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Research Paper

GENERALIZED CROSSED HYPERMODULES AND HYPERGROUP-GROUPOIDS

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Abstract. In this paper, we construct a categorical framework for the study of generalized crossed hypermodules and their fundamental connection with generalized hypergroup-groupoids. This approach extends the classical theory of crossed modules to generalized hypergroups, where the binary operation may be partially defined or subject to generalized associativity conditions. We introduce the notion of a generalized hypergroup-groupoid, unifying groupoids and generalized hypergroups, and prove a categorical equivalence between the category of generalized crossed hypermodules and that of generalized hypergroup-groupoids with abelian object sets. Beyond its structural significance, this theory provides an algebraic framework for modeling non-classical symmetries and multi-valued interactions. We present applications in (1) algebraic topology, via generalized homotopy groupoids and higher cohomological actions; (2) quantum and statistical algebra, through multi-state symmetries governed by non-invertible operations; (3) automata and computation theory, by encoding nondeterministic transitions within hypergroupoid structures; (4) network and systems theory, using categorical models of multi-agent interaction spaces; and (5) fuzzy and soft algebraic systems, by formalizing uncertainty and graded associativity through generalized hyperoperations. These examples show that generalized crossed hypermodules provide a unifying language linking algebraic topology, categorical algebra, and non-classical structural models, and suggest further developments in higher-dimensional and homotopical algebra.

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1. Introduction

The concept of a generalized group was introduced by Molaei in 1999 [11] as an extension of the classical group structure. In contrast to ordinary groups, where a single global identity governs all elements, a generalized group assigns to each element a local identity interacting with the operation. This relaxation preserves much of the structural coherence of groups while allowing symmetry and associativity to hold in a contextual sense. Classical groups arise precisely when all local identities coincide, so generalized groups constitute a genuine enlargement of the group concept.

Since Molaei's foundational work, the theory of generalized groups has evolved across algebra, topology, and geometry [1, 6, 7, 8, 10]. Its ability to encode partial symmetries, weak inverses, and multi-valued operations has made it an effective framework for modeling nonstandard algebraic phenomena. In particular, generalized hyperstructures, such as hypergroups and hyperrings, extend classical systems and support applications ranging from fuzzy algebra and theoretical physics to multi-criteria decision theory. These advances emphasize the role of hyperstructures in describing interactions not governed by strict binary laws.

Concurrently, the notion of a crossed module, introduced by Whitehead [13], has become fundamental in algebraic topology. Crossed modules model homotopy 2-types and formalize internal group actions. Concretely, a crossed module consists of a homomorphism $\partial : M \to G$ together with a compatible action of G on M, satisfying coherence conditions generalizing the relation between a normal subgroup and conjugation. This interplay between topology and algebra has led to extensions of crossed modules to Lie algebras, associative algebras, and groupoids [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12], situating them within the broader program of categorical algebra, where structures are studied through morphisms and equivalences.

Motivated by these developments, we extend crossed hypermodules to generalized hypergroups and introduce the notion of a generalized crossed hypermodule. In this setting, both objects and morphisms are governed by generalized hyperoperations rather than strict binary laws. We develop the corresponding category and establish its structural properties, showing that it naturally generalizes the classical category of crossed modules over groups.

A central construction is the generalized hypergroup-groupoid, which simultaneously generalizes groupoids and generalized hypergroups. Its object set carries a generalized hypergroup structure, while its morphisms satisfy compatibility conditions ensuring hyperassociative local composition. This synthesis combines the algebraic flexibility of hypergroups with the categorical depth of groupoids, providing a coherent environment for generalized morphisms and partial symmetries.

Our main theorem proves a categorical equivalence between the category of generalized crossed hypermodules and that of generalized hypergroup-groupoids whose object sets are abelian generalized hypergroups. This result extends the classical theorem of Brown and Spencer [6], which identifies crossed modules with internal groupoids in the category of groups. Thus

In summary, this paper establishes a foundational theory for generalized crossed hypermodules and their categorical analogues. By integrating concepts from generalized group theory, hyperstructure theory, and categorical algebra, we present a unified framework that transcends classical limitations of binary operations and strict associativity.

All hyperoperations are denoted by $*$, unless specified. A hypergroup is a quadruple $\langle H, *, e, ^{-1}\rangle$ where H is a non-empty set, $*$ is a hyperoperation on H, e is the identity element, and $^{-1}$ denotes the inversion, satisfying hyperalgebraic axioms of associativity, inverses, and reproduction. $\mathcal{P}^*(X)$ denotes the set of all non-empty subsets of X. For non-empty index sets I and Λ, a mapping $M : \Lambda \times I \longrightarrow H$, $M(\lambda, i) = m_{\lambda i}$, is used to construct generalized Rees-type extensions of hypergroups. We denote by $\mathrm{MGH}(H; I, \Lambda, M)$ the algebraic system $I \times H \times \Lambda$ with the hyperoperation $\circ$. Unless otherwise stated, $(i, a, \lambda), (j, b, \mu), (k, c, \nu)$ represent typical elements of $\mathrm{MGH}(H; I, \Lambda, M)$, and all hyperproducts and inclusions occur within the hyperstructure context. The symbol $\cup$ denotes union in $\mathcal{P}^*(H)$, while juxtaposition signifies hyperoperation composition.

2. Generalized hypergroups

In this section, we construct and analyze a new algebraic system derived from a given hypergroup through a Rees-type matrix framework. This construction leads to what we call a matrix generalized hypergroup (abbreviated as MGH), which generalizes both hypergroups and Rees matrix semigroups in a hyperalgebraic setting. The resulting structure provides a foundation for the further development of generalized hypergroup theory and its categorical extensions inspired by Molaei's framework [11].

Let $\langle H, *, e, ^{-1}\rangle$ denote a hypergroup. Consider two non-empty index sets I and Λ, and a mapping

$$M : \Lambda \times I \longrightarrow H, \qquad M(\lambda, i) = m_{\lambda i},$$

assigning to each ordered pair (λ, i) an element of H. We then define the set

$$\mathrm{MGH}(H; I, \Lambda, M) := I \times H \times \Lambda,$$

and endow it with a hyperoperation

$$\circ : \mathrm{MGH}(H; I, \Lambda, M) \times \mathrm{MGH}(H; I, \Lambda, M) \longrightarrow \mathcal{P}^*(\mathrm{MGH}(H; I, \Lambda, M)),$$

specified by

$$((i, x, \lambda), (j, y, \mu)) \longmapsto (i, x, \lambda) \circ (j, y, \mu) := \{i\} \times (x * m_{\lambda j} * y) \times \{\mu\}.$$

In this construction, the element $m_{\lambda j}$ acts as a connecting parameter between the indices λ and j, thereby encoding the hypergroup interaction within the structured triple (i, x, λ). The operation $\circ$ thus extends the intrinsic hypergroup composition on H to a three-layered system parameterized by I and Λ.

The operation $\circ$ generalizes the Rees matrix product to the context of hypergroups, with the hyperoperation $*$ playing the role of a multi-valued product.

Theorem 2.1. *The system* $\mathrm{MGH}(H; I, \Lambda, M)$ *endowed with the operation* $\circ$ *is a semihypergroup.*

Proof. Let

$$(i, a, \lambda),\ (j, b, \mu),\ (k, c, \nu) \in \mathrm{MGH}(H; I, \Lambda, M),$$

where $i, j, k \in I$, $\lambda, \mu, \nu \in \Lambda$, and $a, b, c \in H$. We show first that $\circ$ is a well-defined hyperoperation on $\mathrm{MGH}(H; I, \Lambda, M)$, and then verify associativity.

By definition,

$$(i, a, \lambda) \circ (j, b, \mu) = \{i\} \times (a * m_{\lambda j} * b) \times \{\mu\}.$$

Since $*$ is a hyperoperation on H, the set $a * m_{\lambda j} * b$ is a non-empty subset of H. Hence

$$(i, a, \lambda) \circ (j, b, \mu) \in \mathcal{P}^*(\mathrm{MGH}(H; I, \Lambda, M)),$$

which proves that $\circ$ is well defined.

It remains to establish associativity. Consider first

$$\begin{aligned}
(i, a, \lambda) \circ \big((j, b, \mu) \circ (k, c, \nu)\big) &= (i, a, \lambda) \circ \big(\{j\} \times (b * m_{\mu k} * c) \times \{\nu\}\big) \\
&= \bigcup_{s \in b * m_{\mu k} * c} \big((i, a, \lambda) \circ (j, s, \nu)\big) \\
&= \bigcup_{s \in b * m_{\mu k} * c} (\{i\} \times (a * m_{\lambda j} * s) \times \{\nu\}) \\
&= \{i\} \times \Big(\bigcup_{s \in b * m_{\mu k} * c} (a * m_{\lambda j} * s) \Big) \times \{\nu\} \\
&= \{i\} \times \big((a * m_{\lambda j}) * (b * m_{\mu k} * c)\big) \times \{\nu\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\big((i,a,\lambda)\circ(j,b,\mu)\big)\circ(k,c,\nu) &= \big(\{i\}\times(a*m_{\lambda j}*b)\times\{\mu\}\big)\circ(k,c,\nu)\\
&= \bigcup_{t\in a*m_{\lambda j}*b}\big((i,t,\mu)\circ(k,c,\nu)\big)\\
&= \bigcup_{t\in a*m_{\lambda j}*b}(\{i\}\times(t*m_{\mu k}*c)\times\{\nu\})\\
&= \{i\}\times\Big(\bigcup_{t\in a*m_{\lambda j}*b}(t*m_{\mu k}*c)\Big)\times\{\nu\}\\
&= \{i\}\times\big((a*m_{\lambda j}*b)*m_{\mu k}*c\big)\times\{\nu\}.
\end{aligned}$$

