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Research Paper

HYPEGROUPS OBTAINED FROM BCK-ALGEBRAS WITH CONDITION (S)

SAEED MIRVAKILI*, TAYEBE GOLCHIN AND MOHAMMAD HAMIDI

Abstract. In this paper, we construct a novel hyperoperation derived from a (BCI-algebra)BCK-algebra with condition (S) and present several significant findings. We establish that this construction leads to the formation of commutative hypergroups and join spaces. Furthermore, we demonstrate that when the BCK-algebra satisfies condition (S), the resulting hyperstructure constitutes H_b-monoids. Moreover, we show that every proper subhypergroup of this hypergroup cannot be closed, invertible, ultraclosed and conjugable. Finally, we have derived all the hypergroups obtained from BCK-algebras with condition (S) of order less than 5.

1. Introduction

In 1934, Frederick Marty introduced the concept of hyperoperations and defined hypergroups, employing the axioms of associativity and reproductivity.

The connection between hyperstructures and ordered (semi)groups has been studied by several mathematicians [3, 4, 9, 10, 22]. Some hyperstructures determined by binary relations

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*Corresponding author

are known as EL-hyperstructures. This concept was first introduced by Chvalina [2] when she investigated quasi-ordered sets and hypergroups. Also, Rosenberg in [22], Hoskova in [11], and Novak in [19, 20, 21] studied some results on the ordered semigroups and ordered groups connected with EL-hyperstructures. Ghazavi et al. introduced a new class of EL-hyperstructures called EL^2-hyperstrucutres in [8]. EL^2-hyperstructures are hyperstructures based on (partially) quasi ordered (semi)hypergroup instead of a (partially) quasi ordered (semi)group.

BCI-algebras with condition (S) represent a significant category of BCI-algebras that are closely linked to the theory of pomonoids, playing a vital role in the advancement of BCI-algebra theory. The concept of BCK-algebras with condition (S) was initially proposed by K. Iséki in 1977 [13], and he later extended this idea to BCI-algebras in 1980. Since then, these algebras have been the focus of extensive study by various researchers[12, 14]. In recent years, in the field of the relationship between hyperstructures and BCI-algebras and P-semisimple algebar, one can refer to references [1, 17]. Mirvkili et al. have recently studied fuzzy ideals in the BCI-algebra[16].

In this paper, we will investigate the relationships between BCK-algebras with condition (S) and hypergroups and H_b-monoids. We construct a novel hyperoperation derived from a BCK-algebra under the condition (S), leading to significant findings. Our construction results in the formation of commutative hypergroups and join spaces. We have provided examples to better illustrate the concepts discussed in this paper. Furthermore, we prove that if the BCK-algebra satisfies condition (S), then the induced hyperstructure is both a commutative hypergroup and an H_b-monoid. We also show that any proper subhypergroup of this hypergroup fails to be closed, invertible, ultraclosed, or conjugable.

2. Basic Concepts

A hypergroupoid $(H, \circ)$ is a non-empty set H together with a map $\circ : H \times H \to P^*(H)$ called (binary) hyperoperation, where $P^*(H)$ denotes the set of all non-empty subsets of H. The image of the pair (x, y) is denoted by $x \circ y$.

If A and B are non-empty subsets of H and $x \in H$, then by $A \circ B$, $A \circ x$, and $x \circ B$ we mean $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$.

Definition 2.1. ([7, 6]) Let $\circ : H \times H \to P^*(H)$ be a hyperoperation. Then

(1) A hypergroupoid $(H, \circ)$ is called a semihypergroup if the associativity law of $\circ$ holds: $a \circ (b \circ c) = (a \circ b) \circ c$, for all $a, b, c \in H$.

(2) A hypergroupoid $(H, \circ)$ is called a commutative if it satisfies the following: $a \circ b = b \circ a$, for all $a, b \in H$.

(3) A semihypergroup $(H, \circ)$ is called a Hypergroup if it satisfies the following:

The condition $a \circ H = H \circ a = H$ for all $a \in H$, is called the reproduction axiom.

(4) A hypergroup is called regular if it has at least one identity and each element has at least one inverse.

(5) A hypermonoid $(H, \circ)$ is called H_b-monoid if there exists a monoid $(H, \cdot)$ such that for every $x, y \in H$, $x \cdot y \in x \circ y$.

(6) A non-empty subset K of a hypergroup $(H, \circ)$ is called a subhypergroup if $(K, \circ)$ is a hypergroup.

(7) The subhypergroup $(K, \circ)$ of hypergroup $(H, \circ)$ is called closed if for all x, y of K and z of H, from $x \in (z \circ y)$ (or $x \in (y \circ z)$ it follows that $z \in K$.

