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Research Paper

ON DOUBLE CYCLIC, CONSTACYCLIC AND LINEAR COMPLEMENTARY DUAL CODES OVER $\mathbf{R_q^n}$

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Abstract. In this paper, we study double cyclic and double constacyclic code over the field $\mathbf{R_q^n}$ and a non-chain ring $\mathbf{R_q^n} = \mathbb{F}_q^n + u\mathbb{F}_q^n$, $u^2 = u$. We determine the generator polynomials and minimum spanning sets for $\mathbf{R_q^n}$ and the image of double cyclic code over $\mathbf{R_q}$ is a double cyclic code over $\mathbf{R_q^n}$. Also we discuss the linear complementary dual codes(LCD) over $\mathbf{R_q^n}$ has the application in a multi secret-sharing scheme.

1. Introduction

Let $\mathbb{F}_q^n$ denote the set of all codes C of length n over $\mathbb{F}_q$. The study of codes over finite rings began in the early 1970s, and have recently attracted a lot of attention that began popularizing following a seminal paper by Hammons et al. [1]. Further they have shown that good non linear binary codes exist via $\mathbb{Z}_4$.

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In 2010, Borges et al. [6] introduced $\mathbb{Z}_2\mathbb{Z}_4$-linear dual codes, where some coordinates are binary and the rest is quaternary. In the same year, Fernandez-Cordoba et al. [13] determined the rank and kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and these codes found an engineering application in the area of steganography [21]. Additive codes over the mixed alphabet $\mathbb{Z}_2\mathbb{Z}_{2^s}$ were considered in [3] and then, the mixed alphabet $\mathbb{Z}_p\mathbb{Z}_{p^s}$ more generally $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ was studied in [24]. On the other hand in 2014, Abualrub et al. [2] defined $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes as $\mathbb{Z}_4[x]$-submodule of $\mathbb{Z}_2[x]/\langle x^r-1\rangle \times \mathbb{Z}_4[x]/\langle x^s-1\rangle$ and derived generator set, and minimal spanning set. Also, Borges et al. [7] found generator polynomials and duals for $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. After introducing the new mixed alphabets $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes, where $u^2=0$ in [4], Aydogdu et al. [5] investigated constacyclic codes over mixed alphabets by defining them as $\mathbb{Z}_2[u][x]$-submodules of $\mathbb{Z}_2[x]/\langle x^\alpha-1\rangle \times \mathbb{Z}_2[u][x]/\langle x^\beta-(1+u)\rangle$ and obtained some optimal binary linear codes as the Gray images of $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic codes. Meanwhile, the algebraic properties of $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive cyclic and constacyclic codes with the unit $1+u$ studied in [23]. In 2015, Joaquim et al. [8] determined the structure of $\mathbb{Z}_2$-double cyclic codes, derived the generator polynomials of these codes and gave polynomial representation for its dual codes. Subsequently, Gao et al. [14] analyzed $\mathbb{Z}_2$-double cyclic codes and obtained some optimal codes. Whereas, S. Bathala et al. [9] studied double cyclic code over a non chain ring $\mathbb{F}_4+u\mathbb{F}_4$ where $u^2=u$ and obtained some good examples in 2021.

Linear Complementary Dual Codes (LCD) was introduced by Massey J. L [16] in 1992 and it has theoretical and effective importance against side-channel attacks [10, 11]. The condition for a cyclic code to have a complementary dual is given by Yang and Massey [17]. Moreover LCD codes meets the Gilbert Varshamov bound over the finite field by Sendrier [22]. Subsequently, there are many articles on the LCD codes over finite fields [12, 18, 19, 20]. Nowadays many researchers contributed the working LCD codes over small finite fields [15]. Inorder to protecting data from decoherence through Error correcting codes, in which LCD codes plays an important role. Hence, the above studies motivates us for extensive work on mixed alphabets codes.

In this paper, we have constructed double cyclic and double Constacyclic code over the field $\mathbb{F}_q^n$ and a non-chain ring $\mathbf{R_q^n}=\mathbb{F}_q^n+u\mathbb{F}_q^n$ where $u^2=u$ in Section 3. We determine the generating polynomial and minimal spanning set for R in Section 4. Also we show that the image of Double cyclic code over $\mathbf{R_q^n}$ is a Double cyclic code over $\mathbf{R}_q^n$. In Section 5 we discuss the double constacyclic codes over $\mathbf{R_q^n}$. The LCD codes have been discussed for $\mathbf{R_q^n}$ in the last section.

2. Basic definitions and preliminaries

Let $\mathbf{R_q}$ denote the ring $\mathbb{Z}_p^q$ for any prime p and integer $k > 1$. Then $\mathbf{R_q}$ is a finite commutative principal ideal ring and any element $r \in \mathbf{R_q}$ can be written uniquely as $r = \sum_{i=0}^{k-1} p^i a_i$, where $a_i \in \mathbf{Z_p}$.

Recall that a $\mathbf{R_q}$-submodule of $\mathbf{R_q^n}$ is a linear code D of length n over $\mathbf{R_q}$. By identifying each codeword $d = (d_0, d_1, \ldots, d_{n-2}, d_{n-1})$ in D to a polynomial $d(y) = d_0 + d_1 y + \ldots + d_{n-1} y^{n-1}$ in $\frac{\mathbf{R_q}[y]}{<y^n-1>}$, one can simply examined that D is a cyclic code of length n if and only if D is an ideal of the quotient ring $\frac{\mathbf{R_q}[y]}{<y^n-1>}$. Further if a polynomial $l(x) \in \mathbf{R_q}[y]$ is not a zero divisor, then it is said to be regular.

3. Double cyclic code over $\mathbf{R_q}$

Definition 3.1. Consider C be a linear code of n over $\mathbf{R_q}$ and it is called a double Cyclic code (DC code) of length (g, h) if any

$$c = (c_{1,0}, c_{1,1}, \ldots, c_{1,g-2}, c_{1,g-1} \mid c_{2,0}, c_{2,1}, \ldots, c_{2,h-2}, c_{2,h-1}) \in C$$

implies the cyclic shift

$$T(c) = (c_{1,g-1}, c_{1,0}, c_{1,1}, \ldots, c_{1,g-2} \mid c_{2,h-1}, c_{2,0}, c_{2,1}, \ldots, c_{2,h-2}) \in C.$$

Using the above definition, we can see that the double cyclic code C can be viewed as an $\mathbf{R_q}$-submodule of ${\mathbf{R_q}}^g * {\mathbf{R_q}}^h$, where g, h are any two positive integer such that $n = g + h$.