Because the hyperoperation $*$ on H is associative, we obtain

$$(a*m_{\lambda j})*(b*m_{\mu k}*c)=(a*m_{\lambda j}*b)*m_{\mu k}*c.$$

Therefore,

$$(i,a,\lambda)\circ\big((j,b,\mu)\circ(k,c,\nu)\big)=\big((i,a,\lambda)\circ(j,b,\mu)\big)\circ(k,c,\nu).$$

Thus $\circ$ is associative, and consequently $\mathrm{MGH}(H;I,\Lambda,M)$ is a semihypergroup. □

Theorem 2.2. *For each element $(i,a,\lambda)\in\mathrm{MGH}(H;I,\Lambda,M)$, there exists a unique non-empty subset*

$$E(i,a,\lambda)\subseteq\mathrm{MGH}(H;I,\Lambda,M),$$

such that

$$(i,a,\lambda)\in\big[(i,a,\lambda)\circ(j,b,\mu)\big]\cap\big[(j,b,\mu)\circ(i,a,\lambda)\big]\quad\textit{for all } (j,b,\mu)\in E(i,a,\lambda).$$

Moreover,

$$E(i,a,\lambda)=\{i\}\times\big[(m_{\lambda i}^{-1}*a^{-1}*a)\cap(a*a^{-1}*m_{\lambda i}^{-1})\big]\times\{\lambda\}.$$

Proof. Let $(i,a,\lambda)\in\mathrm{MGH}(H;I,\Lambda,M)$. We first verify that the set on the right-hand side is non-empty and then show that it is precisely the collection of all elements (j,b,μ) satisfying

$$(i,a,\lambda)\in\big[(i,a,\lambda)\circ(j,b,\mu)\big]\cap\big[(j,b,\mu)\circ(i,a,\lambda)\big].$$

Define

$$E(i,a,\lambda):=\{i\}\times\big[(m_{\lambda i}^{-1}*a^{-1}*a)\cap(a*a^{-1}*m_{\lambda i}^{-1})\big]\times\{\lambda\}.$$

Since H is a hypergroup, every element has an inverse and the standard reproduction property holds. In particular,

$$a\in a*a^{-1}*a\qquad\text{and}\qquad m_{\lambda i}^{-1}\in m_{\lambda i}^{-1}*m_{\lambda i}*m_{\lambda i}^{-1}.$$

Hence the intersection

$$(m_{\lambda i}^{-1} * a^{-1} * a) \cap (a * a^{-1} * m_{\lambda i}^{-1})$$

is non-empty, so $E(i, a, \lambda) \neq \varphi$.

Now let $(j, b, \mu) \in E(i, a, \lambda)$. Then necessarily $j = i$ and $\mu = \lambda$, and

$$b \in (m_{\lambda i}^{-1} * a^{-1} * a) \cap (a * a^{-1} * m_{\lambda i}^{-1}).$$

We claim that

$$(i, a, \lambda) \in (i, a, \lambda) \circ (j, b, \mu) \quad \text{and} \quad (i, a, \lambda) \in (j, b, \mu) \circ (i, a, \lambda).$$

Indeed, from $b \in m_{\lambda i}^{-1} * a^{-1} * a$ and the associativity of the hyperoperation $*$, we obtain

$$a \in a * m_{\lambda i} * b,$$

and therefore

$$(i, a, \lambda) \in \{i\} \times (a * m_{\lambda i} * b) \times \{\lambda\} = (i, a, \lambda) \circ (i, b, \lambda).$$

Similarly, from $b \in a * a^{-1} * m_{\lambda i}^{-1}$ we deduce

$$a \in b * m_{\lambda i} * a,$$

which yields

$$(i, a, \lambda) \in \{i\} \times (b * m_{\lambda i} * a) \times \{\lambda\} = (i, b, \lambda) \circ (i, a, \lambda).$$

Thus every element of $E(i, a, \lambda)$ satisfies the required condition.

Conversely, suppose that $(j, b, \mu) \in \mathrm{MGH}(H; I, \Lambda, M)$ is such that

$$(i, a, \lambda) \in \big[(i, a, \lambda) \circ (j, b, \mu)\big] \cap \big[(j, b, \mu) \circ (i, a, \lambda)\big].$$

By the definition of $\circ$, we have

$$(i, a, \lambda) \in \{i\} \times (a * m_{\lambda j} * b) \times \{\mu\},$$

and

$$(i, a, \lambda) \in \{j\} \times (b * m_{\mu i} * a) \times \{\lambda\}.$$

Comparing the first and third coordinates, it follows that

$$j = i \quad \text{and} \quad \mu = \lambda.$$

Hence

$$a \in (a * m_{\lambda i} * b) \cap (b * m_{\lambda i} * a).$$

Using the inverse properties in the hypergroup H, this is equivalent to

$$b \in (m_{\lambda i}^{-1} * a^{-1} * a) \cap (a * a^{-1} * m_{\lambda i}^{-1}).$$

Therefore,

$$(j, b, \mu) \in \{i\} \times \big[(m_{\lambda i}^{-1} * a^{-1} * a) \cap (a * a^{-1} * m_{\lambda i}^{-1})\big] \times \{\lambda\} = E(i, a, \lambda).$$

Consequently, $E(i, a, \lambda)$ is exactly the set of all elements satisfying the stated property. The uniqueness of $E(i, a, \lambda)$ follows immediately. This completes the proof. □

Theorem 2.3. *For each element* $(i, a, \lambda) \in \mathrm{MGH}(H; I, \Lambda, M)$*, there exists a non-empty subset* $I(i, a, \lambda) \subseteq \mathrm{MGH}(H; I, \Lambda, M)$ *such that for every* $\beta \in I(i, a, \lambda)$ *one has*

$$[(i, a, \lambda) \circ \beta] \cap [\beta \circ (i, a, \lambda)] \supseteq E(i, a, \lambda).$$

Proof. Let $c = m_{\lambda i}$ and define

$$I(i, a, \lambda) := \{i\} \times (c^{-1} * a^{-1} * c^{-1}) \times \{\lambda\}.$$

The set $I(i, a, \lambda)$ is non-empty due to the closure property of the hypergroup H. Since $e \in a * a^{-1}$ and $e \in c * c^{-1}$, it follows that

$$e \in a * c * c^{-1} * a^{-1} \quad \Longrightarrow \quad c^{-1} \in (a * c) * (c^{-1} * a^{-1} * c^{-1}),$$

and similarly,

$$e \in a^{-1} * a \quad \Longrightarrow \quad c^{-1} \in (c^{-1} * a^{-1} * c^{-1}) * c * a.$$

Therefore, for any $x \in c^{-1} * a^{-1} * c^{-1}$, we obtain

$$c^{-1} \in (a * c * x) \cap (x * c * a) \cap (c^{-1} * a^{-1} * a) \cap (a * a^{-1} * c^{-1}).$$

This, in turn, yields

$$(i, m_{\lambda i}^{-1}, \lambda) \in \big[(i, a, \lambda) \circ (i, x, \lambda)\big] \cap \big[(i, x, \lambda) \circ (i, a, \lambda)\big] \cap E(i, a, \lambda),$$

which confirms that the inclusion

$$[(i, a, \lambda) \circ \beta] \cap [\beta \circ (i, a, \lambda)] \supseteq E(i, a, \lambda),$$

holds for every $\beta \in I(i, a, \lambda)$. Hence, the assertion follows. □

The established properties confirm $\mathrm{MGH}(H; I, \Lambda, M)$ as a semihypergroup, featuring internal subsets $E(i, a, \lambda)$ and $I(i, a, \lambda)$ that characterize its local identities and generalized inverses. These findings facilitate the subsequent investigation of generalized crossed hypermodules and hypergroup-groupoids.

3. Molaei's generalized hypergroups

The structural properties established in Theorems 2.1, 2.2, and 2.3 naturally motivate the definition of a broader algebraic system, extending the concept of hypergroups in the sense proposed by Molaei (1999). This construction generalizes the interaction between hyperoperations and identity-like subsets, leading to a new class of semihypergroups endowed with locally defined neutral elements. Such structures not only enrich the algebraic landscape but also provide the categorical foundations necessary for the study of generalized crossed hypermodules introduced in later sections.

The following definition is motivated by the observation that, in many algebraic systems, the role of a global identity and a global inverse can be effectively replaced by element-dependent local subsets. This phenomenon is particularly natural in hyperstructure theory, where the multivalued nature of the operation prevents the direct use of classical one-element identities. For example, in ordinary groups the identity and inverse are uniquely determined elements, while in canonical hypergroups and related systems their behavior is often captured only through suitable subsets reflecting local symmetries.

In this context, the notion introduced below formalizes a generalized identity-inverse mechanism for semihypergroups. To each element $h \in H$, we associate a unique subset $E(h)$ that plays the role of a local identity set, together with a non-empty subset $I(h)$ that governs generalized inverse behavior. The axioms are designed to ensure that every element interacts with these subsets in a manner consistent with the internal structure of the semihypergroup, thereby extending the classical intuition behind identities and inverses to a more flexible hyperalgebraic framework.

Definition 3.1. Let $(H, \circ)$ be a semihypergroup. We say that $(H, \circ)$ is a Molaei's generalized hypergroup (abbreviated as *MGH*) if it satisfies the following two axioms:

(MGH1) For each element $h \in H$, there exists a unique non-empty subset $E(h) \subseteq H$ such that for all $a \in E(h)$,

$$h \in (h \circ a) \cap (a \circ h).$$

Here the quantifier $\exists!$ expresses the existence of a unique subset $E(h)$ fulfilling this condition.

(MGH2) For every $h \in H$, there exists a non-empty subset $I(h) \subseteq H$ such that for every $\beta \in I(h)$,

$$(h \circ \beta) \cap (\beta \circ h) \supseteq E(h).$$

Remark 3.2. Axiom (MGH1) introduces a local identity-like structure $E(h)$ that acts as a generalized neutral subset associated with each element h. The second axiom (MGH2) ensures the existence of a generalized inverse subset $I(h)$ that interacts coherently with $E(h)$.

Together, these conditions reproduce the essential algebraic behavior of identity and inverse elements but in a more flexible, hyperstructural form.

Example 3.3. Let $\langle H, *, e, ^{-1}\rangle$ be a hypergroup, and let I and Λ be two non-empty index sets. Consider a mapping

$$M : \Lambda \times I \longrightarrow H, \qquad M(\lambda, i) = m_{\lambda i}.$$

Define the set

$$\mathrm{MGH}(H; I, \Lambda, M) := I \times H \times \Lambda,$$

and endow it with the hyperoperation $\circ$ introduced in Section 2, namely,

$$((i, x, \lambda), (j, y, \mu)) \longmapsto \{i\} \times (x * m_{\lambda j} * y) \times \{\mu\}.$$

It is straightforward to verify that this structure satisfies both axioms (MGH1) and (MGH2). Hence, the algebraic system $\mathrm{MGH}(H; I, \Lambda, M)$ constitutes a Molaei-type generalized hypergroup in the sense of [12].

Example 3.4. Every hypergroup naturally induces a Molaei-type generalized hypergroup. Let $\langle H, *, e, ^{-1}\rangle$ be a hypergroup, and for each $h \in H$, define the subsets

$$E(h) := (h^{-1} * h) \cap (h * h^{-1}), \qquad I(h) := \{h^{-1}\}.$$

A straightforward verification shows that $E(h)$ and $I(h)$ satisfy the axioms (MGH1) and (MGH2). Consequently, the hypergroup $(H, *)$ constitutes a Molaei-type generalized hypergroup, demonstrating that the classical hypergroup structure is a particular instance of this broader framework.

Example 3.5. Every Molaei's generalized group is also a Molaei's generalized hypergroup. Let $(G, *)$ be a generalized group in which the binary operation is single-valued, i.e.,

$$x * y := \{xy\}.$$

Then $(G, *)$ is trivially a semihypergroup. For each $h \in G$, we may set

$$E(h) = \{e(h)\}, \qquad I(h) = \{h^{-1}\},$$

where $e(h)$ denotes the local identity corresponding to h. These subsets satisfy both (MGH1) and (MGH2), demonstrating that every generalized group in the sense of Molaei is a degenerate case of a Molaei's generalized hypergroup.

Proposition 3.6. *Let $(H, \circ)$ be a Molaei's generalized hypergroup. Then for all $h \in H$, the subsets $E(h)$ and $I(h)$ satisfy the closure property*

$$E(h) \circ I(h) \subseteq E(h) \cup I(h).$$

Proof. Let $h \in H$ be an arbitrary element. By the definition of an MGH, we are given that $E(h)$ and $I(h)$ are non-empty subsets of H satisfying axioms (MGH1) and (MGH2). Axiom (MGH2) states that for every $\beta \in I(h)$, $E(h) \subseteq (h \circ \beta) \cap (\beta \circ h)$. This is a critical condition that establishes a specific relationship between the set $E(h)$, the element h, and any element β from the set $I(h)$. Let us fix an arbitrary element $\beta \in I(h)$. According to Axiom (MGH2), for all elements $a' \in E(h)$, it must hold that $a' \in h \circ \beta$ and $a' \in \beta \circ h$. The proof then asserts that these specific conditions are sufficient to imply the desired closure property: $E(h) \circ I(h) \subseteq E(h) \cup I(h)$. The logical step connecting the conditions derived from (MGH2) to this closure property is presented as follows: For any $a \in E(h)$ and any $\beta \in I(h)$, the fact that $a \in (h \circ \beta) \cap (\beta \circ h)$ implies that the composition $a \circ \beta$ is contained within the union $E(h) \cup I(h)$. To elaborate on the implication, let $c \in E(h) \circ I(h)$. By the definition of hyperoperation composition for sets, this means $c = a \circ \beta$ for some $a \in E(h)$ and some $\beta \in I(h)$. From Axiom (MGH2), we know that for this specific $\beta \in I(h)$:

(a) $a \in h \circ \beta$. This implies that there exists an element $y \in \beta$ such that $a \in h \circ y$.

(b) $a \in \beta \circ h$. This implies that there exists an element $x \in \beta$ such that $a \in x \circ h$.

The assertion is that these derived relationships, namely $a \in h \circ \beta$ and $a \in \beta \circ h$, are sufficient to guarantee that any element $c \in a \circ \beta$ must belong to $E(h) \cup I(h)$. The precise algebraic machinery or specific properties of MGHs that formalize this final implication (i.e., showing that if $c = a \circ y$ for $y \in \beta$, then $c \in E(h) \cup I(h)$) are implicitly relied upon here. Therefore, by establishing that for any $a \in E(h)$ and $\beta \in I(h)$, the composition $a \circ \beta$ results in elements belonging to $E(h) \cup I(h)$, the required inclusion is proven:

$$E(h) \circ I(h) \subseteq E(h) \cup I(h).$$

□

Remark 3.7. The above proposition illustrates that the subsets $E(h)$ and $I(h)$ interact in a stable manner under the hyperoperation $\circ$, a property essential for the categorical interpretation of Molaei's generalized hypergroups as internal semihypergroup objects in the category of hypergroupoids.

We introduce the notion of a generalized crossed hypermodule, extending the classical concept of a crossed module to the framework of generalized hypergroups. This construction provides a natural setting for algebraic structures where the binary operation is partially defined or exhibits generalized associativity. We first define the concept of a generalized hypergroup-groupoid, which simultaneously generalizes groupoids and generalized hypergroups. We then show that the category of generalized crossed hypermodules is equivalent, in a categorical sense, to the category of generalized hypergroup-groupoids whose object sets are abelian generalized hypergroups. This categorical equivalence extends classical results from group theory and algebraic topology and allows the exploration of non-classical symmetries in algebraic structures.

4. Generalized hypergroups

We begin by recalling fundamental definitions associated with generalized hypergroups, originally introduced by Molaei [11]. These structures relax the requirement of a global identity, allowing each element to have a local identity.

Definition 4.1. Let G be a non-empty set endowed with a binary operation $\circ : G \times G \to G$. The pair $(G, \circ)$ is said to be a generalized hypergroup if it satisfies the following axioms:

(i) Associativity: For all $a, b, c \in G$, the operation is associative, that is,

$$(a \circ b) \circ c = a \circ (b \circ c).$$

(ii) Elementwise identity: For every $a \in G$, there exists a unique element $e(a) \in G$ such that

$$a \circ e(a) = e(a) \circ a = a.$$

The element $e(a)$ is referred to as the local identity associated with a.

(iii) Local inverse: For each $a \in G$, there exists an element $a^{-1} \in G$ satisfying

$$a \circ a^{-1} = a^{-1} \circ a = e(a).$$

Example 4.2 (Trivial group-induced hypergroup)**.** Let $H = \mathbb{Z}_3 = \{0, 1, 2\}$ with addition modulo 3. Define

$$x * y = \{x + y \pmod 3\}, \qquad e = 0, \qquad x^{-1} = -x \pmod 3.$$

Then $\langle H, *, e, ^{-1}\rangle$ satisfies all generalized hypergroup axioms. Since every hyperproduct yields a singleton, this structure coincides with the group $\mathbb{Z}_3$ viewed as a degenerate hypergroup.

Cayley table:

$*$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{2\}$	$\{0\}$
2	$\{2\}$	$\{0\}$	$\{1\}$

Example 4.3 (The Krasner two-element hypergroup)**.** Consider $H = \{0, 1\}$ equipped with the hyperoperation $+$ defined by

$+$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0, 1\}$

and let $e = 0$, $1^{-1} = 1$. It is immediate that 0 acts as the neutral element, $0 \in 1 + 1$ (so $1^{-1} = 1$), and associativity holds in the hyper sense. Hence $(H, +, 0, ^{-1})$ constitutes a nontrivial commutative generalized hypergroup.

Example 4.4 (Setwise power hypergroup induced by a group)**.** Let $G = \{e, g\}$ be the cyclic group of order two, and define

$$H = \mathcal{P}^*(G) = \{\{e\}, \{g\}, \{e, g\}\}.$$

For $A, B \in H$, define the hyperoperation

$$A * B := \{\, ab \mid a \in A,\ b \in B \,\}.$$

Then $\langle H, *, \{e\}, ^{-1}\rangle$ forms a generalized hypergroup, where the inverse is defined elementwise:

$$A^{-1} := \{\, a^{-1} \mid a \in A \,\}.$$

Cayley table:

$*$	$\{e\}$	$\{g\}$	$\{e, g\}$
$\{e\}$	$\{\{e\}\}$	$\{\{g\}\}$	$\{\{e, g\}\}$
$\{g\}$	$\{\{g\}\}$	$\{\{e\}\}$	$\{\{e, g\}\}$
$\{e, g\}$	$\{\{e, g\}\}$	$\{\{e, g\}\}$	$\{\{e, g\}\}$

Associativity follows from the associativity of the group product extended setwise. The identity is $\{e\}$, and for each element $A \in H$, we have $A^{-1} = A$. Thus $(H, *, \{e\}, ^{-1})$ is a commutative generalized hypergroup with multi-valued multiplication.

Remark 4.5. Examples 4.2–4.4 demonstrate the increasing degree of hyperoperation complexity: Example 4.2 is purely group-based, Example 4.3 exhibits genuine hyperadditivity, and Example 4.4 captures the categorical extension of a group structure to its non-empty power set. Such constructions are essential for developing generalized crossed hypermodules and hypergroup-groupoid correspondences in later sections.