(8) The subhypergroup $(K, \circ)$ of hypergroup $(H, \circ)$ is called invertible if for all x, y of H, from $x \in (K \circ y) \cap (y \circ K)$, it follows that $y \in (K \circ x) \cap (x \circ K)$.

(9) The subhypergroup $(K, \circ)$ of hypergroup $(H, \circ)$ is called ultraclosed if for x of H, we have $(K \circ x) \cap (H \setminus K \circ x) = 0$ and $(x \circ K) \cap x \circ (H \setminus K) = 0$.

(10) The subhypergroup $(K, \circ)$ of hypergroup $(H, \circ)$ is called conjugable if it is closed nand for all $x \in H$, there exists $y \in H$ such that $(y \circ x) \subseteq K$ and $(x \circ y) \subseteq K$.

Definition 2.2. [4] Let $(H, \circ)$ be a commutative hypergroup. Set $a/b := \{\, x \mid a \in x \circ b \,\}$. The hypergroup is called a *join space* if for all $a, b, c, d \in H$ we have

$$b/a \ \cap \ c/d \neq \varnothing \implies (a \circ d) \cap (b \circ c) \neq \varnothing.$$

In the next theorem, the relationship between the types of subhypergroups is stated.

Theorem 2.3. [5] *Let K be a subhypergroup of a hypergroup $(H, \circ)$.*

(1) *If K is ultraclosed, then it is closed and invertible.*

(2) *If K is invertible, then it is closed.*

(3) *If K is conjugable, then it is ultraclosed.*

We briefly recall the following definitions and results from References [12, 14].

Definition 2.4. (BCI**-Algebra**) A BCI*-algebra* is a structure $(X, *, 0)$ where X is a non-empty set, $* : X \times X \to X$ is a binary operation, and $0 \in X$ satisfies the following axioms for any $x, y, z \in X$,

(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$.

(BCI-2) $x * 0 = x$.

(BCI-3) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We often write X instead $(X, *, 0)$ for a BCI-algebra in brevity.

Suppose that $(X, *, 0)$ is a BCI-algebra . Define binary relation $\leq$ by which $x \leq y$ if and only if $x * y = 0$, for any $x, y \in X$. Then $(X, \leq)$ is a partially ordered set with 0 as a minimal element.

Below we bring some important properties of BCI-algebras. The reader is referred to Book [12] for more properties.

Let $(X, *, 0)$ be a BCI-algebra.

(I) For all $x \in X$, $x * x = 0$.

(II) For all $x, y, z \in X$, $(x * y) * z = (x * z) * y$. For convenience, we call it the head-fixed commutative law.

(III) For all $x \in X$, $x * (x * (x * y)) = x * y$.

(IV) For all $x \in X$, $0 * (x * y) = (0 * x) * (0 * y)$.

Minimal elements play an important role in the research of BCI-algebras and BCK-algebras. An element a in a BCI-algebra X is called minimal if $x * a = 0$ (i.e., $x \leq a$) implies $x = a$ for any $x \in X$. It is called least if $a * x = 0$ (i.e., $a \leq x$) for all $x \in X$.

Given a minimal element a of a BCI-algebra X, the set

$$V(a) = \{x \in X | x \geq a\} (\text{ i.e.,} V(a) = \{x \in X | a * x = 0\}),$$

is called the branch of X generated by a. Obviously, the BCK-part B of X is just the branch $V(0)$. We explain that in order to avoid making confusion, if a is not a minimal element of X, we do not adopt the symbol $V(a)$ to denote a branch of X.

Let x and y be two elements in a BCI-algebra X. Consider

$$A(x, y) = \{t \in X | t * x \leq y\}.$$

Let X be a BCI-algebra. Set

$$a = 0 * ((0 * x) * y).$$

Since

$$a * x = (0 * ((0 * x) * y)) * x = (0 * x) * ((0 * x) * y) \leq y,$$

it follows that $a \in A(x, y)$. Furthermore, the inequality $a * x \leq y$ implies

$0 * a = ((a * x) * y) * a = (0 * x) * y$, by the head-fixed commutative law.

Now let $t \in A(x, y)$. Since $t * x \leq y$,

$$(t * x) * y = 0.$$

Applying the head-fixed commutative law yields

$$0 * t = ((t * x) * y) * t = (0 * x) * y.$$

Consequently,

$$0 * a = 0 * t.$$

Moreover, since a is a minimal element of X,

$$a * t = (0 * t) * (0 * a) = 0.$$

Hence, for every $t \in A(x,y)$, $a \leq t$ and so a is the least element of $A(x,y)$. Moreover, $t \in V(a)$ and therefore

$$A(x,y) \subseteq V(a),$$

where $V(a)$ is a branch of X.

Therefore, the above arguments establish that

Proposition 2.5. [12, 14] *Let $(X,*)$ be a BCI-algebra. Set $a = 0*((0*x)*y)$. Then a is the least element of $A(x,y)$ and $A(x,y) \subseteq V(a)$. Especially, if B is the BCK-part of X, then $\{0,x,y\} \subseteq A(x,y) \subseteq B$ for any $x,y \in B$.*

Definition 2.6. [12, 14](**BCK-Algebra**) A *BCK-algebra* is a structure $(X,*,0)$, which is a BCI-algebra, that satisfies the additional condition:

(BCK-1) For all $x \in X$, we have $0*x = 0$.

Proposition 2.7. [12, 14] *Let X be a BCK-algebra. Then the following hold: for any $x,y \in X$,*

(1) *$(x*y)*x = 0$, that is, $x*y \leq x$.*
(2) *$x*(x*y) = 0$ if and only if $x = x*y$.*

Definition 2.8. [12, 14](**BCI-Algebra with Condition** (S)) A *BCI-algebra with condition* (S) is a BCI-algebra X such that set $A(x,y) = \{t \in X | t*x \leq y\}$ has the greatest element for all $x,y \in X$.

In similar fashion, we can define the notion of **BCK-algebras with condition** (S).

Proposition 2.9. [14] *Let x and y be two elements in a BCI-algebra X. Denote $a = 0*((0*x)*y)$. Then a is the least element of $A(x,y)$. If X is a BCK-algebra then $a = 0$. Moreover, $A(x,y)$ has the greatest element usually denoted by $x \circ yv$ and for every $x,y \in X$ we have the following two conditions:*

(1) *$(x \circ y)*x \leq y$;*
(2) *$t*x \leq y$ implies $t \leq x \circ y$ for any $t \in X$.*

Theorem 2.10. [12] *If X is a BCI-algebra with condition (S), then $(X,\circ,0)$ is a commutative monoid with 0 as the unit element, satisfying the isotonic property, where the operation $\circ$ on X is defined by*

$$x \circ y = \sup\{t \in X | t*x \leq y\}.$$

The following establishes a connection between $*$ and $\circ$.

Proposition 2.11. [12] *Let X be a BCI-algebra with condition (S). Then*

$$(x*y)*z = x*(y \circ z),$$

for any $x,y,z \in X$.

3. Construction hypergroups from BCK-algebras with condition (S)

In this section, we first construct a novel hyperoperation using a BCK-algebra equipped with condition (S). This construction leads to significant findings; in particular, the resulting hypergroups are commutative and form join spaces. We also present some examples.

We define a hyperoperation $\bullet$ on BCK algebra X (with condition (S)) as follows:

$$x \bullet y = \{t \in X | t \leq x \circ y\},$$

for any $x, y \in X$.

Lemma 3.1. *Let X be a BCK-algebra. Then $0 \in A(x, y) \subseteq x \bullet y$.*

Proof. Let $t \in A(x, y)$ then $t * x \leq y$. By (2) we have $t \leq x \circ y$. Then $x \in x \bullet y$. Moreover, $0 \leq x \circ y$ if and only if $0 * x \leq y$. Then $0 \in A(x, y) \subseteq x \bullet y$. □

Theorem 3.2. *If X is a BCK-algebra with condition (S), then for every $x, y \in X$ we have $\{0, x, y\} \subseteq x \bullet y$.*

Proof. By the Lemma 3.1, $0 \in x \bullet y$. For every $x, y \in X$, $x * x = 0 \leq y$ and by part (2) of Proposition 2.9, $x \leq x \circ y$. This implies $x \in x \bullet y$. Moreover, in BCK-algebra with condition (S), $x * y \leq z$ if and only if $x \leq y \circ z$. Hence, by head-fixed commutative law, $0 = y * y \leq x$ implies $y * x \leq y$ and so $y \leq x \circ y$. and Therefore $y \in x \bullet y$. □

For a BCI-algebra with condition (S), For every $x, y \in X$, in Theorem 3.2, $0, x \in x \bullet y$ is not true

Example 3.3. Let $X = \{0, 1, 2, a, b\}$ be a BCI-algebra with the operation $*$ on X by:

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	b	a	2	0

$\circ$	0	1	2	a	b
0	0	1	2	a	b
1	1	1	2	a	b
2	2	2	2	b	b
a	a	a	b	2	2
b	b	b	b	2	2

$$\begin{tikzcd} 2 \arrow[d, dash] & \\ 1 \arrow[d, dash] & b \arrow[d, dash] \\ 0 & a \end{tikzcd}$$

Then we have the hypergroupoid $(X, \bullet)$ as follows:

$\bullet$	0	1	2	a	b
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\{a\}$	$\{a,b\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{a\}$	$\{a,b\}$
2	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{a,b\}$	$\{a,b\}$
a	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{0,1,2\}$	$\{0,1,2\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{0,1,2\}$	$\{0,1,2\}$

It is clear that $\{0,2\} \cap 2 \bullet a = \emptyset$ and $\{0,2\} \cap a \bullet 2 = \emptyset$.

Theorem 3.4. *Let X be a BCK-algebra with condition (S). Then $(X, \bullet)$ is a hypergroup.*

Proof. Let $x, y, z \in X$. By Theorem 2.10, the binary operation $\circ$ is associative and so $x \circ (y \circ z) = (x \circ y) \circ z$. Now, we have

$$\begin{aligned}
x \bullet (y \bullet z) &= \bigcup_{u \in y \bullet z} x \bullet u \\
&= \bigcup_{u \in X, u \leq y \circ z} \{v \in X | v \leq x \circ u\} \\
&= \{v \in X | u \in X, u \leq y \circ z \& v \leq x \circ u\} \\
&= \{v \in X | v \leq x \circ (y \circ z)\} \\
&= \{v \in X | v \leq (x \circ y) \circ z\} \\
&= \{v \in X | w \in X, w \leq x \circ y \& v \leq w \circ z\} \\
&= \bigcup_{w \in X, w \leq x \circ y} \{v \in X | v \leq w \circ z\} \\
&= \bigcup_{w \in x \bullet y} w \bullet z \\
&= (x \bullet y) \bullet z.
\end{aligned}$$

Therefore the hyperoperation $\bullet$ is associative.

For every $a, x \in X$, we have $x \in x \bullet a$ and $x \in a \bullet x$. It follows that $X \bullet a = X = a \bullet X$, and hence the reproduction axiom holds. $\square$

Corollary 3.5. *Let X be a BCK-algebra with condition (S). Then $(X, \bullet)$ is a commutative hypergroup.*

Proof. Let $x, y \in X$, then

$$\begin{aligned} t \in x \bullet y &\Leftrightarrow t \leq x \circ y \\ &\Leftrightarrow t * x \leq y \text{ (head-fixed commutative law)} \\ &\Leftrightarrow t * y \leq x \\ &\Leftrightarrow t \leq y \circ x \\ &\Leftrightarrow t \in y \bullet x. \end{aligned}$$

Therefore $x \bullet y = y \bullet x$. □

Theorem 3.6. *Let X be a BCK-algebra with condition (S). Then $(X, \bullet)$ is a commutative H_b-monoid.*

Proof. By Theorem 2.10, the structure $(X, \circ)$ is a commutative monoid and $x \circ y \in x \bullet y$ for all $x, y \in X$. Since $x \in 0 \bullet x \cap x \bullet 0$ so $(X, \bullet)$ has a identity 0. By Definition 2.1 part (7), Theorem 3.4 we have $(X, \bullet)$ is a commutative H_b-monoid. □

Theorem 3.7. *Let X be a BCK-algebra with condition (S). Then $(X, \bullet)$ is a join space.*

Proof. Let $a, b, c, d \in X$ and $a/b \approx c/d$. Then $0 \in a \bullet d \cap b \bullet c$ and proof is complete. □

Theorem 3.8. *Let X be a BCK-algebra with condition (S). Then $A(x, y) = x \bullet y$.*

Proof. By Lemma 3.1, we have $A(x, y) \subseteq x \bullet y$. Now, let $t \leq x \circ y$. so $t * x \leq y$. Then $t \in A(x, y)$.
□

Applying the construction described above to several well-known BCK-algebras, we explicitly calculate the induced hyperoperations in the examples below.

Example 3.9. Let $X = \{0, 1, 2, 3, 4\}$ and X is a BCK-algebra with the operation $*$ on X by:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	3	3	1	0

then we have the commutative hypergroup $(X, \bullet)$ as follows:

$\bullet$	0	1	2	3	4
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,3\}$	$\{0,1,2,3,4\}$
1	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
2	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2.3.4\}$	$\{0,1,2,3,4\}$
3	$\{0,1,3\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$
4	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$

Example 3.10. Let $(X, \leq)$ be a partially ordered set and 0 be the least element of $(X, \leq)$. Define the operation $*$ on X by

$$x * y = \begin{cases} 0, & \text{if } \ x \leq y, \\ x, & \text{if } x \not\leq y. \end{cases}$$

By Example 1.1.3 of [12] we know $(X, *, 0)$ is a BCK-algebra. Now, by Theorem 3.8, we have

$$\begin{aligned} x \bullet y &= \{t \in X | t * x \leq y\} \\ &= (\{t \in X | t * x = 0\} \cup \{t \in X | t * x = t \leq y\}) \\ &= (\{t \in X | t \leq x\} \cup \{t \in X | t \not\leq x \ \& \ t \leq y\}) \\ &= \{t \in X | t \leq x \vee y\}. \end{aligned}$$

Example 3.11. The set $X = \mathbb{N}$ of nonnegative integers with binary operation $*$ on X by

$$x * y = \max\{0, x - y\},$$

forms a BCK-algebra (By Example 1.2.1 of [12]). Now, we have

$$\begin{aligned} x \bullet y &= \{t \in X | t \leq x \circ y\} \\ &= \{t \in X | \max\{0, t - x\} \leq y\} \\ &= (\{t \in X | t \leq x\} \cup \{t \in X | t \not\leq x \ \& \ t - x \leq y\}) \\ &= \{0, 1, 2, \ldots, x + y\}. \end{aligned}$$

Theorem 3.12. *Let X be a BCK-algebra with condition (S). Then*

(1) *Every element of $(X, \bullet)$ is an identity element.*
(2) *The hypergroup $(X, \bullet)$ is regular.*
(3) *The hypergroup $(X, \bullet)$ has a scalar identity if and only if $|X| = 1$.*
(4) *The hypergroup $(X, \bullet)$ is a (poly)group if and only if $|X| = 1$.*
(5) *The fundamental group X/β^* is trivial; that is, $X/\beta^* \cong \{0\}$.*

Proof. (1) For every $a \in X$ we have $x \in A(x,a) \subseteq x \bullet a = a \bullet x$, for all $x \in X$. Then a is an identity element.

(2) By the Theorem 3.12 we have 0 is an identity element. Moreover for every $x, y \in X$ we have $0 \in x \bullet y$ and so y is an inverse of x. Then $(X, \bullet)$ is a regular hypergroup.

(3) If $x \in X$ and $x \neq 0$ then for every $y \in X$, $\{0, x\} \in x \bullet y$. Then $x \bullet y$ is not singleton. Therefore, X does not admit a scalar identity.

(4) It obtains from part (3).

(5) Since 0 belongs to every finite product of elements in the hypergroup $(X, \bullet)$, then we have one equivalence class. Then $X/\beta^* \cong \{0\}$. □

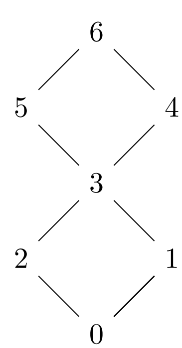
Theorem 3.13. *Let A be a subhypergroup of X with $A \neq X$. Then A cannot be closed, ultraclosed, invertible, or conjugable.*

Proof. If A is a subhypergroup of X and $A \neq X$, then for every $x \in X \setminus A$ we have $0 \in 0 \bullet x \cap x \bullet 0$ but $x \notin A$. Hence A is not closed. By Theorem 2.3, A is not ultraclosed, not invertible, and not conjugable. □

Example 3.14. In the tables below, the Cayley table of the BCK-algebra $(X, *)$ with condition (S) is presented, and based on it, the Cayley tables of the monoid $(X, \circ)$ and hypergroup $(X, \bullet)$ are derived and included. Let $X = \{0, 1, \ldots, 6\}$.

$*$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0
2	2	2	0	0	0	0	0
3	3	2	1	0	0	0	0
4	4	4	4	4	0	4	0
5	5	5	5	5	5	0	0
6	6	6	6	6	5	4	0

$\circ$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	3	3	4	5	6
2	2	3	2	3	4	5	6
3	3	3	3	3	4	5	6
4	4	4	4	4	4	5	6
5	5	5	5	5	6	5	6
6	6	6	6	6	6	6	6



$\bullet$	0	1	2	3	4	5	6
0	$\{0\}$	$\{0,1\}$	$\{0,2\}$	Y	Z	W	X
1	$\{0,1\}$	$\{0,1\}$	Y	Y	Z	W	X
2	$\{0,2\}$	Y	$\{0,2\}$	Y	Z	W	X
3	Y	Y	Y	Y	Z	W	X
4	Z	Z	Z	Z	Z	X	X
5	W	W	W	W	X	W	X
6	X	X	X	X	X	X	X

Where $Y = \{0,1,2,3\}$, $Z = \{0,1,\ldots,4\}$ and $W = \{0,1,2,3,5\}$.

$A = \{0,1\}$ is a subhypergroup of $(X,\bullet)$. $1 \in 4 \bullet 1$ but $4 \notin A$. So A is not closed.

In the following, the definition EL-hypergroup is introduced, and finally we present another proof of the associativity of the hyperoperation $\bullet$. For more on EL-hypergroups, refer to [15, 18, 19, 20, 21].

By a quasi (partially) ordered (semi)group, we mean a triple $(S,\cdot,R)$, where $(S,\cdot)$ is a (semi)group and R is a quasi (partially) order relation on S such that for all $x,y,z \in S$ with the property xRy there holds $(x\cdot z)R(y\cdot z)$ and $(z\cdot x)R(z\cdot y)$. This property is known as monotone condition. Moreover, the notation $[x)_R$ used below stands for the set $[x)_R = \{s \in S; xRs\}$ and also $[A)_R = \bigcup\limits_{x\in A}[x)_R$. Similarly, $(x]_R = \{s \in S; sRx\}$ and $(A]_R = \bigcup\limits_{x\in A}(x]_R$. The EL-hyperstructures or *Ends lemma* based hyperstructures are hyperstructures constructed from a quasi (partially) ordered (semi)groups using "Ends lemma". This concept was first introduced by Chvalina in 1995 [2]. In particular, Chvalina proved that:

Theorem 3.15. *([2], Theorem 1.3) Let $(S,\cdot,R)$ be a partially ordered semi(group). Binary hyperoperation $\diamond : S\times S \longrightarrow P^*(S)$ defined by $a\diamond b = [a\cdot b)_R = \{x \in S, a\cdot bRx\}$ is associative.*

The semi(hypergroup) in Theorem 3.15 is called an EL-semihypergroup(EL-hypergroup).

Theorem 3.16. *The hyperstructure $(X,\bullet)$ is an EL-hypergroup.*

Proof. By Theorem 2.10 $(X,\circ)$ is a monoid, then by Theorem 3.15, $x\diamond y = [x\circ y)_{\leq} = \{t | t \leq x\circ y\} = x\bullet y$. It shows that the hyperoperation $\bullet$ is associative and $(X,\bullet)$ is an EL-hypergroup. □

4. Classification of hypergroups obtained from BCK-Algebras with condition (S)

In this section, using the classification of BCK-algebras from Appendix B of reference [12], we obtain the Cayley tables of the hypergroups constructed from these BCK-algebras. Moreover, for those BCK-algebras that do not admit a hypergroup, we provide reasons why this is not possible.

There is a unique BCK-algebra of order two. According to [12], we call it B_{2-1-1}.

Example 4.1. (B_{2-1-1}) In the tables below, the Cayley table of the BCK-algebra $(X, *)$ with condition (S) is presented, and based on it, the Cayley tables of the monoid $(X, \circ)$ and hypergroup $(X, \bullet)$ are derived and included. Let $X = \{0, 1\}$.

$*$	0	1
0	0	0
1	1	0

$\circ$	0	1
0	0	1
1	1	1

$\bullet$	0	1
0	$\{0\}$	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$

Therefore $(B_{2-1-1}, \bullet)$ is the only hypergroup of order 2 obtained from a BCK-algebra with condition (S) of order 2.

According to [12], we have three BCI-algebras with the following names B_{3-1-1}, B_{3-2-1} and B_{3-1-2}. B_{3-1-1}, B_{3-2-1} are BCK-algebras with condition (S). But B_{3-1-2} is not a BCK-algebra with condition (S).

Example 4.2. (B_{3-1-2}) In the tables below, the Cayley table of the BCK-algebra $(X, *)$ is presented, and based on it, the Cayley tables of the monoid $(X, \circ)$ and hypergroup $(X, \bullet)$ are derived and included. Let $X = \{0, 1, 2\}$. Let $X = \{0, 1, 2\}$.

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

$\circ$	0	1	2
0	0	1	2
1	1	1	
2	2		2

In fact $A(1, 2) = A(2, 1) = \{0, 1, 2\}$ and this set has not supremum. Therefore in the Cayley table $(X, \circ)$, do we leave the entries corresponding to $1 \circ 2$ and $2 \circ 1$ blank.

Therefore $(B_{2-1-1}, *)$ is not a BCK-algebra with condition (S).

Example 4.3. (B_{3-1-1}) In the tables below, the Cayley table of the BCK-algebra $(X, *)$ with condition (S) is presented, and based on it, the Cayley tables of the monoid $(X, \circ)$ and hypergroup $(X, \bullet)$ are derived and included. Let $X = \{0, 1, 2\}$.

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

$\circ$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\bullet$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$
2	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Therefore $(B_{3-1-1}, \bullet)$ is a hypergroup.

Example 4.4. (B_{3-2-1}) In the tables below, the Cayley table of the BCK-algebra $(X, *)$ with condition (S) is presented, and based on it, the Cayley tables of the monoid $(X, \circ)$ and hypergroup $(X, \bullet)$ are derived and included. Let $X = \{0, 1, 2\}$.

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

$\circ$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

$\bullet$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$
1	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$
2	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Therefore $(B_{3-2-1}, \bullet)$ is a hypergroup.

According to [12], we have 14, BCK-algebras with the following names B_{4-3-1}, B_{4-3-2} , B_{4-3-3}, B_{4-2-1}, B_{4-2-2}, B_{4-2-3}, B_{4-1-1}, B_{4-1-2}, B_{4-1-3}, B_{4-1-4}, B_{4-1-5}, B_{4-1-6}, B_{4-1-7} and B_{4-1-8}.

BCK-algebras B_{4-3-1}, B_{4-3-2}, B_{4-2-1}, B_{4-2-2}, B_{4-2-3}, B_{4-1-1} and B_{4-1-4} have condition (S). But BCK-algebras B_{4-1-2}, B_{4-1-3}, B_{4-1-5}, B_{4-1-6}, B_{4-1-7}, B_{4-1-8} and B_{4-3-3} do not satisfy condition (S).

Example 4.5. In the tables below, we present seven BCK-algebras B_{4-3-3}, B_{4-1-2}, B_{4-1-3}, B_{4-1-5}, B_{4-1-6}, B_{4-1-7} and B_{4-1-8}, along with their corresponding Cayley tables.

B_{4-1-2}

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	3	3	0

B_{4-1-3}

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

B_{4-1-5}

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	2
3	3	3	3	0

B_{4-1-6}

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

B_{4-1-7}

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

B_{4-1-8}

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

B_{4-3-3}

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	1	1	0

The BCK-algebras with the above Cayley tables do not satisfy condition (S). For each of the following BCI-algebras, there exist $a, b \in X$ such that the set $A(a, b)$ does not possess a supremum (i.e., a least upper bound) in X. In particular:

B_{4-1-2}: $A(1,3) = \{0,1,2,3\}$ has no supremum in X.
B_{4-1-3}: $A(1,3) = \{0,1,3\}$ has no supremum in X.
B_{4-1-5}: $A(2,3) = \{0,1,2,3\}$ has no supremum in X.
B_{4-1-6}: $A(2,3) = \{0,1,2,3\}$ has no supremum in X.
B_{4-1-7}: $A(1,2) = \{0,1,2\}$ has no supremum in X.
B_{4-1-8}: $A(1,2) = \{0,1,2\}$ has no supremum in X.
B_{4-3-3}: $A(1,1) = \{0,1,2,3\}$ has no supremum in X.

Example 4.6. (B_{4-3-1}) In the tables below, the Cayley table of the BCK-algebra $(X, *)$ is presented, and based on it, the Cayley table of the hypergroup $(X, \bullet)$ is derived and included. Let $X = \{0, 1, 2, 3\}$.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	1	1	0

$\bullet$	0	1	2	3
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
1	$\{0,1\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
2	$\{0,1,2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
3	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

Therefore $(B_{4-3-1}, \bullet)$ is a hypergroup.

Example 4.7. (B_{4-3-2}) In the tables below, the Cayley table of the BCK-algebra $(X,*)$ is presented, and based on it, the Cayley table of the hypergroup $(X,\bullet)$ is derived and included. Let $X = \{0,1,2,3\}$.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

$\bullet$	0	1	2	3
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
1	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
2	$\{0,1,2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
3	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

Therefore $(B_{4-3-2}, \bullet)$ is a hypergroup.

Example 4.8. (B_{4-2-1}) In the tables below, the Cayley table of the BCK-algebra $(X,*)$ is presented, and based on it, the Cayley table of the hypergroup $(X,\bullet)$ is derived and included. Let $X = \{0,1,2,3\}$.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	1	0

$\bullet$	0	1	2	3
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
2	$\{0,1,2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
3	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

Therefore $(B_{4-2-1},\bullet)$ is a hypergroup.

Example 4.9. (B_{4-2-2}) In the tables below, the Cayley table of the BCK-algebra $(X,*)$ is presented, and based on it, the Cayley table of the hypergroup $(X,\bullet)$ is derived and included. Let $X=\{0,1,2,3\}$.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	2	0

$\bullet$	0	1	2	3
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
2	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
3	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

Therefore $(B_{4-2-2},\bullet)$ is a hypergroup.

Example 4.10. (B_{4-1-1}) In the tables below, the Cayley table of the BCK-algebra $(X,*)$ is presented, and based on it, the Cayley table of the hypergroup $(X,\bullet)$ is derived and included. Let $X=\{0,1,2,3\}$.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	3	3	0

$\bullet$	0	1	2	3
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
1	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
2	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
3	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

Therefore $(B_{4-1-1}, \bullet)$ is a hypergroup.

Example 4.11. (B_{4-1-4}) In the tables below, the Cayley table of the BCK-algebra $(X, *)$ is presented, and based on it, the Cayley table of the hypergroup $(X, \bullet)$ is derived and included. Let $X = \{0, 1, 2, 3\}$.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

$\bullet$	0	1	2	3
0	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
2	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2,3\}$
3	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

Therefore $(B_{4-1-4}, \bullet)$ is a hypergroup.

Example 4.12. (B_{4-2-3}) In the tables below, the Cayley table of the BCK-algebra $(X, *)$ is presented, and based on it, the Cayley table of the hypergroup $(X, \bullet)$ is derived and included. Let $X = \{0, 1, 2, 3\}$.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

$\bullet$	0	1	2	3
0	$\{0\}$	$\{0,1\}$	$\{0,2\}$	$\{0,1,2,3\}$
1	$\{0,1\}$	$\{0,1\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$
2	$\{0,2\}$	$\{0,1,2,3\}$	$\{0,2\}$	$\{0,1,2,3\}$
3	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$

Therefore $(B_{4-2-3}, \bullet)$ is a hypergroup.

Theorem 4.13. *The All hypergroups are pairwise non-isomorphic.*

Proof. For hypergroups of the same order, an isomorphism must preserve the cardinality of every hyperproduct. A simple distinguishing invariant is the set of diagonal entries $\{x \bullet x \mid$

$x \in X\}$ and their cardinalities. The table below summarises these invariants for the seven hypergroups of order 4 (the only order with more than one hypergroup). Recall that $0 \bullet 0 = \{0\}$ in every case.

Hypergroup	$\lvert 1 \bullet 1 \rvert$	$\lvert 2 \bullet 2 \rvert$	$\lvert 3 \bullet 3 \rvert$	Additional distinctive feature
B_{4-3-1}	4	4	4	All non-zero elements have full product
B_{4-3-2}	3	4	4	$1 \bullet 1$ has size 3
B_{4-2-1}	2	4	4	$1 \bullet 2 = X$ (full set)
B_{4-2-2}	2	4	4	$1 \bullet 2 = \{0, 1, 2\}$ (size 3)
B_{4-2-3}	2	2	4	$2 \bullet 2 = \{0, 2\}$ (missing 1)
B_{4-1-1}	3	3	4	$1 \bullet 1 = 2 \bullet 2 = \{0, 1, 2\}$
B_{4-1-4}	2	3	4	Sizes $(2, 3, 4)$ appear only here

Since the tuple of cardinalities ($\lvert 1 \bullet 1 \rvert$, $\lvert 2 \bullet 2 \rvert$, $\lvert 3 \bullet 3 \rvert$) together with the actual pattern of products (e.g., whether $1 \bullet 2$ equals the whole set or not) differs for each hypergroup, no two of them can be isomorphic. For order 3 we have two hypergroups: $\lvert 1 \bullet 1 \rvert = 2$ for B_{3-1-1} and $\lvert 1 \bullet 1 \rvert = 3$ for B_{3-2-1}, so they are also non-isomorphic. For order 2 there is only one hypergroup, so no issue. □

Corollary 4.14. *Based on the examples in this section and the Theorem 4.13, all the obtained hypergroups are shown in the table below.*

TABLE 1. Number and names of hypergroups of small orders.

Order	*Number of hypergroups*	*Identifiers*
1	*1*	*Trivial hypergroup*
2	*1*	B_{2-1-1}
3	*2*	B_{3-1-1}, B_{3-2-1}
4	*7*	B_{4-3-1}, B_{4-3-2}, B_{4-2-1}, B_{4-2-2}, B_{4-2-3}, B_{4-1-1}, B_{4-1-4}

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Saeed Mirvakili

Department of Mathematical Sciences,

Yazd University,

Yazd, Iran.

saeedmirvakili@yazd.ac.ir

Tayebe Golchin

Department of Mathematics,

Payame Noor University,

Tehran, Iran.

golchin.ta2020@student.pnu.ac.ir

Mohammad Hamidi

Department of Mathematics,

Payame Noor University,

Tehran, Iran.

m.hamidi@pnu.ac.ir