For two elements $c = (c_{1,0}, c_{1,1}, \ldots, c_{1,g-1} \mid c_{2,0}, c_{2,1}, \ldots, c_{2,h-1}) \in C$ and $c' = (c'_{1,0}, c'_{1,1}, \ldots, c'_{1,g-1} \mid c'_{2,0}, c'_{2,1}, \ldots, c'_{2,h-1}) \in C$ of $\mathbf{R_q^g} \times \mathbf{R_q^h}$, inner product is defined as

$$c.c' = \sum_{i=0}^{g-1} C_{1,i} C'_{1,i} + \sum_{j=0}^{h-1} C_{2,j} C'_{2,j}.$$

Define the dual code of the DC code C as

$$C^{\perp} = \{c' \in \mathbf{R_q^g} \times \mathbf{R_q^h} \mid c.c' = 0, \text{ for all } c \in C\}.$$

Proposition 3.2. *Let C be a DC code of length (g, h) over $\mathbf{R_q}$, then the dual code $C^{\perp}$ is also a DC code of length (g, h) over $\mathbf{R_q}$.*

Proof. Consider $\mathbf{R}_{q_{g,h}}$ is the ring $\frac{\mathbf{R_q}[y]}{<y^g-1>} \times \frac{\mathbf{R_q}[y]}{<y^h-1>}$. Define a map C from $R_q^g \times R_q^h$ to $R_{q_g,h}$ as $C((c_{1,0}, c_{1,1}, \ldots, c_{1,g-1} \mid c_{2,0}, c_{2,1}, \ldots, c_{2,h-1})) = c_{1,0} + c_{1,1} y + \ldots, c_{1,g-1} y^{h-1} \mid$ $c_{2,0} + c_{2,1} y + \ldots, c_{2,h-1} y^{h-1}$, then C is a bijective $\mathbf{R_q}$-module isomorphism. Consider $y * (l(x)|m(x)) = (yl(x)|ym(x))$ and the multiplication $*$ is well-defined. Assume $\mathbf{R}_{q_{g,h}}$ is an $\mathbf{R_q}[y]$-module with respect to this multiplication. Also, for any $c(y) = c_{1,0} + c_{1,1} y + \ldots, c_{1,g-1} y^{g-1} \mid c_{2,0} y + c_{2,1} y^2 + \ldots, c_{2,h-1} y^{h-1} \in \mathbb{R}_{q_{g,h}}$, $y * c(y) = (c_{1,g-1} + c_{1,0} y + \ldots, C_{1,r-2} y^{g-1} | c_{2,h-1} + c_{2,0} y + \ldots, c_{2,h-2} y^{h-1})$. Therefore, $y * c(y)$ is the image of the vector

$(c_{1,g-1}, c_{1,0}, c_{1,1}, \ldots, c_{1,g-2}|c_{2,h-1}, c_{2,0}, c_{2,1} \ldots, c_{2,h-2}y^{h-1})$ which indicates the multiplication of $c(y) \in \mathbf{R_q}_{g,h}$, element by y corresponds to a cyclic shift of the preimage of $c(y)$. Therefore, the DC code of length (g,h) over $\mathbf{R_q}^n$ is able to be viewed as an $\mathbf{R_q}[y]$-submodule of $\mathbf{R_q}_{g,h}$. We found the DC code of length (g,h) with the $\mathbf{R_q}[y]$-submodules of $\mathbf{R_q}_{g,h}$. □

Proposition 3.3. *Let C be cyclic code of length (g,h) over $\mathbf{R_q}$. Then $C = ((l_1(y)|0), (l(y)|l_2(y)))$, where the polynomials $l_1(y), l(y)$ and $l_2(y) \in \mathbf{R_q}[y]$ are monic with $l_1(y)$ $|y^g - 1$ and $l_2(y)|y^h - 1$.*

Proof. Define a map from $\phi' : C \to \frac{\mathbf{R_q}[y]}{<y^h-1>}$ with $\phi((c_1(y)|c_2(y))) = c_2(y)$, where the double cyclic codes C and $\frac{\mathbf{R_q}[y]}{<y^h-1>}$ are $\mathbf{R_q}(\mathbf{y})$-submodules of $\mathbf{R_q}_{g,h}$. If h is an odd positive integer and it is following the concept of cyclic codes over $\mathbf{R_q}(y)$, $\phi(C) = (l_2(y))$ with $l_2(y) \in \mathbf{R_q}[y]$ $l_2(y)| < y^h - 1 >$, then ϕ' is an $\mathbf{R_q}[y]$-module homomorphism and its image is an ideal of $\frac{\mathbf{R_q}[y]}{<y^h-1>}$. Note that $ker(\phi') = \{(c_1(y)|0) \in \mathbf{R_q}_{g,h}|c_1(y) \in C\}$. Define a set

$$I = \{c_1(y) \in \mathbf{R_q}_{g,h}|(c_1(y)|0 \in ker(\phi'))\}.$$

It shows that I is an ideal of $\frac{\mathbf{R_q}[y]}{<y^g-1>}$. Therefore, there exist polynomials $l_1(y) \in \frac{\mathbf{R_q}[y]}{<y^g-1>}$ with $l_1(y)| < y^g - 1 >$ such that $I = (l_1(y))$. Consider $c_1(y)|0 \in ker(\phi')$ implies that $c_1(y) \in I$ such that $c_1(y) = m(y)(l_1(y))$, where $m(y)$ is some polynomial in $\mathbf{R_q}[\mathbf{y}]$. It follows that, $(c_1(y)|0) = m(y) * (l_1(y)|0)$, which implies $ker(\phi')$ is an $\mathbf{R_q}[\mathbf{y}]$-submodule of C generated by $(l_1(y)|0)$. We have $C|ker(\phi') \cong (l_2(y))$, by the first isomorphism theorem.

