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Research Paper

TOPOLOGICAL INDICES OF THE ZERO-DIVISOR GRAPH FOR THE DIRECT PRODUCT OF FINITE COMMUTATIVE RINGS

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Abstract. A topological index is a numerical quantity associated with the topological structure of a graph, used to describe certain of its properties. In this paper, we compute several degree-based topological indices of the zero-divisor graph $\Gamma(R)$ when R is the commutative ring $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_{2p}$ for a prime $p > 2$. For the commutative ring R with the set of zero-divisors $Z(R)$, the simple graph $\Gamma(R)$ contains the vertex set $V(\Gamma(R)) = Z(R) \setminus \{0\}$ and the edge set $E(\Gamma(R)) = \{uv | u, v \in Z(R) \setminus \{0\} \ \& \ uv = vu = 0\}$.

1. Introduction

Let $G = (V, E)$ be a simple graph of order n and size m with vertex set $V(G)$ and edge set $E(G)$. The set $N(u) = \{v \in V : uv \in E\}$ is called the open neighborhood of a vertex u. The number of vertices connected to the vertex u is called the degree of u and is denoted by $d(u)$. The maximum and the minimum degrees of a graph G are denoted by Δ and δ, respectively [11].

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A topological index of a graph is a numerical invariant that gives the topological property of the graph. Based on the structure of the graph the topological index is represented as a mathematical formula associated with the degrees, distances, and eccentricities of the graph. The Sombor index was proposed by Gutman [9] in 2021 and is defined as follows

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2},$$

Several degree-based topological indices of a graph are introduced motivated by the Sombor index and some known topological indices. