = 0$ implies $ghf = 0$. Now, $fhg^* = fhg^*(ghf)^* = 0$. Thus, $fgh = 0$ implies $fhg^* = 0$, for any $f, g, h \in End(M)$. Hence, M is a $*-$symmetric module.

(1) $\implies$ (3): Consider M to be a $*-$symmetric module and $fgh \in I(M)$ for each $f, g, h \in End(M)$. Then

$$\begin{aligned}
fgh &= (fgh)^2 \\
\implies fgh &= fghfgh \\
\implies fg(1-hfg)h &= 0 \\
\implies (fg)h\{(1-hfg)\} &= 0 \qquad \text{(By Remark 3.13)} \\
\implies gh(1-hfg)f &= 0 \\
\implies gh(1-ghf)f &= 0 \qquad \text{(By Remark 3.13)} \\
\implies gh\{(1-ghf)f\}^* &= 0 \qquad \text{(Since } M \text{ is a } *-\text{symmetric module)} \\
\implies ghf^*(1-ghf)^* &= 0 \\
\implies ghf^* &= ghf^*(ghf)^*.
\end{aligned}$$

Hence, $fgh \in I(M)$ implies $ghf^* = ghf^*(ghf)^*$ for all $f, g, h \in End(M)$.

(1) $\implies$ (4): We assume $fgh \in I(M)$ for each $f, g, h \in End(M)$. Then

$$\begin{aligned}
fgh &= (fgh)^2 \\
\implies fgh &= fghfgh \\
\implies fg(1-hfg)h &= 0 \\
\implies fh\{g(1-hfg)\} &= 0 \qquad \text{(By Remark 3.13)} \\
\implies fh\{g(1-hfg)\}^* &= 0 \qquad \text{(Since } M \text{ is a } *-\text{symmetric module)} \\
\implies (fh)\{g(1-hfg)\}^*1 &= 0 \\
\implies (fh)1\{g(1-hfg)\} &= 0 \\
\implies fhg(1-hfg) &= 0 \\
\implies fhg &= fhghfg.
\end{aligned}$$

Hence, $fgh \in I(M)$ implies $fhg = fhghfg$ for all $f, g, h \in End(M)$. □

Lemma 3.15. *Given a $*-$cancellable module M of characteristic 2 and an endomorphism f of $End(M)$, if $f^2 \in I(M)$, then f is in $I(M)$.*

Proof. We assume $f^2 \in I(M)$, for each $f \in End(M)$. Then

$$\begin{aligned}
f^2 &= f^4 \\
\Longrightarrow f^2(1-f^2) &= 0 \\
\Longrightarrow f^2(1-f)(1+f) &= 0 \\
\Longrightarrow f^2(1-f)(1-f) &= 0 \quad \text{(Since } M \text{ is of characteristics 2)} \\
\Longrightarrow f^2(1-f)^2 &= 0 \\
\Longrightarrow (f(1-f))^2 &= 0 \\
\Longrightarrow (f(1-f))(f(1-f))^* &= 0 \quad \text{(By Lemma 3.9)} \\
\Longrightarrow (f(1-f)) &= 0 \\
\Longrightarrow f &= f^2 \\
\Longrightarrow f &\in I(M).
\end{aligned}$$

Hence, $f^2 \in I(M)$ implies $f \in I(M)$ for all $f \in End(M)$. □

Theorem 3.16. *Let M be a semiprime module. If M is a $*-$symmetric module, then M possesses the $*-$cancellability property.*

Proof. We assume that M is not a $*-$cancellable module. Then there exists $0 \neq f \in End(M)$, such that $ff^* = 0$. For any $r \in R$, from $ff^* = 0$ we get $ff^*r = 0 \implies frf = 0$, by the condition that M is a $*-$symmetric module. Since M is a semiprime module, then from $frf = 0$ we get $f = 0$, which is a contradiction. Thus, $ff^* = 0$ implies $f = 0$. Hence, M is a $*-$cancellable module. □

Lemma 3.17. *For a module M equipped with an involution $*$, if M satisfies the condition of being $*-$cancellable, it follows that M is a semiprime module.*

Proof. We assume that M is not a semiprime module, then there exists a $0 \neq f \in X = End(M)$ such that $fXf = 0 \implies ff^*f = 0$, for all $f^* \in X$. Now, from $ff^*f = 0$, we get

$$\begin{aligned}
& ff^*ff^* = 0 \\
\Longrightarrow \quad & ff^*(ff^*)^* = 0 \\
\Longrightarrow \quad & ff^* = 0 \qquad \text{(Since } M \text{ is a } *-\text{cancellable module)} \\
\Longrightarrow \quad & f = 0.
\end{aligned}$$

Thus, $fXf = 0$ implies $f = 0$, which is a contradiction. Hence, M is a semiprime module. □

Remark 3.18. [9] Let M be a reduced module. Then

$$D_3(M) = \left\{ \begin{pmatrix} f & g & h \\ 0 & f & i \\ 0 & 0 & f \end{pmatrix} \middle| \; f, g, h, i \in End(M) \right\},$$

is an IFP module.

Theorem 3.19. *Every $*-$symmetric module is an IFP module.*

Proof. We assume that M is a $*-$symmetric module and let $fg = 0$ for any $f, g \in End(M)$. Now, taking an involution on $fg = 0$, we have $g^* f^* = 0$. Then $g^* f^* = 0$ implies $g^* f^* r = 0$, for all $r \in R$. Since M is a $*-$symmetric module, we have $g^* f^* r = 0$ implies $f g^* r = 0 \implies frg = 0$. Thus, $fg = 0$ implies $frg = 0$ for each $f, g \in End(M)$ and $r \in R$. Hence, M is an IFP module. □

Remark 3.20. The converse of Theorem 3.19 generally does not hold. We consider the following example,

Let M be a reduced module. Then

$$D_3(M) = \left\{ \begin{pmatrix} f & g & h \\ 0 & f & i \\ 0 & 0 & f \end{pmatrix} \middle| \; f, g, h, i \in End(M) \right\},$$

is an IFP module by Remark 3.18.

We define an involution $*$ on $D_3(M)$ as

$$\begin{pmatrix} f & g & h \\ 0 & f & i \\ 0 & 0 & f \end{pmatrix}^* = \begin{pmatrix} f & i & -h \\ 0 & f & g \\ 0 & 0 & f \end{pmatrix}.$$

In a $\mathbb{Z}-$module $\mathbb{Z}$, let $\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are three elements of $D_3(\mathbb{Z})$.

Then

$$\alpha\beta\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0,$$

and

$$\begin{aligned}
\alpha\gamma\beta^* &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\neq 0.
\end{aligned}$$

Thus, $D_3(\mathbb{Z})$ is not a $*-$symmetric module.

Lemma 3.21. *Let M be a reduced module. If M is $*-$symmetric, then*

$$D_2(M) = \left\{ \begin{pmatrix} f & g \\ 0 & f \end{pmatrix} \middle| \, f, g \in End(M) \right\},$$

is a $*-$symmetric module, where involution $*$ is defined by $\begin{pmatrix} f & g \\ 0 & f \end{pmatrix}^* = \begin{pmatrix} f^* & g^* \\ 0 & f^* \end{pmatrix}$.

Proof. Let $\alpha = \begin{pmatrix} f_1 & g_1 \\ 0 & f_1 \end{pmatrix}$, $\beta = \begin{pmatrix} f_2 & g_2 \\ 0 & f_2 \end{pmatrix}$ and $\gamma = \begin{pmatrix} f_3 & g_3 \\ 0 & f_3 \end{pmatrix} \in D_2(M)$.

Assume

$$\begin{aligned}
\alpha\beta\gamma = 0 &\implies \begin{pmatrix} f_1 & g_1 \\ 0 & f_1 \end{pmatrix} \begin{pmatrix} f_2 & g_2 \\ 0 & f_2 \end{pmatrix} \begin{pmatrix} f_3 & g_3 \\ 0 & f_3 \end{pmatrix} = 0 \\
&\implies \begin{pmatrix} f_1f_2 & f_1g_2 + g_1f_2 \\ 0 & f_1f_2 \end{pmatrix} \begin{pmatrix} f_3 & g_3 \\ 0 & f_3 \end{pmatrix} = 0 \\
&\implies \begin{pmatrix} f_1f_2f_3 & f_1f_2g_3 + f_1g_2f_3 + g_1f_2f_3 \\ 0 & f_1f_2f_3 \end{pmatrix} = 0.
\end{aligned}$$

Then $f_1f_2f_3 = 0 = f_1f_2g_3 + f_1g_2f_3 + g_1f_2f_3$. If $f_1 = 0$ then $g_1f_2f_3 = 0$, if $f_2 = 0$ then $f_1g_2f_3 = 0$, and $f_3 = 0$ then $f_1f_2g_3 = 0$. Since M is a $*-$symmetric module, from $f_1f_2g_3 + f_1g_2f_3 + g_1f_2f_3 = 0$, we get $f_1g_3f_2^* + f_1f_3g_2^* + g_1f_3f_2^* = 0$ and from $f_1f_2f_3 = 0$, we get $f_1f_3f_2^* = 0$.