Let $l(y)|(l_2(y)) \in C$ with $\phi(l(y)|(l_2(y))) = l_2(y)$, then any DC code of length (g,h) over $\mathbf{R_q}$ can be generated by $\mathbf{R_q}[y]$-submodule of $\mathbf{R_q}_{g,h}$ by two elements of the form $l_1(y)|0$ and $l(y)|l_2(y)$ with $l_1(y),\ l(y),\ l_2(y) \in \mathbf{R_q}[y]$ and $l_1(y)| < y^g - 1 >, l_2(y)| < y^g - 1 >$. □

Lemma 3.4. *If $C = ((l_1(y)|0)), (l(y)|l_2(y))$ is a DC code of length (g,h) over $\mathbf{R_q}$. Then we may assume that $deg(l(y)) < deg(l_1(y))$.*

Proof. Suppose that $deg(l(y)) \geq deg(l_1(y))$. with $deg(l(y)) - deg(l_1(y)) = i$. Consider another DC code of length (g,h) with generators of

$$D = ((l_1(y)|0), (l(y) + y^i(l_1(y))|l_2(y))).$$

Clearly, $C \subseteq D$. However, we also have that $(l(y)|l_2(y) = (l(y) + y^i(l_1(y))|l_2(y)) - y^i * (l_1(y)|0)$, which implies that $(l(y)|l_2(y)) \in C$. Therefore, $D \subseteq C$ implying that $C = D$. □

4. Minimal generating set

Consider C is a DC code over $\mathbf{R_q^n}$. We will find the minimal generating sets of C in $\mathbf{R_{q}}_{g,h}^{\mathbf{n}}$ in this section. Also, We will evaluate the size and the generator matrix of C can be used by it.

Proposition 4.1. [8] *Let C be an $\mathbf{R_q}$-DC code of length (g,h). Then $C=(b(x)|0)$, $(l(x)|a(x))$, where $l(x)\in\mathbf{R_q}[x], b(x)|(x^g-1)$ and $a(x)|(x^h-1)$. Moreover $deg(l(x))<deg(b(x))$; $b(x)$ divides $\frac{x^h-1}{a(x)}l(x)$*

Proposition 4.2. *Let $C=((l_1(y)|0),(l(y)|l_2(y))$ be a DC code of length (g,h) over $\mathbf{R_q^n}$ with $deg(l_1(y))=g_1$ and $deg(l_2(y))=t_1$. Consider $S_1=\bigcup_{j=0}^{g-g_1-1}\{y^j*(l_1(y)|0)\}$, $S_2=\bigcup_{j=0}^{h-h_1-1}\{y^j*(l(y)|l_2(y))\}$.*

Then $S_1\cup S_2$ forms a minimal generating set for C of $\mathbf{R_{q}}_{g,h}$.

Proof. The proof can be done by using the proof of Proposition 3.3 □

Example 4.3. Let $q=9$ and C be a (6,6) DC code generated by $b(x)=(x+1)(x-1)^2, a(x)=x-1, l(x)=1,\ C=(b(x)|0),(l(x)|a(x))$, where $l(x)\in F_9[x], b(x)|(x^g-1)$and $a(x)|(x^h-1)$. Moreover $\deg(l(x))<\deg(b(x))$; $b(x)$ divides $\frac{x^h-1}{a(x)}l(x)$. Thus we obtain a code with parameter $[12,8,3]$ which is close to optimal.

Proposition 4.4. *Let $C=((l_1(y)|0),(l(y)|l_2(y))$ be a DC code of length (g,h) over $\mathbf{R_q}$ with $deg(l_1(y))=r_1$ and $deg(l_2(y))=t_1$. Consider $S_1=\bigcup_{j=0}^{g-r_1-1}\{y^j*(l_1(y)|0)\}$, $S_2=\bigcup_{j=0}^{h-t_1-1}\{y^j*(l(y)|l_2(y))\}$.*

Then $S_1\cup S_2$ forms a minimal generating set for C of $\mathbf{R_{q}}_{g,h}$.

Proof. The proof can be done by using the proof of Proposition 3.3.

It is easy to check that the codewords of $S_1\cup S_2$ are linearly independent.

Let $c(y)=p_1(y)*(l_1(y)|0)+p_2(y)*(l(y)|l_2(y))\in C$. We have to check that $c(y)\in<S_1\cup S_2>$.

If $deg(p_1(y))\le g-deg(l_1(y))-1$, then $p_1(y)*(l_1(y)|0)\in<S_1>$. Otherwise, using the division algorithm, we compute $p_1(y)=q_1(y)\frac{y^g-1}{l_1(y)}+r_1(y)$ with $deg(r_1(y))\le g-deg(l_1(y))-1$, hence

$$\begin{aligned} p_1(y)*(l_1(y)|0) &= (q_1(y)\frac{y^q-1}{l_1(y)}+r_1(y))*(l_1(y)|0) \\ &= p_1(y)*(l_1(y)|0)+r_1(y)*(l_1(y)|0). \end{aligned}$$

It follows that $c(y)\in S_1\cup S_2$ if $p_2(y)*(l(y)|l_2(y))\in<S_1\cup S_2>$. If $deg(p_2(y))\le t-deg(l_2(y))-1$, then $p_2(y)*(l_y|l_2(y))\in<S_2>$. If not, using the division algorithm, consider

$p_2(y) = q_2(y)\frac{y^t-1}{l_2(y)} + r_2(y)$, where $deg(r_2(y)) \leq t - deg(l_2(y)) - 1$. Then,

$$\begin{aligned} p_2(y) * (l(y)|l_2(y)) &= (q_2(y)\frac{y^t-1}{l_2(y)}) + r_2(y) * (l(y)|l_2(y)) \\ &= ((q_2(y)\frac{y^t-1}{l_2(y)}) * (l(y)|l_2(y)) + r_2(y)) * (l(y)|l_2(y)) \\ &= (q_2(y)\frac{y^t-1}{l_2(y)}) * (l(y)|0) + r_2(y) * (l(y)|l_2(y)) \end{aligned}$$

To prove that $p_2(y) * (l(y)|l_2(y)) \in S_1 \cup S_2$ first note that $r_2(y) * (l(y)|l_2(y)) \in S_2$. Finally, by proposition , $l_1(y)$ divides $\frac{y^t-1}{l_1(y)}l(y)$ and it follows that $(q_2(y)\frac{y^t-1}{l_1(y)}|0) \in < S_1 >$. Thus, $c(y) \in < S_1 \cup S_2 >$. □

Example 4.5. Let $q = 9$ and C be a (8,8) DC code $b(x) = x^2 + w^3x + w^3, a(x) = x+1, l(x) = x+1,\ C = (b(x)|0), (l(x)|a(x))$, where $l(x) \in F_9[x], b(x)|(x^g - 1)$and $a(x)|(x^h - 1)$. Moreover $\deg(l(x)) < \deg(b(x))$; $b(x)$ divides $l(x)\frac{x^h-1}{a(x)}$. Thus we obtain a code with parameter $[16, 13, 3]$ which is close to optimal.