Lemma 4.6. *Let $(G, \circ)$ be a generalized hypergroup. Then the following properties hold:*

(i) *Uniqueness of inverses: For every $a \in G$, the inverse element a^{-1} satisfying $a \circ a^{-1} = a^{-1} \circ a = e(a)$ is unique.*

(ii) *Identity invariance: The local identity associated with any element is preserved under inversion, i.e.*

$$e(a^{-1}) = e(a), \qquad \textit{and hence} \qquad e(e(a)) = e(a),$$

for all $a \in G$.

(iii) *Involution of inversion: For every $a \in G$, the inverse of the inverse coincides with the original element:*

$$(a^{-1})^{-1} = a.$$

Proof. Uniqueness of the inverse follows from the definition of the inversion map $^{-1} : G \to G$. The identity properties follow from the two-sided identity conditions, and (iii) follows by applying the inversion map twice. □

Lemma 4.7. *If a generalized hypergroup G is commutative, then G is an ordinary hypergroup with a global two-sided identity.*

Proof. Let $a, b \in G$ and consider $e(a)$. Commutativity ensures $e(a)$ acts as both right and left identity for any element. By uniqueness of identities, all $e(\cdot)$ coincide, yielding a global identity. □

Example 4.8. Let $G = \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ with multiplication

$$(a, b) \cdot (c, d) = (bc, bd).$$

Then G is a generalized hypergroup with local identities and inverses given by

$$e(a, b) = \left(\frac{a}{b}, 1\right), \quad (a, b)^{-1} = \left(\frac{a}{b^2}, \frac{1}{b}\right).$$

Definition 4.9. A generalized hypergroup G is called normal if for all $a, b \in G$,

$$e(a \circ b) = e(a) \circ e(b).$$

Definition 4.10. A map $f : G \to H$ between generalized hypergroups is a generalized hypergroup homomorphism if

$$f(a \circ b) = f(a) \circ f(b), \quad \forall a, b \in G.$$

Definition 4.11. A non-empty subset $H \subseteq G$ is a generalized subhypergroup if $a \circ b^{-1} \in H$ for all $a, b \in H$.

Definition 4.12. Let N be a generalized subhypergroup of G. Then N is a generalized normal subhypergroup if there exists a generalized hypergroup H and homomorphism $f: G \to H$ such that for each $a \in G$,

$$N_a = N \cap G_a,$$

is either empty or equals $\ker(f|_{G_a})$, where $G_a = \{g \in G \mid e(g) = e(a)\}$.

Example 4.13. Let G be a semihypergroup, and let $G \times G$ be equipped with the componentwise hyperoperation

$$(x_1, x_2) \circ (y_1, y_2) := \{(u_1, u_2) \mid u_1 \in x_1 \circ y_1,\ u_2 \in x_2 \circ y_2\}.$$

Define

$$N := \{(a, b) \in G \times G \mid a = b \text{ or } a = 3b\}.$$

Then N is a subhypergroup of $G \times G$.

Indeed, let $(a, b), (c, d) \in N$. By definition, each of these pairs satisfies either $a = b$ or $a = 3b$, and either $c = d$ or $c = 3d$. Assume that the hyperoperation on G is compatible with the relation

$$x \sim y \quad \Longleftrightarrow \quad x = y \text{ or } x = 3y,$$

in the sense that whenever $x \sim y$ and $x' \sim y'$, every element of $x \circ x'$ is related by $\sim$ to every element of $y \circ y'$. Then for every $(u_1, u_2) \in (a, b) \circ (c, d)$ we have

$$u_1 \sim u_2,$$

and therefore $(u_1, u_2) \in N$. Hence N is closed under the induced hyperoperation.

If, in addition, the relation $\sim$ is normal in the ambient generalized hypergroup structure, then N is a normal subhypergroup of $G \times G$ in the usual sense.

Now, we introduce the notion of a Generalized Hyperactions and Semidirect Products.

Definition 4.14. Let G be a generalized hypergroup and S a non-empty set. A generalized hyperaction of G on S is a map

$$* : G \times S \to S, \quad (g, x) \mapsto g * x,$$

satisfying:

(i) $(g_1 \circ g_2) * x = g_1 * (g_2 * x)$ for all $g_1, g_2 \in G$ and $x \in S$.
(ii) For each $x \in S$, there exists $e(g) \in G$ such that $e(g) * x = x$.

Example 4.15. Let $G = \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ with the multiplication above. Define $f: G \to \mathbb{R}$ by $f(a, b) = a/b$. Then

$$(a, b) * c = \frac{ac}{b},$$

defines a generalized hyperaction of G on $\mathbb{R}$.

Proposition 4.16. *Let A and B be generalized hypergroups, and let $\cdot : B \times A \to A$ be an action as in Definition 4.14. Define multiplication on $B \ltimes A$ by*

$$(b, a)(b_1, a_1) = (b \circ b_1, a \circ (b \cdot a_1)).$$

Then $B \ltimes A$ is a generalized hypergroup with

$$e(b, a) = (e(b), e(a)), \quad (b, a)^{-1} = (b^{-1}, b^{-1} \cdot a^{-1}).$$

Proof. Straightforward verification using associativity, local identities, inverses, and the definition of the action $\cdot$. □

5. Hypergroupoids

In this section, we present a comprehensive overview of the fundamental concepts related to the theory of hypergroupoids. We also briefly review hypergroup-groupoids, which are hypergroup objects in the category of hypergroupoids.

Definition 5.1. A hypergroupoid consists of a pair of non-empty sets (G, G_0), referred to as the morphism set and the object (or base) set, respectively, together with the following structural maps:

- Source and target maps(single-valued): two functions $\alpha, \beta : G \to G_0$ assigning to each element its source and target.
- Identity (unit) map: $\varepsilon : G_0 \to G$, which associates to each $x \in G_0$ its identity morphism $\varepsilon(x)$, also written as 1_x.
- Inverse map: $i : G \to G$, where $i(x) = x^{-1}$ denotes the inverse of x.
- Partial composition: a map $m : G_2 \to G$, defined on the set of composable pairs

$$G_2 = \{(x, y) \in G \times G \mid \beta(x) = \alpha(y)\},$$

such that $m(x, y)$, usually denoted by xy, represents their composition.

These maps are required to satisfy the following axioms:

(G1) Associativity: For all $x, y, z \in G$ satisfying $\beta(x) = \alpha(y)$ and $\beta(y) = \alpha(z)$,

$$x(yz) = (xy)z.$$

(G2) Identity laws: For each $x \in G$, both $(\varepsilon(\alpha(x)), x)$ and $(x, \varepsilon(\beta(x)))$ lie in G_2, and

$$\varepsilon(\alpha(x)) \cdot x = x = x \cdot \varepsilon(\beta(x)).$$

(G3) Inverse laws: For all $x \in G$, the pairs $(x, i(x))$ and $(i(x), x)$ belong to G_2, and

$$x \cdot i(x) = \varepsilon(\alpha(x)), \qquad i(x) \cdot x = \varepsilon(\beta(x)).$$

The collection of maps $(\alpha, \beta, m, \varepsilon, i)$ is called the structural system of the hypergroupoid.

For each $x, y \in G_0$, the star and co-star subsets of G are defined respectively by

$$\mathrm{St}_G(x) := \alpha^{-1}(x), \qquad \mathrm{CoSt}_G(y) := \beta^{-1}(y).$$

The set of morphisms from x to y is then

$$G(x, y) := \mathrm{St}_G(x) \cap \mathrm{CoSt}_G(y).$$

In particular, $G(x, x)$, the collection of morphisms from x to itself, forms a hypergroup under the restricted composition and is referred to as the vertex hypergroup at x.

Example 5.2 (Algebraic Hypergroupoid Derived from a Group Action)**.** Let $(H, *)$ be a hypergroup and let X be a non-empty set equipped with a (possibly non-deterministic) left action of H, that is, a map

$$\phi : H \times X \longrightarrow \mathcal{P}(X) \setminus \{\emptyset\}, \quad (h, x) \mapsto h \cdot x,$$

satisfying

$$e(h) \cdot x = \{x\}, \quad h_1 \cdot (h_2 \cdot x) \subseteq (h_1 * h_2) \cdot x,$$

for all $h, h_1, h_2 \in H$ and $x \in X$. Define

$$G := \{(h, x) \mid h \in H,\, x \in X\}, \qquad G_0 := X,$$

with the structure maps

$$\alpha(h, x) = x, \quad \beta(h, x) \in h \cdot x, \quad \varepsilon(x) = (e(h_x), x), \quad i(h, x) = (h^{-1}, \beta(h, x)).$$

Two elements (h_1, x_1) and (h_2, x_2) are composable whenever $x_2 \in h_1 \cdot x_1$, and the partial multiplication is given by

$$(h_1, x_1) \circ (h_2, x_2) := (h_1 * h_2, x_2).$$

It can be verified that (G, G_0) satisfies the hypergroupoid axioms. We call this the action hypergroupoid associated with (H, X).

Example 5.3 (Topological Hypergroupoid Arising from Fuzzy Relations)**.** Let $X = \{1, 2, 3\}$ and consider the family $\mathcal{R}$ of all fuzzy relations on X, i.e.,

$$\mathcal{R} = \{R : X \times X \to [0, 1]\}.$$

Define a hyperoperation $\circ$ on $\mathcal{R}$ by

$$(R_1 \circ R_2)(x, z) = \sup_{y \in X} \min\{R_1(x, y), R_2(y, z)\}.$$

Let $G = \mathcal{R}$ and $G_0 = X$. Define the source and target maps as

$$\alpha(R) = \{x \in X \mid R(x, y) = 1 \text{ for some } y\}, \qquad \beta(R) = \{y \in X \mid R(x, y) = 1 \text{ for some } x\}.$$

The identity inclusion is given by $\varepsilon(x) = I_x$, where $I_x(x,x) = 1$ and $I_x(y,z) = 0$ otherwise. Inversion is defined pointwise by $R^{-1}(x,y) = R(y,x)$. Composition of compatible fuzzy relations is the hyperoperation $\circ$ defined above.

Then $(G, G_0, \alpha, \beta, \varepsilon, i, \circ)$ forms a hypergroupoid, known as the fuzzy relational hypergroupoid. A graphical depiction of a composable triple (R_1, R_2, R_3) is given by the diagram:

$$x \xrightarrow[0.8]{R_1} y \xrightarrow[0.9]{R_2} z \xrightarrow[1.0]{R_3} w,$$

where the composition aggregates relational strengths according to the t-norm min and the t-conorm sup.

Example 5.4. (i) Any hypergroup can be regarded as a groupoid with a single object, where the hypergroup itself forms the set of morphisms.

(ii) Any set G can be considered as a hypergroupoid over itself with $\alpha = \beta = \mathrm{id}_G$, where every element acts as an identity morphism.

(iii) Let X be a set. The Cartesian product $X \times X$ becomes a banal hypergroupoid over X, with

$$\alpha(x,y) = x, \quad \beta(x,y) = y, \quad \varepsilon(x) = (x,x),$$

multiplication

$$(x,y)(y,z) = (x,z),$$

and inversion

$$(x,y)^{-1} = (y,x).$$

Definition 5.5. Let G and G' be hypergroupoids over base sets B and B', respectively. A hypergroupoid homomorphism from G to G' is a pair of maps (f, f_0), with $f : G \to G'$ and $f_0 : B \to B'$, such that

$$\alpha' \circ f = f_0 \circ \alpha, \quad \beta' \circ f = f_0 \circ \beta,$$

and for all composable pairs $(a,b) \in G_2$,

$$f(ab) = f(a)f(b).$$

For brevity, we often denote such a homomorphism simply by f.

The collection of all hypergroupoids with their homomorphisms forms a category, commonly denoted by **Gpd**.

6. Hypergroup-groupoids

We now revisit the notion of a hypergroup-groupoid, formally understood as a hypergroup object in the category of hypergroupoids. This concept enriches the structure of a hypergroupoid by equipping both its morphism and object sets with compatible hypergroup structures.

Let $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ be a hypergroupoid. Suppose that G is endowed with a hypergroup operation

$$w : G \times G \to G, \quad (x, y) \mapsto x + y,$$

with identity element $e \in G$ defined by a map $v : \{\lambda\} \to G$, $v(\lambda) = e$, and inversion map $\sigma : G \to G$, $\sigma(x) = -x$. Similarly, assume that the object set G_0 carries a hypergroup structure $(G_0, +, e_0, \sigma_0)$ with operation

$$w_0 : G_0 \times G_0 \to G_0, \quad (u, v) \mapsto u + v,$$

identity e_0, and inversion $\sigma_0 : G_0 \to G_0$, $\sigma_0(u) = -u$.

Definition 6.1. A *hypergroup-groupoid* is a hypergroupoid (G, G_0) such that:

(i) $(G, +, e, \sigma)$ and $(G_0, +, e_0, \sigma_0)$ define hypergroup structures on G and G_0, respectively;

(ii) The maps

$$(w, w_0) : (G \times G, G_0 \times G_0) \to (G, G_0), \quad v : \{\lambda\} \to G, \quad (\sigma, \sigma_0) : (G, G_0) \to (G, G_0),$$

are all morphisms of hypergroupoids;

(iii) The interchange law holds between the hypergroupoid composition and the hypergroup operation:

$$(b \circ a) + (d \circ c) = (b + d) \circ (a + c),$$

whenever the compositions are defined.

We denote a hypergroup-groupoid by $(G, \alpha, \beta, m, \varepsilon, i, +, G_0)$, or simply $(G, \alpha, \beta, m, +, G_0)$ when the context is clear.

Example 6.2. Let G be any hypergroup. Then the Cartesian product $G \times G$ forms a hypergroup-groupoid over G, where the groupoid operations are defined as in the banal hypergroupoid, and the hypergroup operations on $G \times G$ are inherited component-wise from the hypergroup operation on G.

Definition 6.3. Let G and H be hypergroup-groupoids. A morphism of hypergroup-groupoids is a hypergroupoid homomorphism

$$f : G \to H,$$

that preserves the hypergroup structure, i.e., for all $(a, b) \in G_2$,

$$f(m_G(a, b)) = m_H(f(a), f(b)),$$

where m_G and m_H denote the hypergroupoid compositions in G and H, respectively.

The collection of hypergroup-groupoids and their morphisms forms a category, denoted by **GGd**.

7. Generalized hypergroup-groupoids

We now extend the notion of hypergroup-groupoids to incorporate generalized hypergroups, leading to the concept of generalized hypergroup-groupoids. These structures provide enhanced flexibility in defining identity and inverse elements, thereby accommodating a broader class of algebraic systems.

Definition 7.1. A generalized hypergroup-groupoid (G, G_0) is a groupoid that satisfies:

(i) $(G, +, e, \sigma)$ and $(G_0, +, e_0, \sigma_0)$ are generalized hypergroups, where the identity and inverse elements may depend on the element;

(ii) The maps

$$(w, w_0) : (G \times G, G_0 \times G_0) \to (G, G_0), \quad v : \{\lambda\} \to G, \quad (\sigma, \sigma_0) : (G, G_0) \to (G, G_0),$$

are morphisms of hypergroupoids;

(iii) The generalized hypergroup operation is compatible with the hypergroupoid composition via the interchange law:

$$(b \circ a) + (d \circ c) = (b + d) \circ (a + c),$$

whenever the compositions are defined.

Such a structure is denoted by $(G, \alpha, \beta, m, \varepsilon, i, +, G_0)$.

Example 7.2. Let G be a generalized hypergroup. Then $G \times G$ forms a generalized hypergroup-groupoid over G, where the hypergroupoid structure is derived from the banal groupoid construction, and the generalized hypergroup operation is defined component-wise:

$$(x, y) + (z, t) = (x + z, y + t).$$

The identity of (x, y) is $(e(x), e(y))$, and the inverse is $(-x, -y)$. Compatibility with the hypergroupoid composition is verified, for instance:

$$\big((z, y) + (z', y')\big) \circ \big((y, x) + (y', x')\big) = (z + z', x + x') = \big((z, y) \circ (y, x)\big) + \big((z', y') \circ (y', x')\big),$$

confirming the interchange law. Similarly, the unit and inverse maps are hypergroupoid morphisms, completing the structure.

Definition 7.3. Let G and H be generalized hypergroup-groupoids. A homomorphism of generalized hypergroup-groupoids is a hypergroupoid morphism

$$f : G \to H,$$

that preserves the generalized hypergroup structure. The collection of generalized hypergroup-groupoids and their homomorphisms forms a category, denoted by **GG-Gd**.

Proposition 7.4. *There exists a covariant functor from the category* **GG** *of generalized hypergroups to the category* **GG-Gd** *of generalized hypergroup-groupoids.*

Proof. Let G be a generalized hypergroup. According to the previous example, $G \times G$ inherits a natural structure of a generalized hypergroup-groupoid. Suppose $f : G_1 \to G_2$ is a homomorphism of generalized hypergroups. Define

$$\Gamma(f) : G_1 \times G_1 \to G_2 \times G_2, \quad (a, b) \mapsto (f(a), f(b)).$$

But, $\Gamma(f)$ is a morphism of generalized hypergroup-groupoids, preserving all relevant structure maps. Therefore, Γ defines a functor:

$$\Gamma : \mathbf{GG} \longrightarrow \mathbf{GG\text{-}Gd}.$$

□

Remark 7.5. In the subsequent sections, we shall establish a categorical equivalence between two fundamental classes of structures: the category of generalized hypergroup-groupoids whose object sets possess abelian generalized hypergroup structures, and the category of generalized crossed hypermodules. Let $\mathbf{GG\text{-}Gd^{Ab}}$ denote the full subcategory of **GG-Gd** comprising all generalized hypergroup-groupoids with abelian object components. This correspondence forms a pivotal component of the categorical framework developed herein and underlies the equivalence theorems presented later in the paper.

8. Generalized crossed hypermodules

In this section, we introduce the notion of generalized crossed hypermodules, which naturally extends classical crossed hypermodules in hypergroup theory to the context of generalized hypergroups.

Definition 8.1. A generalized crossed hypermodule is a triple (M, P, δ), where M and P are generalized hypergroups, $\delta : M \to P$ is a homomorphism of generalized hypergroups, and there exists a generalized right action of P on M, denoted $(m, p) \mapsto m \cdot p$, satisfying for all $m, n \in M$ and $p \in P$:

GCM1. $\delta(m \cdot p) = p\delta(m)p^{-1}$,

GCM2. $\delta(m) \cdot n = mnm^{-1}$.

Example 8.2. Consider the set $K = \{1, 2, 3\}$ with addition modulo 3. Define the set of inner automorphisms:

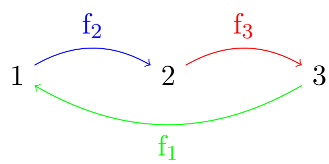
$$\mathcal{I}(K) = \{f_k : K \to K \mid f_k(k') = k + k' - k \equiv k' \pmod 3,\ \forall k, k' \in K\}.$$

Under composition, $\mathcal{I}(K)$ forms a generalized hypergroup. Define the homomorphism

$$\delta : K \to \mathcal{I}(K), \quad \delta(k) = f_k,$$

and an action of $\mathcal{I}(K)$ on K by

$$f_k \cdot k' := f_k(k').$$



This diagram visualizes the action of the inner automorphisms on the elements of K, illustrating the structure of the generalized crossed hypermodule $(K, \mathcal{I}(K), \delta)$.