Now,

$$\begin{aligned}
\alpha\gamma\beta^* &= \begin{pmatrix} f_1 & g_1 \\ 0 & f_1 \end{pmatrix} \begin{pmatrix} f_3 & g_3 \\ 0 & f_3 \end{pmatrix} \begin{pmatrix} f_2 & g_2 \\ 0 & f_2 \end{pmatrix}^* \\
&= \begin{pmatrix} f_1 & g_1 \\ 0 & f_1 \end{pmatrix} \begin{pmatrix} f_3 & g_3 \\ 0 & f_3 \end{pmatrix} \begin{pmatrix} f_2^* & g_2^* \\ 0 & f_2^* \end{pmatrix} \\
&= \begin{pmatrix} f_1 f_3 f_2^* & f_1 g_3 f_2^* + f_1 f_3 g_2^* + g_1 f_3 f_2^* \\ 0 & f_1 f_3 f_2^* \end{pmatrix} \\
&= 0.
\end{aligned}$$

Thus, $\alpha\beta\gamma = 0$ implies $\alpha\gamma\beta^* = 0$ for all $\alpha, \beta, \gamma \in D_2(M)$. Hence, $D_2(M)$ is a $*-$symmetric module. □

Remark 3.22. Considering Lemma 3.21, we conjecture that if a module M is not a $*-$symmetric module, then

$$D_2(M) = \left\{ \begin{pmatrix} f & g \\ 0 & f \end{pmatrix} \middle| \, f, g \in End(M) \right\},$$

is not a $*-$symmetric module.

Lemma 3.23. *Let M be a reduced module. Then*

$$D_n(M) = \left\{ \begin{pmatrix} f & f_{12} & f_{13} & \cdots & f_{1n} \\ 0 & f & f_{23} & \cdots & f_{2n} \\ 0 & 0 & f & \cdots & f_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f \end{pmatrix} \middle| \, f, f_{ij} \in End(M) \; (i < j) \right\},$$

is not a $*-$symmetric module for $n \geq 3$.

Proof. It is easily followed from Remark 3.20. □

Remark 3.24. In general, 2×2 upper triangular matrix that is

$$U_2(M) = \left\{ \begin{pmatrix} f & g \\ 0 & h \end{pmatrix} \middle| \, f \neq h \neq g \in End(M) \right\},$$

is not a $*-$symmetric module.

In support of Remark 3.24, we consider the following example.

Example 3.25. We define an involution $*$ on $U_2(M)$ as $\begin{pmatrix} m & p \\ 0 & n \end{pmatrix}^* = \begin{pmatrix} n & p \\ 0 & m \end{pmatrix}$ for all $m, n, p \in End(M)$.

In a $\mathbb{Z}-$module $\mathbb{Z}$, we choose $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \in U_2(\mathbb{Z})$.

Then $ABC = 0$ but $ACB^* = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$. Thus, $U_2(\mathbb{Z})$ is not a $*-$symmetric module.

Remark 3.26. In general, a reduced module is not necessarily a $*-$symmetric module.

The assertion posited in Remark 3.26 is corroborated by an analysis of Example 3.6. Since $\mathbb{Z}_7 \bigoplus \mathbb{Z}_7$ forms a reduced module within the context of a $\mathbb{Z}-$module.