$(b(x)\|0), (l(x)\|a(x))$	(h, g)	$[n, k, d]$
$(x^3 + w^{14}x^2 + w^{22}x + 1\|0), (x + 1\|x + 1)$	$(6, 6)$	$[12, 8, 4]_{5^2}$
$(x^3 + wx^2 + w^{17}x + 4\|0), (x + 4\|x + 4)$	$(7, 7)$	$[14, 10, 4]_{5^2}$
$(x^2 + w^{13}x + w^9\|0), (x + w^3\|x + 1)$	$(8, 4)$	$[12, 9, 3]_{5^2}$
$(x^2 + w^{27}x + 1\|), (x + 6\|x + 6)$	$(5, 5)$	$[10, 7, 3]_{7^2}$
$(x^2 + w^{47}x + w^6\|0), (x + w^6\|w + 2)$	$(8, 6)$	$[14, 11, 3]_{7^2}$
$(x^3 + x^2 + 2x + 2\|0), (x + 1\|x^2 + x + 2)$	$(6, 8)$	$[14, 9, 3]_3$
$(x^3 + 2x^2 + 2x + 1\|0), (x^2 + x + 1\|x + 2)$	$(6, 8)$	$[14, 10, 3]_5$
$(x^3 + w^4x^2 + 4x + w^{16}\|0), (x + w^4\|x + 3)$	$(6, 4)$	$[10, 6, 3]_{5^2}$
$(x^3 + w^3x^2 + 4x + w^{15}\|0), (x + w^3\|x + w^4)$	$(8, 6)$	$[14, 10, 3]_{5^2}$

$\mathbf{R_q^n}$-double cyclic codes

We followed the similar concept to study $\mathbf{R_q^n}$- Double Cyclic Codes ($\mathbf{R_q^n}$- DC code) and their duals as direct sum of $\mathbf{R_q^n}$- DC code in this section. Let C be an $\mathbf{R_q^n}$- DC code and C_1, C_2 be the corresponding $\mathbf{R_q^n}$ linear code. The following results shows that, by the use of $\mathbf{R_q^n}$- DC codes C_1 and C_2 is completely determined for the code C.

Theorem 4.6. *Let $C = (1 + r)C_1 \oplus rC_2$ be a linear code of length $g + h$ over $\mathbf{R_q^n}$. Then C is an $\mathbf{R_q^n}$-DC code of length (g, h) if and ony if C_1 and C_2 are $\mathbf{R_q^n}$- DC code of length (g, h).*

Proof. Let $y = (a_1, a_2, \ldots, a_g|b_1, b_2, \ldots, b_h) \in C_1$. Then $y + rz \in C$ for some $y = (a'_1, a'_2, \ldots, a'_g|b'_1, b'_2, \ldots, b'_h) \in \mathbf{R_q^{g^n}R_q^{h^n}}$. If C is an $\mathbf{R_q^n}$ cyclic code, then $\tau(y + rz) = (a_g + ra'_g, a_1 + ra'_1, \ldots, a_{g-1} + ra'_{g-1}|b_s + rb'_h, b_1 + rb'_1, \ldots, b_{h-1} + rb'_{h-1}) \in C$. Hence $(a_g, a_1, \ldots, a_{g-1}|b_h, b_1, \ldots, b_{h-1}) \in C_1$ and proved that, C_1 is an $\mathbf{R_q^n}$- DC code. Similarly we can show that C_2 is an $\mathbf{R_q^n}$- DC code. Consider C_1 and C_2 are $\mathbf{R_q^n}$- DC code for the converse part. Further for any $y + rz \in C$, we have $\tau(x) \in C_1$ and $\tau(x + y) \in C_2$. Using the unique representation of C, we get $(1 + r)\tau(x) + r\tau(y + z) = \tau(y + rz) \in C$. Hence C is an $\mathbf{R_q^n}$- double cyclic code. □

Remark 4.7. For any $k(x) \in \mathbf{R_q}[x]$, there exist some $g(x)$, $g'(x) \in \mathbf{R_q}^n[x]$ such that $(1 + r)k(x) = (1 + r)g(x)$ and $rk(x) = vg'(x)$.

The following theorem gives the form of generators of an $\mathbf{R_q}^n$-double cyclic code of length (g, h).

Theorem 4.8. *Let* $C_1 =< (b_1(x)|0), (l_1(x)|a_1(x)) >$ *and* $C_2 =< (b_2(x)|0, l_2(x)|a_2(x)) >$ *be two* $\mathbf{R_q}$*-DC code of length* (g, h) *as specified in Proposition 3.3 and Lemma 3.4. If* $C = (1+r)C_1 \oplus vC_2$ *is an* $\mathbf{F_q}^n$*-DC code of length* (g, h)*, then* $C =< ((1+r)b_1(x)|0, (1+r)l_1(x)|(1+r)a_1(x)), (rb_2(x)|0), (rl_2(x)|ra_2(x)) >$.

Proof. Consider $C = (1 + r)C_1 + rC_2$ by the unique representation of C, we have $C \subseteq< (1 + r)(b_1(x)|0), (1 + r)(l_1(x)|a_1(x)), r(b_2(x)|0), r(l_2(x)|a_2(x))) >$. Conversely, let $c(x) \in< ((1 + r)b_1(x)|0), ((1 + r)l_1(x)|(1 + r)a_1(x)), (rb_2(x)|0), (rl_2(x)|ra_2(x)) >$, so $c(x) = (1 + r)k_1(x)(b_1(x)|0) + (1 + r)k_2(x)(l_1(x)|a_1(x)) + rk_3(x)(b_2(x)|0) + rk_4(x)(l_2(x)|a_2(x))$ for some $k_1(x)$, $k_2(x)$, $k_3(x)$ and $k_4(x)$ in $\mathbf{R_q^n}[x]$. Whereas from the Remark 4.7 , there exists $r_1(x)$, $r_2(x)$, $r_3(x)$ and $r_4(x)$ in $\mathbf{R_q^n}[x]$ such that $c(x) = (1 + r)[r_1(x)(b_1(x)|0) + r_2(x)(l_1(x)|a_1(x))] + r[r_3(x)(b_2(x)|0) + r_4(x)(l_2(x)|a_2(x))] \in c(x) \in C$. As a result $< (1 + r)(b_1(x)|0), (1 + r)(l_1(x)|a_1(x)), r(b_2(x)|0), r(l_2(x)|a_2(x)) >\subseteq C$. □

Theorem 4.9. *Let* $C_1 =< (b_1(x)|0), (l_1(x)|a_1(x)) >$ *and* $C_2 =< (b_2(x)|0), (l_2(x)|a_2(x)) >$ *be two* $\mathbf{R_q^n}$*-DC code of length* (g, h) *as specified in Proposition 3.3 and Lemma 3.4. If* $C = (1+r)C_1 \oplus rC_2$ *is an* $\mathbf{R_q}$*-DC code of length* (g, h)*, then* $C =< ((1+r)b_1(x) + rb_2(x)|0), ((1 + r)l_1(x) + rl_2(x)|(1 + r)a_1(x) + ra_2(x))) >$.