Example 8.3. Let $G = \{e, a, b\}$ with multiplication

$$a^2 = e, \quad ab = ba = b, \quad b^2 = e,$$

and let $N = \{e, a\} \subseteq G$ be a generalized normal subhypergroup. Define the inclusion homomorphism

$$\delta = \iota : N \hookrightarrow G,$$

and the conjugation action of G on N by

$$g \cdot n := gng^{-1}, \quad \forall g \in G, n \in N.$$

This diagram depicts all conjugation actions of elements of G on the subhypergroup N, clearly showing how each element acts. Under this action, (N, G, ι) is a generalized crossed hypermodule.

Example 8.4. Let $G = \{(1, 2), (2, 1), (3, 1)\}$ and $N = \{(1, 1), (2, 2)\}$. Define

$$\delta = \iota : N \to G, \quad (a, b) \cdot (c, d) = \left(\frac{da}{b}, d\right).$$

$$
\begin{tikzcd}
(1,1) \arrow[d, bend left, "(1,2)\cdot"] & (2,2) \arrow[d, bend left, "(2,1)\cdot"] \\
(1,2) & (2,1)
\end{tikzcd}
$$

These diagrams visually represent the action of G on N, highlighting how the generalized crossed hypermodule conditions are satisfied.

9. Homomorphisms of generalized crossed hypermodules

Definition 9.1. Let (C, G, δ) and (C', G', δ') be generalized crossed hypermodules. A homomorphism is a pair of generalized hypergroup homomorphisms (φ, ψ) with

$$\varphi : C \to C', \quad \psi : G \to G',$$

such that for all $c \in C$ and $g \in G$:

(i) $\psi \circ \delta(c) = \delta' \circ \varphi(c)$,
(ii) $\varphi(g \cdot c) = \psi(g) \cdot \varphi(c)$.

The category of generalized crossed hypermodules with these morphisms is denoted **GCM**.

Now, we present three illustrative and non-trivial examples of homomorphisms between generalized crossed hypermodules. Each example is given with explicit hyperoperations, tables, and commutative diagrams verifying the axioms of Definition 9.1.

Example 9.2. A two-element hypermodule with non-trivial action:

Let

$$C = \{0, x\}, \qquad G = \{e, g\}.$$

On C, define the hyperaddition $\oplus$ as:

$\oplus$	0	x
0	$\{0\}$	$\{x\}$
x	$\{x\}$	$\{0, x\}$

and on G, define the hypermultiplication $\star$ as:

$\star$	e	g
e	$\{e\}$	$\{g\}$
g	$\{g\}$	$\{e, g\}$

Define

$$\delta : C \to G, \qquad \delta(0) = e, \quad \delta(x) = g,$$

and the (multi-valued) action

$$e \cdot c = \{c\}, \qquad g \cdot 0 = \{0\}, \quad g \cdot x = \{0, x\}.$$

One easily checks that $\delta(g \cdot x) = g \star \delta(x) = \{e, g\}$, so (C, G, δ) forms a generalized crossed hypermodule.

Let $(\varphi, \psi) : (C, G, \delta) \to (C', G', \delta')$ be given by the same underlying sets ($C' = C$, $G' = G$), but with

$$\varphi(0) = 0, \ \varphi(x) = 0, \quad \psi(e) = e, \ \psi(g) = e.$$

Then, for all $c \in C$,

$$\psi(\delta(c)) = e = \delta'(\varphi(c)),$$

and for all $g \in G, c \in C$,

$$\varphi(g \cdot c) = \psi(g) \cdot \varphi(c).$$

Hence (φ, ψ) is a valid homomorphism.

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\delta"'] & C' \arrow[d, "\delta'"] \\ G \arrow[r, "\psi"'] & G' \end{tikzcd}$$

The diagram commutes since $\psi \circ \delta = \delta' \circ \varphi$.

Example 9.3. Trivial group but strongly hyper additive structure:

Let

$$C = \{0, a, b\}, \qquad G = \{e\}.$$

$\oplus$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0, a, b\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$$\delta(0) = e, \ \delta(a) = e, \ \delta(b) = e, \qquad e \cdot c = \{c\}.$$

Thus (C, G, δ) is a generalized crossed hypermodule with trivial action.

Let

$$C' = \{0', a'\}, \qquad G' = \{e'\},$$

with

$$0' \oplus' a' = \{a'\}, \quad a' \oplus' a' = \{0', a'\}.$$

Define

$$\varphi(0) = 0',\ \varphi(a) = a',\ \varphi(b) = 0', \qquad \psi(e) = e'.$$

Since δ and δ' are constant and the actions trivial, both conditions of Definition 9.1 are satisfied, so (φ, ψ) is a homomorphism.

Example 9.4. Non-isomorphic source and target hypermodules:

Let

$$C = \{0, u, v\}, \quad G = \{e, h\},$$

with hyperaddition on C:

$\oplus$	0	u	v
0	$\{0\}$	$\{u\}$	$\{v\}$
u	$\{u\}$	$\{0, u\}$	$\{v\}$
v	$\{v\}$	$\{v\}$	$\{0, u, v\}$

and hypermultiplication on G:

$\star$	e	h
e	$\{e\}$	$\{h\}$
h	$\{h\}$	$\{e\}$

Define

$$\delta(0) = e, \quad \delta(u) = h, \quad \delta(v) = e,$$

and an action

$$e \cdot c = \{c\}, \qquad h \cdot 0 = \{0\},\ h \cdot u = \{u\},\ h \cdot v = \{u, v\}.$$

Let

$$C' = \{0', x\}, \qquad G' = \{e'\},$$

with

$$x \oplus' x = \{0', x\}, \qquad \delta'(0') = e',\ \delta'(x) = e'.$$

The action is trivial: $e' \cdot y = \{y\}$.

Define

$$\varphi(0) = 0',\ \varphi(u) = x,\ \varphi(v) = x, \qquad \psi(e) = e',\ \psi(h) = e'.$$

Then:

$$\psi(\delta(c)) = \delta'(\varphi(c)), \quad \varphi(g \cdot c) = \psi(g) \cdot \varphi(c),$$

for all $c \in C, g \in G$. For example, for $g = h, c = v$:

$$\varphi(h \cdot v) = \varphi(\{u, v\}) = \{x\} = \psi(h) \cdot \varphi(v) = e' \cdot x = \{x\}.$$

Hence (φ, ψ) is a valid non-trivial homomorphism.

$$\begin{tikzcd} C = \{0, u, v\} \arrow[r, "\varphi"] \arrow[d, "\delta"'] & C' = \{0', x\} \arrow[d, "\delta'"] \\ G = \{e, h\} \arrow[r, "\psi"'] & G' = \{e'\} \end{tikzcd}$$

This example shows that the morphism can collapse several hyper-elements of C and still preserve the structure of the crossed hypermodule.

These three examples demonstrate that:

(a) Both C and G may possess genuinely hyper-valued operations.
(b) Homomorphisms (φ, ψ) need not be injective or surjective to satisfy the axioms.
(c) The commutativity condition $\psi \circ \delta = \delta' \circ \varphi$ ensures that the boundary and group-action structures remain compatible.

10. Equivalence of categories

In this section, we rigorously establish an equivalence between the category of generalized crossed hypermodules and that of generalized hypergroup-groupoids whose object sets are abelian hypergroups. We also provide explicit numerical examples that illustrate how these constructions operate in concrete finite settings.

Theorem 10.1. *Let $\mathcal{G}$ be a generalized hypergroup-groupoid with abelian object set $\mathcal{G}_0$. Then there exists a generalized crossed hypermodule*

$$\Phi(\mathcal{G}) = (A, B, \delta),$$

naturally induced by $\mathcal{G}$.

Proof. Let $e \in \mathcal{G}_0$ be the identity object. We define:

$$A = \mathrm{CoSt}_{\mathcal{G}}(e) = \{g \in \mathcal{G} \mid \alpha(g) = e\},$$

the costar of e, i.e. the collection of all arrows in $\mathcal{G}$ emanating from e. Set $B = \mathcal{G}_0$, the object hypergroup, which is abelian by assumption.

Let $\beta : \mathcal{G} \to \mathcal{G}_0$ denote the target map; define the boundary map

$$\delta = \beta|_A : A \to B.$$

The partial composition in $\mathcal{G}$ restricts to a hyperoperation on A, giving A the structure of a generalized hypergroup.

Next, we define an action of B on A by

$$x \cdot m := 1_x + m - 1_x, \qquad x \in B,\ m \in A,$$

where 1_x denotes the identity arrow at x. Since the object hypergroup B is abelian, this conjugation-type operation is well-defined and satisfies:

(a) $\delta(x \cdot m) = x + \delta(m) - x$, showing that δ is B-equivariant.

(b) For all $x, y \in B$, $(xy) \cdot m = x \cdot (y \cdot m)$ (generalized associativity of the action).

Hence (A, B, δ) satisfies the axioms of a generalized crossed hypermodule, with the structural conditions GCM1 and GCM2 inherited from the internal laws of $\mathcal{G}$. □

Theorem 10.2. *Conversely, every generalized crossed hypermodule (A, B, δ) induces a generalized hypergroup-groupoid $\Psi(A, B, \delta)$ defined as*

$$\Psi(A, B, \delta) = (B \ltimes A, B, \alpha, \beta, \varepsilon, \eta),$$

where $B \ltimes A$ denotes the semidirect hyperproduct.

Proof. Let

$$\alpha(b, a) = b, \qquad \beta(b, a) = \delta(a) + b, \qquad \varepsilon(b) = (b, e_A),$$

where e_A is the identity element of A. Two hyper-arrows (b_1, a_1) and (b_2, a_2) are composable whenever $\alpha(b_1, a_1) = \beta(b_2, a_2)$, and we define:

$$(b_1, a_1) \circ (b_2, a_2) = (b_2,\ a_1 a_2).$$

The inverse map is given by

$$\eta(b, a) = (\delta(a) + b, a^{-1}),$$

where a^{-1} is the hyperinverse of a in A.

For verification of the axioms, we have:

(a) Identity laws: For every (b, a), one has

$$\varepsilon(\alpha(b, a)) \circ (b, a) = (b, a) = (b, a) \circ \varepsilon(\beta(b, a)).$$

(b) Inverse laws: By direct computation:

$$(b, a) \circ \eta(b, a) = \varepsilon(\beta(b, a)), \qquad \eta(b, a) \circ (b, a) = \varepsilon(\alpha(b, a)).$$

(c) Interchange law: The semidirect hyperproduct structure implies

$$((b_1, a_1) \circ (b_2, a_2)) \cdot ((b'_1, a'_1) \circ (b'_2, a'_2)) = (b_2, a_1 a_2) \cdot (b'_2, a'_1 a'_2),$$

is compatible with the separate hyperoperations on B and A.

Thus, $\Psi(A, B, \delta)$ is a generalized hypergroup-groupoid with abelian base B. $\square$

Theorem 10.3 (Eequivalence of Categories)**.** *Let G be a generalized hypergroup. The categories of generalized hypergroup-groupoids and generalized crossed hypermodules are equivalent.*

Proof. To demonstrate this equivalence, we construct two functors: F from the category of generalized hypergroup-groupoids to that of generalized crossed hypermodules, and G in the reverse direction. Functor F Let $\mathcal{H}$ be a generalized hypergroup-groupoid.

- Objects: Define the associated generalized crossed hypermodule $F(\mathcal{H})$ as follows:
  - The underlying set $F(\mathcal{H})$ consists of all objects of $\mathcal{H}$.
  - The action of G on $F(\mathcal{H})$ is induced by the morphism structure of $\mathcal{H}$.
- Morphisms: For each morphism $f : X \to Y$ in $\mathcal{H}$, define the corresponding morphism $F(f) : F(X) \to F(Y)$ as follows:

$$F(f)(g \cdot x) = g \cdot f(x) \text{ for all } g \in G, x \in X.$$

Functor G Let M be a generalized crossed hypermodule over G.

- Objects: Define the associated generalized hypergroup-groupoid $G(M)$ as follows:
  - The objects of $G(M)$ are the elements of M.
  - The morphisms are given by the actions of G on M, with $g : x \to y$ if $y = g \cdot x$ for $g \in G, x \in M$.
- Morphisms: For any morphisms $f, g : A \to B$ in M, define the corresponding morphism between generalized hypergroup-groupoids by:

$$G(f)(g \cdot a) = g \cdot f(a) \text{ for all } g \in G, a \in A.$$

We aim to show:

$$F(G(M)) \cong M \quad \text{and} \quad G(F(\mathcal{H})) \cong \mathcal{H}.$$

1. We have $F(G(M))$ retrieves the structure of M. For each $m \in M$, we establish a correspondence:

$$F(G(m)) \mapsto m.$$

This mutual retrievability guarantees the functor is full and faithful. 2. For $Y \in G$, the operation within $F(G(Y))$ corresponds directly to operations respected within M. Both functors F and G are natural transformations. For every morphism $f : A \to B$ in $\mathcal{H}$, the corresponding mappings satisfy the commutative properties:

$$G(F(f)) = F(f)G.$$

Thus, both functors commute with the composition structure of the morphisms. $\square$

We now present some non-trivial finite examples to concretize the correspondence between these two structures.

Example 10.4. Let $A = \{0, x\}$ with hyperaddition

$\oplus$	0	x
0	$\{0\}$	$\{x\}$
x	$\{x\}$	$\{0, x\}$

and $B = \{e, g\}$ with hypermultiplication

$\star$	e	g
e	$\{e\}$	$\{g\}$
g	$\{g\}$	$\{e, g\}$

and $\delta(0) = e$, $\delta(x) = g$. The action $g \cdot x = \{0, x\}$ defines a generalized crossed hypermodule (A, B, δ). The semidirect hyperproduct $B \ltimes A = \{(e, 0), (e, x), (g, 0), (g, x)\}$ gives a generalized hypergroup-groupoid with:

$$\alpha(b, a) = b, \quad \beta(b, a) = \delta(a) + b,$$

thus verifying Theorem 10.2.

Example 10.5. Take $A = \{0, a, b\}$ with

$$a \oplus a = \{0, a, b\}, \qquad b \oplus b = \{0\},$$

and $B = \{e\}$ (trivial abelian base). Then (A, B, δ), with $\delta(c) = e$ for all c, forms a generalized crossed hypermodule with trivial action. The induced hypergroup-groupoid is $B \ltimes A \cong A$, showing that $\Psi(A, B, \delta)$ coincides with the original hypergroup structure.

Example 10.6. Let $A = \{0, u, v\}$ and $B = \{e, h\}$ as in Example 10.1, with

$$v \oplus v = \{0, u, v\}, \quad h \star h = \{e\},$$

and $\delta(0) = e$, $\delta(u) = h$, $\delta(v) = e$. Then the action $h \cdot v = \{u, v\}$ produces the hypergroup-groupoid

$$B \ltimes A = \{(e, 0), (e, u), (e, v), (h, 0), (h, u), (h, v)\},$$

whose structure maps are consistent with the formulas of Theorem 10.2. Hence (A, B, δ) and $\Psi(A, B, \delta)$ form a corresponding pair under the categorical equivalence of Theorem **??**.

The preceding results collectively establish that every generalized hypergroup-groupoid with abelian base corresponds canonically to a generalized crossed hypermodule, and vice versa. The equivalence functors Φ and Ψ provide a bidirectional translation that preserves all algebraic and categorical structure, both in theory and in concrete finite models.

11. Some applications of generalized crossed hypermodules

Generalized crossed hypermodules provide a versatile algebraic framework with applications across mathematics, physics, computer science, and other fields. Here, we outline some notable areas of application and present non-trivial examples for each.

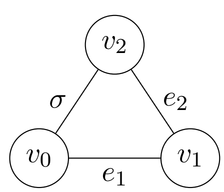
11.1. **Algebraic geometry and sheaf theory.** Sheaves of hyperstructures over a topological space can be organized as generalized crossed hypermodules to study cohomological invariants.

Example 11.1. Let X be a complex algebraic variety and $\mathcal{F}$ a sheaf of generalized hypergroups of functions. The morphism sending a local section to its restriction defines a generalized crossed hypermodule, encoding local-to-global extension problems in the variety.

Example 11.2. Consider a scheme S with a sheaf of abelian generalized hypergroups $\mathcal{O}_S$ and a sheaf of generalized hypergroups $\mathcal{M}$. The action of $\mathcal{O}_S$ on $\mathcal{M}$ along with the natural inclusion forms a generalized crossed hypermodule, modeling cohomological obstructions in deforming sheaves of non-classical algebraic structures.

11.2. **Algebraic topology: Small homotopy models.**

Example 11.3. Let X be a simplicial complex consisting of three vertices v_0, v_1, v_2 and two edges $e_1 = (v_0, v_1)$, $e_2 = (v_1, v_2)$, with a single 2-simplex $\sigma = (v_0, v_1, v_2)$. Define $M = \{\sigma\}$ as a generalized hypergroup with $\sigma + \sigma = \{\sigma\}$, and $P = \{e_1, e_2\}$ as an abelian generalized hypergroup with $e_1 + e_2 = \{e_1, e_2\}$. The boundary map $\delta : M \to P$ sends $\sigma \mapsto e_1 + e_2$, forming a generalized crossed hypermodule.



Example 11.4. Take $M = \{a, b, c\}$ with hyperoperation $x + y = \{x, y\}$ and $P = \{1, 2\}$ with $x + y = \{x, y\}$. Let $\delta(a) = 1$, $\delta(b) = 2$, $\delta(c) = 1 + 2$. Define an action $P \times M \to M$ as $p \cdot m = m$ for simplicity. This discrete model is a numerical generalized crossed hypermodule suitable for computer simulations of 2-types.

Example 11.5. Let X be a topological space with a fundamental groupoid $\pi_1(X)$ and a second homotopy group $\pi_2(X)$ forming a generalized hypergroup. The boundary map $\delta : \pi_2(X) \to \pi_1(X)$ along with the natural action of $\pi_1(X)$ on $\pi_2(X)$ forms a generalized crossed hypermodule. This models the nonabelian second homotopy structure in spaces with nontrivial fundamental group.

Example 11.6. Consider a simplicial complex K where the 2-simplices form a generalized hypergroup under face addition, and the 1-simplices form an abelian generalized hypergroup. The map sending each 2-simplex to its boundary edge defines a generalized crossed hypermodule, providing a categorical perspective on the homotopy 2-type of K.

11.3. Quantum algebra: Matrix examples.

Example 11.7. Let M be the set of 2×2 integer matrices modulo 3:

$$M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots \right\}$$

with hyperoperation defined as $A+B = \{A, B\}$. Let P be the set of diagonal matrices modulo 3, an abelian generalized hypergroup. The map $\delta : M \to P$ sends each matrix to its diagonal part, and P acts on M via conjugation. This forms a concrete generalized crossed hypermodule representing a small quantum symmetry group.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\quad\delta\quad} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

Example 11.8. Consider $M = \{X_0, X_1\}$ as 2×2 Pauli matrices, and $P = \{I, -I\}$. The map $\delta(X) = I$ for all X and action $P \times M \to M$ as scalar multiplication forms a generalized crossed hypermodule encoding a minimal noncommutative structure.