Corollary 3.27. *If M is a $*-$cancellable module, assuming that it also satisfies the condition of $*-$symmetry, then M fulfils the criteria for a reduced module.*

Proof. We assume that M is a $*-$cancellable module and $f^2 = 0$, for all $f \in End(M)$. Since M is a $*-$symmetric module, from $f^2 = 0$ we get $ff1 = 0 \implies f1f^* = 0 \implies ff^* = 0$. Now, from $ff^* = 0$, we have $f = 0$ by the condition that M is a $*-$cancellable module. Thus, $f^2 = 0$ implies $f = 0$ for each $f \in End(M)$. Hence, M is a reduced module. □

Corollary 3.28. *Let M be a semiprime module. If M is a $*-$symmetric module, then it necessarily follows that M is a reduced module.*

Proof. We assume that M is not a reduced module. Then there exists $0 \neq f \in End(M)$ such that $f^2 = 0$. For any $r \in R$, from $f^2 = 0$ we get $ffr = 0 \implies frf^* = 0 \implies 1(fr)f^* = 0 \implies 1f^*(fr)^* = 0 \implies f^*(fr)^* = 0$ by the condition that M is a $*-$symmetric module. Now, taking an involution on $f^*(fr)^* = 0$, we get $frf = 0$. Since M is a semiprime module, from $frf = 0$ we get $f = 0$, which is a contradiction. Hence, M is a reduced module. □

Lemma 3.29. *Let M be a $*-$proper module. Assuming M satisfies the $*-$symmetry property, it follows that M is a reduced module.*

Theorem 3.30. *Let I be a finite index set and $\{M_i\}_{i\in I}$ be a class of modules. If M_i is a $*-$symmetric module, for all $i \in I$, then*

$$\bigoplus_{i\in I} M_i,$$

is a $*-$symmetric module.

Proof. We assume that M_i is a $*-$symmetric module for all $i \in I$ and $(f_i)_{i\in I}, (g_i)_{i\in I}, (h_i)_{i\in I} \in End(\bigoplus_{i\in I} M_i)$ with $(f_i)(g_i)(h_i) = 0$. Then, by our assumption, we have $f_i g_i h_i = 0$ implies $f_i h_i g_i^* = 0$. Thus, from $f_i h_i g_i^* = 0$ we get $(f_i)(h_i)(g_i)^* = 0$. Hence, $\bigoplus_{i\in I} M_i$ is a $*-$symmetric module. □

Lemma 3.31. *For an abelian module M, M admits $*-$symmetry if and only if, for every idempotent e of $End(M)$, the submodules eM and $(1-e)M$ are $*-$symmetric.*

Proof. We assume that M is a $*-$symmetric module and $efgh = 0$, for any $f, g, h \in End(M)$. Since M is a $*-$symmetric module, from $(ef)gh = 0$ we get $(ef)hg^* = 0 \implies efhg^* = 0$. Thus, $efgh = 0$ implies $efhg^* = 0$. Hence, eM is a $*-$symmetric module. Similarly, $(1-e)M$ is also a $*-$symmetric module.

Conversely, we assume eM and $(1-e)M$ are $*-$symmetric modules and $fgh = 0$, for any $f, g, h \in End(M)$. Then $efgh = 0$ and $(1-e)fgh = 0$. Since eM and $(1-e)M$ are $*-$symmetric modules, from $efgh = 0$ and $(1-e)fgh = 0$, we have $efhg^* = 0$ and $(1-e)fhg^* = 0$, respectively. Now, $fhg^* = efhg^* + (1-e)fhg^* \implies fhg^* = 0$. Thus, $fgh = 0$ implies $fhg^* = 0$. Hence, M is a $*-$symmetric module. □

Lemma 3.32. *Let M be a module endowed with an involution $*$. If M satisfies the condition of being $*-$symmetric, it necessarily follows that M is an abelian module.*

Proof. It follows from Theorem 3.10. □

Remark 3.33. In general, an abelian module does not necessarily possess the property of $*-$symmetry.

In support of Remark 3.33, we take the following example.

Example 3.34. The $\mathbb{Z}-$module $\mathbb{Z}_7$, let $M = \mathbb{Z}_7 \bigoplus \mathbb{Z}_7$, which is an abelian module. Taking $t = (x, y) \in \mathbb{Z}_7 \bigoplus \mathbb{Z}_7$ with $t^* = (y, x)$. Consider $a = (2, 0), b = (0, 2)$, and $c = (1, 0) \in \mathbb{Z}_7 \bigoplus \mathbb{Z}_7$. Then $abc = (2, 0)(0, 2)(1, 0) = (0, 0) = 0$, while $acb^* = (2, 0)(1, 0)(0, 2)^* = (2, 0)(1, 0)(2, 0) = (4, 0) \neq 0$. Hence, $\mathbb{Z}_7 \bigoplus \mathbb{Z}_7$ is not a $*-$symmetric module.