Proof. Assume $C =< (1 + r)(b_1(x)|0, (1 + r)(l_1(x)|a_1(x)), r(b_2(x)|0), r(l_2(x)|a_2(x)) >$, from Theorem 4.8.

Clearly, $< ((1 + r)b_1(x) + rb_2(x)|0), ((1 + r)l_1(x) + rl_2(x)|(1 + r)a_1(x) + ra_2(x))) >\subset C$. Meanwhile $c(x) \in< (1 + r)(b_1(x)|0, (1 + r)(l_1(x)|a_1(x)), r(b_2(x)|0), r(l_2(x)|a_2(x)) >$. so there

exists $r_1(x),\ r_2(x),\ r_3(x)$ and $r_4(x)$ in ${\mathbf{R_q}}^n[x]$ such that

$$\begin{aligned} c(x) &= r_1(x)[(1+r)(b_1(x)|0)] + r_2(x)[(1+r)(l_1(x)|a_1(x))] + r_3(x)[rb_2(x)|0] \\ &\quad + r_4(x)[rl_2(x)|a_2(x)] \\ &= r_1(x)[(1+r)(1+r)b_1(x) + rb_2(x)|0)] + r_2(x)[(1+r)(1+r)l_1(x) \\ &\quad + rl_2(x)|(1+r)a_1(x) + ra_2(x))] + r_3(x)[r((1+r)b_1(x) + rb_2(x)|0)] \\ &\quad + r_4(x)[r((1+r)l_1(x) + rl_2(x)|(1+r)a_1(x) + ra_2(x))] \\ &= (1+r)b_1(x) + rb_2(x)|0)[(1+r)r_1(x) + rr_3(x)] + ((1+r)l_1(x) \\ &\quad + rl_2(x)|(1+r)a_1(x) + ra_2(x))[(1+r)r_2(x) + rr_4(x)]. \end{aligned}$$

Finally, $c(x) \in < (1+r)(b_1(x)|0), (1+r)(l_1(x)|a_1(x)), r(b_2(x)|0), r(l_2(x)|a_2(x)) >$. □

In the following theorem, if $\pi'(C)$ and K has no monic polynomials of least degree, then we have given the form of generator polynomials of an $\mathbf{R_q^n}$- DC by their definitions of $\pi'(C)$, $ker\pi'(C)$ and K.

Theorem 4.10. *Consider C is an $\mathbf{R_q^n}$- DC of length (g,h) satisfies $\pi'(C) =< ra(x) >$ and $Ker\pi'(c) =< rb(x)|0) >$, where $a(x),\ b(x) \in \mathbf{R_q^n}[x]$. Then $C =< (rb(x)|0),$ $(rl(x)|ra(x)) >$ for some $l(x) \in \mathbf{R_q^n}[x]$. Furthermore, If $\pi'(C) =< (1+r)a(x) >$ and $Ker(\pi')_C =< ((1+r)b(x)|0) >$, then $C =< ((1+r)b(x)|0), ((1+r)l(x)|(1+r)a(x)) >$.*

Proof. Assume C is an $\mathbf{R_q^n}$- DC code with $\pi'(C) =< ra(x) >$ and $Ker(\pi')C =< (rb(x)|0) >$ Then $C =< (rb(x)|0), (1+r)l' + rl(x) + ra(x)) >$. Obviously, $(1+r)((1+r))l' + rl(x)|ra(x) = ((1+r)l'|0) \in C$ which implies that $(1+r)l' \in K$. As K does not contain a polynomial with leading coefficient as $(1+r)$, we have $l' = 0$. Therefore $C =< (rb(x)|0), (rl(x)|ra(x)) >$. Similarly we have proved the second part of the theorem. □

5. Double constacyclic codes over $\mathbf{R_q}$

We define an $\mathbf{R_q^n}$- double constacyclic code ($\mathbf{R_q^n}$- DCC code) of length (g,h) in this section. Generalization of cyclic codes is known as constacyclic codes and also called multi-twisted cyclic codes (generalisation of quasi-twisted cyclic codes) of index two over $\mathbf{R_q^n}$. With the help of Gray images of those codes is important because its have clear algebraic structure and its applications. Recently, In finite fields Aydin and Halilovic presented a few construction methods for double Constacyclic family of codes. We remember that, an (α,β)-DCC code of length (g,h) over $\mathbf{R_q^n}$ is an $\mathbf{R_q^n}[x]$-submodule of $\frac{{\mathbf{R_q}}^n[x]}{<x^g-\alpha>} \times \frac{{\mathbf{R_q}}^n[x]}{<x^h-\beta>}$, where $\alpha,\ \beta$ are units in $\mathbb{R}_q^n$.

Now, we have follow the methods of Aydin and Halilovic, we extend $\mathbf{R_q}$-DC code with help of DCC code over finite fields.

Definition 5.1. Let λ_1 and λ_2 be two unit elements in $\mathbf{R_q^n}$ and $c = (c_0, c_1, \ldots, c_{g-1}|$ $c'_1, c'_2, \ldots, c'_{h-1}) \in \mathbf{R_q^{g^n}} \times \mathbf{R_q^{h^n}}$. Then the double constacyclic shift $\sigma_{\lambda_1\lambda_2}$ of c is defined by

$$\sigma_{\lambda_1\lambda_2}(C) = (\lambda_1 c_{g-1}, c_0, \ldots, c_{g-2}|\lambda_2 c'_{h-1}, c'_0, \ldots, c'_{h-2}).$$

Let $\sigma_{\lambda_1\lambda_2}$ be the double constacyclic shift operators and consider a linear code C of length $g+h$ over $\mathbf{R_q^n}$ if C is invariant under the double constacyclic shift operators, then C is called a (λ_1, λ_2)- DCC code of length (g, h). We can simply find that C is a (λ_1, λ_2)- DC code code of length (g, h) with the help of polynomial representation over $\mathbf{R_q^n}$ if and only if C is an $\mathbf{R_q^n}[x]$-submodule of $\frac{\mathbf{R_q^n}[x]}{<x^g-\lambda_1>} \times \frac{\mathbf{R_q^n}[x]}{<x^h-\lambda_2>}$. If we take $\lambda_1 = \lambda_2 = 1$, then C is an $\mathbf{R_q^n}$- DC code as a special case. We address C as a λ- DCC code over $\mathbf{R_q^n}$, when $\lambda_1 = \lambda_2 = \lambda$.