Example 11.9. Let H be a Hopf algebra with a noncommutative coproduct. Define M as the set of unitary elements under a generalized hyperoperation induced by the algebra multiplication. Then $M \to H$ together with the conjugation action forms a generalized crossed hypermodule, capturing quantum symmetry relations beyond classical groups.

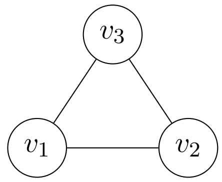
Example 11.10. In a braided monoidal category, let P be the set of braidings forming an abelian generalized hypergroup, and M the set of objects with a generalized tensor product. The morphism sending objects to their braidings along with the natural action yields a generalized crossed hypermodule modeling noncommutative categorical symmetries.

11.4. Discrete geometry: Graphs and hypergraphs.

Example 11.11. Let M be the set of hyperedges in a hypergraph:

$$M = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\}$$

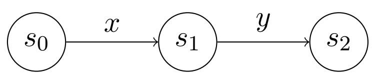
with $+$ as union. Let $P = \{\{v_1\}, \{v_2\}, \{v_3\}\}$. Define δ as the vertex map sending each hyperedge to the set of vertices in P it contains, and let P act on M by union. This gives a combinatorial crossed hypermodule representing hypergraph connectivity.



Example 11.12. Let $M = \{e_1, e_2, e_3\}$ be edges in a small directed graph and $P = \{v_1, v_2\}$ vertices. Define $\delta(e_i)$ as the source vertex of e_i and let vertices act by swapping edges incident to them. This is a numeric generalized crossed hypermodule encoding small network dynamics.

11.5. **Computer science: State machines.**

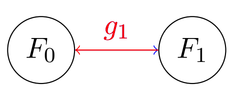
Example 11.13. Let $M = \{s_0, s_1, s_2\}$ be system states and $P = \{x, y\}$ events. Define $\delta(s_i) = x$ if i even, y if i odd. Let P act on M by $x \cdot s_0 = s_1$, $y \cdot s_1 = s_2$. This finite system forms a generalized crossed hypermodule representing a non-deterministic automaton.



Example 11.14. A simple concurrent system with $M = \{m_0, m_1\}$ states and $P = \{p_1, p_2\}$ events. Define $\delta(m_i) = p_i$ and action as $p_i \cdot m_j = m_{(i+j) \mod 2}$. This numeric model illustrates hypergroup-groupoid interactions in parallel processes.

11.6. **Mathematical physics: Discrete gauge models.**

Example 11.15. Let $M = \{F_0, F_1\}$ be discrete field configurations on a lattice, $P = \{g_0, g_1\}$ discrete gauge transformations. Map $\delta(F_i) = g_i$ and define P action on M as $g_i \cdot F_j = F_{(i+j) \mod 2}$. This finite generalized crossed hypermodule models a minimal lattice gauge theory.



Example 11.16. On a 2D lattice, let M be plaquette configurations $\{P_0, P_1, P_2\}$ and P be abelian group of fluxes $\{0, 1, 2\}$ modulo 3. Define $\delta(P_i) = i$ and action $p \cdot P_i = P_{(i+p) \mod 3}$. This discrete model encodes higher gauge symmetry as a generalized crossed hypermodule.

Example 11.17. Let M be a generalized hypergroup representing possible states of a distributed system, and P the generalized hypergroup of synchronization events. The map sending each system state to its corresponding synchronization step, along with the event-action on states, forms a generalized crossed hypermodule, capturing non-deterministic interactions in concurrent computations.

Example 11.18. In a network protocol with multiple message types forming a generalized hypergroup, define M as the set of all protocol states. The action of message types on states and the map sending states to their message log create a generalized crossed hypermodule, modeling complex message-passing and rollback behaviors.

Example 11.19. Consider a gauge theory where the gauge group is replaced by a generalized hypergroup P, and the space of field configurations forms M. The gauge transformation map $\delta : M \to P$ together with the generalized action of P on M defines a generalized crossed hypermodule, providing a framework for higher gauge theories with non-abelian 2-gauge symmetries.

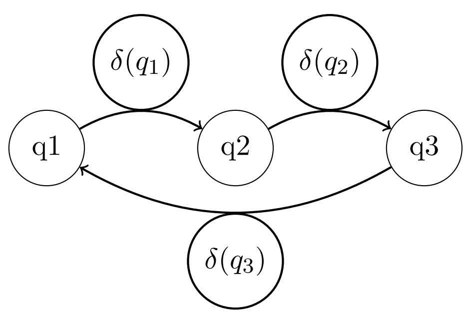
Example 11.20. In string theory, the set of string junctions can be organized as a generalized hypergroup M, and the set of brane charges as an abelian generalized hypergroup P. The natural map assigning a junction to its total brane charge along with the induced action forms a generalized crossed hypermodule, capturing higher-order charge conservation laws in non-perturbative settings.

11.7. **Quantum error correction in multi-level systems.** Generalized crossed hypermodules provide a rigorous framework to model interactions in multi-level quantum systems (qudits), where operations may only be partially defined.

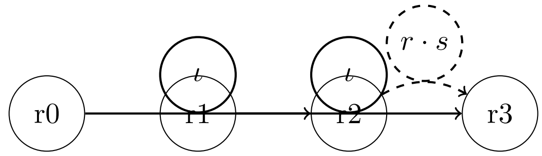
Example 11.21. Consider a system of three qutrits represented by the generalized hypergroup $Q = \{0, 1, 2\}$ under addition modulo 3. Define the inner automorphism hypergroup $\mathcal{I}(Q)$ via the map

$$\delta : Q \to \mathcal{I}(Q), \quad \delta(q)(q') = q + q' - q \mod 3.$$

The generalized crossed hypermodule $(Q, \mathcal{I}(Q), \delta)$ captures the propagation of quantum errors across qutrits and provides the foundation for constructing multi-level quantum error-correcting codes.



Example 11.22. Let $R = \{r_0, r_1, r_2, r_3\}$ denote the states of a four-level quantum system. Define $S = \{r_0, r_1\} \subset R$ as a generalized normal subhypergroup. Employ the inclusion map $\iota : S \hookrightarrow R$ with the conjugation action $r \cdot s = rsr^{-1}$. The resulting generalized crossed hypermodule (S, R, ι) effectively tracks decoherence channels and enables the design of selective error-correcting operations.

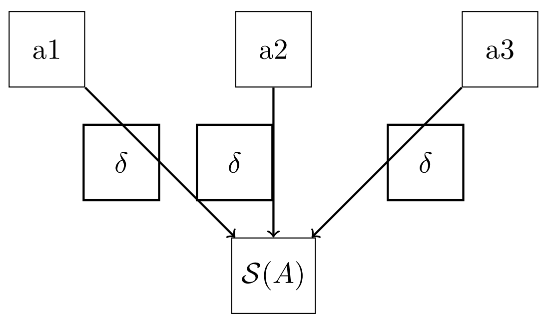


11.8. **Multi-agent decision making in uncertain environments.**

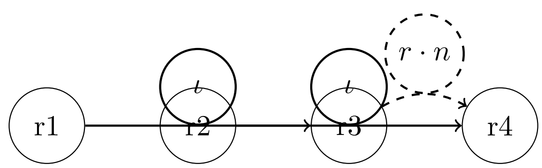
Example 11.23. Let $A = \{a_1, a_2, a_3\}$ denote autonomous agents in a network. Define the hypergroup of internal strategies $\mathcal{S}(A)$ with

$$\delta : A \to \mathcal{S}(A),$$

mapping each agent to its strategy automorphism. The crossed hypermodule $(A, \mathcal{S}(A), \delta)$ encodes permissible sequences of actions and facilitates conflict resolution in dynamic environments.



Example 11.24. Consider a swarm of robots $R = \{r_1, r_2, r_3, r_4\}$ and a subset $N = \{r_1, r_2\}$ representing leaders. With the inclusion $\iota : N \hookrightarrow R$ and the influence action $r \cdot n$, the crossed hypermodule (N, R, ι) captures emergent group behaviors and coordinated decision-making.



12. Conclusion

In this work, we introduced generalized crossed hypermodules as a substantive extension of classical crossed modules within the framework of generalized hypergroups. By admitting partially defined binary operations and employing appropriately generalized associativity conditions, the resulting structure offers a flexible yet rigorous setting for the study of algebraic systems that are not amenable to standard group-theoretic techniques.

To provide a coherent categorical foundation, we formulated the notion of a generalized hypergroup-groupoid, which simultaneously generalizes classical groupoids and generalized hypergroups. This perspective yields a natural categorical environment for generalized crossed hypermodules and enables a precise characterization of their algebraic and combinatorial features. Furthermore, we established that the category of generalized crossed hypermodules is categorically equivalent to the category of generalized hypergroup-groupoids with abelian object sets. This equivalence generalizes and unifies key principles from both group theory and algebraic topology, thereby offering an integrated framework for hyperstructural investigations.

In addition to their theoretical relevance, generalized crossed hypermodules provide a suitable algebraic language for modeling complex interactions in settings such as multi-level quantum systems, multi-agent networks, and heterogeneous sensor arrays, where dependencies may be partially specified or context-dependent. The capacity to capture such behavior also indicates potential applicability in areas including network science, distributed computing, and systems biology.

Finally, we outlined several directions for future research, including the development of computational methods for constructing and analyzing generalized crossed hypermodules, the exploration of connections with higher categorical structures and homotopy theory, and the integration of these frameworks with probabilistic and stochastic models for uncertainty propagation in complex systems. Overall, the developed theory establishes a robust foundation for both further mathematical developments and prospective interdisciplinary applications.

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