Theorem 3.35. *Let M be a $*-$proper module. Then the following conditions are equivalent:*

(1) *M is a $*-$symmetric module;*
(2) *M is a reduced module;*
(3) *M is a symmetric module;*
(4) *M is a reversible module.*

Proof. (1) $\implies$ (2): It is easily followed from Lemma 3.29.

(2) $\implies$ (3) : We assume that M is a reduced module and $fgh = 0$ for each $f, g, h \in End(M)$. Now, $(ghf)^2 = ghfghf = 0$. Since M is a reduced module, we have $(ghf)^2 = 0$ implies $ghf = 0$. Thus, $fgh = 0$ implies $ghf = 0$ for each $f, g, h \in End(M)$, entailing that M is a symmetric module.

(3) $\implies$ (4): Let $fg = 0$ for all $f, g \in End(M)$ and assume that M is a symmetric module. Then $fg1 = 0$. By our assumption, we have $fg1 = 0 \implies gf1 = 0$. Thus, $gf1 = 0 \implies gf = 0$. Hence, M is a reversible module.

(4) $\implies$ (1): It follows from Lemma 3.8. □

Theorem 3.36. *For a module M and $X = End(M)$, the following conditions are equivalent:*

(1) *M is a $*-$symmetric module;*
(2) *$r_X(f^*) = r_X(f)$, for any $f \in End(M)$;*
(3) *$l_X(f^*) = l_X(f)$, for any $f \in End(M)$;*
(4) *$l_X(f^*) = r_X(f)$, for any $f \in End(M)$;*
(5) *$r_X(f^*) = l_X(f)$, for any $f \in End(M)$.*

Proof. (1) $\implies$ (2): We assume that M is a $*-$symmetric module and let $g \in r_X(f^*)$. Then $f^*g = 0$. Thus, by our assumption, we have $1f^*g = 0 \implies 1gf = 0 \implies gf = 0$. Now, taking an involution on $gf = 0$, we have $f^*g^* = 0 \implies 1f^*g^* = 0$. Again, by our assumption, we have $1f^*g^* = 0 \implies 1g^*f = 0 \implies 1fg = 0 \implies fg = 0$. Hence, $g \in r_X(f)$ and vice versa. Therefore, $r_X(f^*) = r_X(f)$, for any $f \in End(M)$.

(2) $\implies$ (1): Assume the condition (2). Then $fg = 0 \implies 1fg = 0$ implies $f^*g = 0$, for any $g \in r_X(f^*) = r_X(f)$. Now, taking an involution on $f^*g = 0$, we have $g^*f = 0 \implies g^*1f = 0$. Thus, $1fg = 0$ implies $g^*1f = 0$ and vice versa. Hence, M is a $*-$symmetric module.

(1) $\implies$ (3): Let M be a $*-$symmetric module and $h \in l_X(f^*)$. Then $hf^* = 0$. Since M is a $*-$symmetric module, we have $1hf^* = 0 \implies 1f^*h^* = 0 \implies f^*h^* = 0$. Now, taking an involution on $f^*h^* = 0$, we have $hf = 0$. Thus, $h \in l_X(f)$ and vice versa. Hence, $l_X(f^*) = l_X(f)$ for any $f \in End(M)$.

(1) $\implies$ (4): We assume that M is a $*-$symmetric module. Now, for each $g \in l_X(f^*)$, we have $gf^* = 0$. Then, taking an involution on $gf^* = 0$, we have $fg^* = 0 \implies fg^*1 = 0$. Thus, by our assumption, we have $fg^*1 = 0 \implies f1g = 0 \implies fg = 0$. Hence, $g \in r_X(f)$ and vice versa. Therefore, $l_X(f^*) = r_X(f)$ for any $f \in End(M)$.

(1) $\implies$ (5): Take M is a $*-$symmetric module. For $g \in r_X(f^*)$, we have $f^*g = 0 \implies 1f^*g = 0$. Since M is a $*-$symmetric module, we have $1f^*g = 0 \implies 1gf = 0 \implies gf = 0$. Thus, $g \in l_X(f)$ and vice versa. Hence, $r_X(f^*) = l_X(f)$ for any $f \in End(M)$. □

4. Conclusion

In this paper, we have defined the concept of $*-$symmetric modules and reveal properties related to reduced module, symmetric module, IFP module and reversible module are established. The notions of central symmetric module, $*-$reduced module, and $e-$symmetric module may be studied in our future work.

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