Let $\pi'_i = \mathbf{R_q^n} \times \mathbf{R}_q^n \longrightarrow \mathbf{R_q^n}$ be a projection map such that $\pi'_i(a_1|a_2) = a_i$, $i = 1, 2$. Also let $L_n = [\frac{\mathbf{R_q^n}[x]}{<x^g-\pi'_1(\phi'(\lambda_1))>} \times \frac{\mathbf{R_q^n}[x]}{<x^h-\pi'_1(\phi'(\lambda_2))>}] \times [\frac{\mathbf{R_q^n}[x]}{<x^g-\pi'_1(\phi'(\lambda_1))>} \times \frac{\mathbf{R_q^n}[x]}{<x^h-\pi'_1(\phi'(\lambda_2))>}]$, where $\phi'(\lambda_1)$ and $\phi'(\lambda_2)$ are Gray images of λ_1 and λ_2. Now,we extend the Gray map π' to ϕ' as follows: $\phi' : \mathbf{R_q^n}[\mathbf{x}] < x^g - \lambda_1 > \times \frac{\mathbf{R_q^n}[x]}{<x^h-\lambda_2>} \longrightarrow L_n$ such that

$$\begin{aligned}\phi'(\sum_{i=0}^{g} a_i x^i | \sum_{j=0}^{h} b_i x^j) \quad &\mapsto \quad (\sum_{i=0}^{g} \pi'_1(\phi'(a_i))x^i, (\sum_{i=0}^{g} \pi'_2(\phi'(b_j))x^j, \\ &\quad (\sum_{j=0}^{h} \pi'_1(\phi'(b_j))x^j, (\sum_{j=0}^{h} \pi'_2(\phi'(b_j))x^j).\end{aligned}$$

Clearly, ϕ' is a modular isomorphism, and, therefore,

$$\begin{aligned}\phi' : \frac{\mathbf{R_q^n}[x]}{< x^g - \lambda_1 >} \times \frac{\mathbf{R_q^n}[x]}{< x^h - \lambda_2 >} \quad &\cong \quad \phi'(\sum_{i=0}^{g} a_i x^i | \sum_{j=0}^{h} a_i x^j) \mapsto (\sum_{j=0}^{g} \pi'_1(\phi'(a_i))x^i, \\ &\quad \sum_{j=0}^{g} \pi'_2\phi'(a_i))x^j, \sum_{j=0}^{h} \pi'_1\phi'(b_j))x^j, \sum_{j=0}^{h} \pi'_2\phi'(b_j))x^j).\end{aligned}$$

Clearly, ϕ' is a modular isomorphism, and, therefore,

$$\begin{aligned}\frac{\mathbf{R_q}}{x^g - \lambda_1} \times \frac{\mathbf{R_q}}{x^h - \lambda_2} \quad &\cong \quad \frac{\mathbf{R_q^n}[x]}{< (x^g - \pi_1(\phi'(\lambda_1))) >} \times \frac{\mathbf{R_q^n}[x]}{< (x^h - \pi'_1(\phi'(\lambda_2))) >} \\ &\quad \times \frac{\mathbf{R_q^n}[x]}{< (x^g - \pi'_2(\phi'(\lambda_1))) >} \times \frac{\mathbf{R_q^n}[x]}{< (x^h - \pi'_2(\phi'(\lambda_2))) >}\end{aligned}$$

With this we have the following result.

Theorem 5.2. *Let $C = (1 + r)C_1 + rC_2$ be a linear code of length $g + h$ over $\mathbf{R_q^n}$, and λ_1, λ_2 be two unit elements in $\mathbf{R_q^n}$. Then C is a (λ_1, λ_2)-constacyclic code of length (g, h) over $\mathbf{R_q^n}$ if and only if C_1 and C_2 is a $(\pi_1(\phi'(\lambda_1)), \pi_2(\phi'(\lambda_1)))$ and $(\pi_1(\phi'(\lambda_1)), \pi_2(\phi'(\lambda_2)))$-DCC code respectively of length (g, h) over $\mathbf{R_q^n}$.*

Proof. Let $\lambda_1 = \lambda_{11} + r\lambda_{12}$ and $\lambda_2 = \lambda_{21} + r\lambda_{22}$ be two unit elements in $\mathbf{R_q^n}$. Let $c = (c_0, c_1, \ldots, c_{g-1}|c'_0, c'_1, c'_{h-1}) \in C$, where $c_i = (1+r)x_i + ry_i$, $c'_j = (1+r)x'_j + ry'_j$ for $0 \leq i \leq g-1$ and $0 \leq j \leq h-1$. Then we have

$$\begin{aligned}
\sigma_{\lambda_1\lambda_2}(c) &= (\lambda_1 c_{g-1}, c_0, \ldots, c_{g-2}|\lambda_2 c'_{h-1}, c'_0, \ldots, c'_{h-2}) \\
&= ([\lambda_{11} + r\lambda_{12}][(1+r)x_{g-1} + ry_{g-1}], (1+r)x_0 + ry_0, \ldots, (1+r)x_{g-2} \\
&\quad + ry_{g-2}|[\lambda_{21} + r\lambda_{22}][(1+r)x'_{h-1} + ry'_{h-1}], (1+r)x'_0 + ry'_0, \ldots, \\
&\quad (1+r)x'_{h-2} + ry'_{h-2}) \\
&= ([(1+r)\lambda_{11} + r(\lambda_{11} + \lambda_{12})][(1+r)x_{g-1} + ry_{g-1}], (1+r)x_0 + ry_0, \ldots, \\
&\quad (1+r)x_{g-2} + ry_{g-2}|[(1+r)\lambda_{21} + r(\lambda_{21} + \lambda_{22})][(1+r)x'_{h-1} + ry'_{h-1}], \\
&\quad (1+r)x'_0 + ry'_0, (1+r)x'_{h-2} + ry'_{h-2}) \\
&= (1+r)(\lambda_{11}x_{g-1}, x_0, \ldots, x_{g-2}|\lambda_{21}x'_{h-1}, x'_0, \ldots, x'_{h-2}) \\
&\quad r((\lambda_{21} + \lambda_{22})y_{g-1}, y_0, \ldots, y_{g-2}|\lambda_{22}y'_{g-1}, y'_0, \ldots, y'_{g-2}) \\
&= (1+r)(\pi_2(\phi(\lambda_1))x_{g-1}, x_0, \ldots, x_{g-2}|(\pi_2(\phi(\lambda_2))x'_{g-1}, x'_0, \ldots, x'_{g-2}) + \\
&\quad v(((\pi_1(\phi(\lambda_1)))y_{g-1}, y_0, \ldots, y_{g-2}|(\pi_1\phi(\lambda_2))y'_{g-1}, y'_0, \ldots, y'_{g-2}).
\end{aligned}$$

Suppose C_1 is a $(\pi'_2(\phi'(\lambda_1)), \pi'_2(\phi'(\lambda_2)))$- DCC code and C_2 is a $(\pi'_1(\phi'(\lambda_1)), (\pi'_1(\phi'(\lambda_2)))$- DCC code of length (g,h) over $\mathbf{R_q^n}$. So for any $c = (c_0, c_1, \ldots, c_{g-1})|c'_0, c'_1, \ldots, c'_{h-1} \in C$, where $c_i = (1+r)x_i + ry_i$, $c'_j = x'_j + r'y_j$ for $0 \leq g-1$ and $0 \leq j \leq h-1$, we have $x = (x_0, x_1, \ldots, x_{g-1}|x'_0, x'_1, \ldots, x'_{h-1}) \in C_1$ and $y = (y_0, y_1, \ldots, y_{g-1}|y'_0, y'_1, \ldots, y'_{h-1}) \in C_2$. Hence $\sigma_{(\pi_2(\phi'(\lambda_1)),(\pi'_2(\phi'(\lambda_2))}(x) \in C_1$ and $\sigma_{(\pi_1(\phi'(\lambda_1)),(\pi'_1(\phi'(\lambda_2))} \in C_2$. In addition to that $(1+r)\sigma_{(\pi'_2(\phi'(\lambda_1)),(\pi_2(\phi'(\lambda_2))}(x) + r\sigma_{(\pi'_1(\phi'(\lambda_1)),(\pi'_1(\phi'(\lambda_2))}(y) = \sigma_{\lambda_1\lambda_2}(c) \in C$. Since x and y are arbitrary elements in C_1 and C_2, respectively, C is a (λ_1, λ_2)-constacyclic code of length (g,h) over $\mathbf{R_q^n}$. Similarly, we can also prove the sufficient part of the theorem. □

Lemma 5.3. *Assume α, β are any two unit elements in $\mathbf{R_q^n}$ and C be an (α, β)- DCC code of length (g,h) over $\mathbf{F_q^n}$. Then $C^\perp$ is an (α', β')- DCC code of length (g,h) over $\mathbf{R_q^n}$.*

Proof. Let $n = lcm\{g,h\}$. We see that for any $a = (a_{11}, a_{12}, \ldots, a_{1g}|a_{11}, a_{12}, \ldots, a_{1h})$, we have $\sigma_{\alpha,\beta}^{n-1}(a) = (\alpha^{\frac{n}{h}}a_{12}, \ldots, \alpha^{\frac{n}{h}}a_{1g}, \alpha^{\frac{n}{h}-1}a_{11}|\beta^{\frac{n}{g}}a_{22}, \ldots, \beta^{\frac{n}{g}}a_{2g}, \beta^{\frac{n}{g}-1}a_{21})$. Also, as multiplicative order of each non-zero element in $\mathbf{R_q^n}$ is $q-1$, we have $\sigma_{\alpha,\beta}^{(q-1)n}(a) = a$. Let $a = (a_{11}, a_{12}, \ldots, a_{1g}|a_{21}, a_{22}, \ldots, a_{2h}) \in C$ and $b = (b_{11}, b_{12}, \ldots, b_{1g}|b_{21}, b_{22}, \ldots, b_{2h}) \in C^\perp$. Then

$$\begin{aligned}
\sigma_{\alpha,\beta}^{(q-1)n-1}(a) \cdot b &= (a_{12}b_{11} + \ldots a_{1g}b_{1g-1} + \alpha^{(q-1)\frac{n}{h}-1}a_{11}b_{1g}) \\
&\quad + (a_{22}b_{21} + \ldots, a_{2h}b_{2h-1} + \beta^{(q-1)(n-1)}a_{21}b_{2h-1})
\end{aligned}$$

$$\begin{aligned}
&= (a_{12}b_{11} + \ldots + a_{1g}b_{1g-1} + \alpha^{(q-1)\frac{n}{h}-1}[\alpha\alpha^{-1}]a_{11}b_{1g}) \\
&\quad +(a_{22}b_{21} + \ldots a_{2h}b_{2h-1} + \beta^{(q-1)\frac{n}{h}-1}[\beta\beta^{-1}]a_{21}b_{2h-1}) \\
&= (a_{12}b_{11} + \ldots + a_{1g}b_{1g-1} + [\alpha^{-1}]a_{11}b_{1g}) \\
&\quad +(a_{22}b_{21} + \ldots + a_{2h}b_{2h-1} + [\beta^{-1}]a_{21}b_{2h-1}) \\
&= (a_{11}[\alpha^{-1}b_{1g}] + a_{12}b_{11} + \ldots + a_{1g}b_{1g-1} + a_{21}[\beta^{-1}b_{2h-1}]) \\
&\quad +(a_{22}b_{21} + \ldots + a_{2h}b_{2h-1}) \\
&= a \cdot \sigma_{\alpha^{-1},\beta^{-1}}(b).
\end{aligned}$$

Since $\sigma_{\alpha,\beta}^{(q-1)n-1}(a) \in C$, $\sigma_{\alpha,\beta}^{n-1}(a) \cdot b = 0$. This deduce that $a \cdot \sigma_{\alpha^{-1},\beta^{-1}}(b) = 0$. Hence $\sigma_{\alpha^{-1},\beta^{-1}}(b) \in C^{\perp}$. As b is an arbitrary element in $C^{\perp}$, we have $C^{\perp}$ an $(\alpha^{-1}, \beta^{-1})$- DCC code over $\mathbf{R_q^n}$. □

Using Lemma 5.3 and Theorem 5.2, we have

Theorem 5.4. *Let $\lambda_1 = (1+r)\lambda_{11} + r\lambda_{12}$, $\lambda_2 = (1+r)\lambda_{21} + r\lambda_{22}$ be two unit elements in $\mathbf{R_q^n}$ and $C = (1+r)C_1 \oplus vC_2$ be a (λ_1, λ_2)- DCC code of length (g, h) over $\mathbf{R_q^n}$. Then $C^{\perp}$ is a $((1+r)\lambda_{11}^{-1} + r\lambda'_{12}, (1+r)\lambda_{21}^{-1} + r\lambda'_{22})$- DCC code of length (g, h) over $\mathbf{R_q^n}$.*

6. Linear complementary dual codes

A linear code C which intersects with its dual trivially is called linear complementary codes (LCD). The reversible code is a code in which a vector obtained by reversing the order of component of a codeword is again a codeword. The ring $\mathbf{R_q^n} = \mathbf{F}_q^n + u\mathbf{F}_q^n$ where $u^2 = u$.

Let $\delta_1 = (1+r)\lambda_{11} + r\lambda_{12}$, $\delta_2 = (1+r)\lambda_{21} + r\lambda_{22}$, where p is prime number and $\lambda \in \mathbf{F_q^n}$ such that $\alpha_1 + r\alpha_2 \equiv 1(\mod p)$, for $\alpha_1 = \lambda_{11} + \lambda_{21}$ and $\alpha_2 = \lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{22}$. It is trivially obtained that $\delta_1 + \delta_2 = 1$, $\lambda_i^2 = \lambda_i$ and $\lambda_i\lambda_j = 0$, for $i \neq j$ where $i, j \in \{1, 2\}$. By the orthogonal idempotent decomposition, we have $\mathbf{R_q} = \delta_1\mathbf{F_q^n} + \delta_2\mathbf{F_q^n}$. We define a Gray map $\varphi : \mathbf{R_q^n} \to \mathbf{F_q^{2n}}$ by

$$\begin{aligned}
\varphi(g) &= \varphi(g_1 + rh_1, g_2 + rh_2, \ldots, g_n + rh_n) \\
&= (g_1 - h_1, g_2 - h_2, \ldots, g_n - h_n, g_1 + h_1, g_2 + h_2, \ldots, g_n + h_n).
\end{aligned}$$

Let C be a linear code of length n over $\mathbf{R_q^n}$. Define

$$C_1 = \{g \in \mathbf{F_q^n} : \text{ there exists } h \in \mathbf{F_q^n} \text{ such that } \delta_1 g + \delta_2 h \in C\},$$

$$C_2 = \{h \in \mathbb{F}_q^n : \text{ there exixts } g \in \mathbf{F_q^n} \text{ such that } \delta_1 g + \delta_2 h \in C\}.$$

We can easily verify that C_1, C_2 are linear codes over $\mathbf{F_q^n}$ of length n and $C = \delta_1 C_1 + \delta_2 C_2$. Also, C is cyclic code if and only if C_1 and C_2 are cyclic codes of length n over $\mathbb{F}_q^n$.

Lemma 6.1. [17] *If $\mathbb{Z}$ is an $[n,k,d]$ cyclic code over $\mathbf{R_q^n}$ with generator polynomial $f(y)$, then it is an LCD code if and only if $f(y)=\widetilde{f}(y)$ and all the monic irreducible factors of $f(y)$ has the same multiplicity in y^n-1 and $f(y)$. In particular, for $\gcd(n,q)=1$, $\mathbb{Z}$ is an LCD code if and only if it is a reversible code.*

Theorem 6.2. *Let $C=(1+r)C_1+rC_2$ be a linear code over $\mathbf{R_q^n}$. Then C is an LCD code if and only if $C_1,\ C_2$ are LCD codes over $\mathbf{R_q^n}$.*

Proof. The proof of the theorem is the same as in Theorem 4.6. □

Theorem 6.3. *Let $C=\delta_1C_1+\delta_2C_2$ be a cyclic code over $\mathbf{R_q^n}$ of length n, relatively prime to q. Then C is LCD code over $\mathbf{R_q^n}$ if and only if C_1, C_2 are reversible codes over $\mathbf{F_q^n}$.*

Proof. Consider C as an LCD over $\mathbf{R_q^n}$. Then C_1 and C_2 are LCD codes over $\mathbf{F^n}_q+v\mathbf{F^n}_q$. which implies C_1 and C_2 are reversible codes over $\mathbf{F_q}$ by the Lemma 6.1. Therefore we have $\delta_1C_1+\delta_2C_2$ is also a reversible code over $\mathbf{R_q^n}$.

Conversely, suppose C is a reversible code over $\mathbf{R_q^n}$. Then, $c=\delta_1x+\delta_2y\in C$ implies that $c^g\in C$, where $x'=(x'_0,x'_1,\ldots,x'_{n-1})\in C_1$ and $y'=(y'_0,y'_1,\ldots,y'_{n-1})\in C_2$.

Consider

$$\begin{aligned} c^g &= (\delta_1x'_0+\delta_2y'_0,\delta_1x'_1+\delta_2y'_1,\ldots,\delta_1x'_{n-1}+\delta_2y'_{n-1})^g \\ &= (\delta_1x'_{n-1}+\delta_2y'_{n-1},\delta_1x'_{n-2}+\delta_2y'_{n-2},\ldots,\delta_1x'_0+\delta_2y'_0) \\ &= \delta_1x'^g+\delta_2y'^g. \end{aligned}$$

This implies that C_1 and C_2 are reversible codes over $\mathbb{F}_q^n$ and hence LCD codes over $\mathbb{F}_q^n$. Hence C is an LCD code over $\mathbf{R_q^n}$. □

Lemma 6.4. *If $C=(1+r)C_1+rC_2$ and let G_1 and G_2 be the generator matrices of C_1 and C_2, respectively. Then $G_{\varphi(C)}=\begin{bmatrix} G_2 & 0 \\ 0 & G_1 \end{bmatrix}$.*

Theorem 6.5. *If $C=(1+r)C_1+rC_2$ is a cyclic LCD code, then $\varphi(C)$ is q-quasi cyclic LCD code.*

Proof. The Gray images of LCD code is LCD code.

$$G_{\varphi(C)}.G^{\perp}_{\varphi(C)}=\begin{bmatrix} G_2 & 0 \\ 0 & G_1 \end{bmatrix}\cdot\begin{bmatrix} G_2\cdot G_2^T & 0 \\ 0 & G_1\cdot G_1^T \end{bmatrix}.$$

We have $G_2\cdot G_2^T$ nonsingular and $G_1\cdot G_1^T$ nonsingular then the matrice $G_{\varphi(C)}.G^{\perp}_{\varphi(C)}$ is a nonsingular. □

7. Conclusion and future work

This paper is presented to the study of double cyclic, constacyclic and linear complementary dual codes over $\mathbf{R_q^n}$. First I have prove the generator polynomials of this family of codes, and give their minimum generating sets. Also I have found double constacyclic codes over $\mathbf{R_q^n}$. Finally, I have constructed LCD codes over $\mathbf{R_q^n}$. The relation between minimal linear codes and LCD codes over $\mathbf{R_q^n}$ is interesting future work.

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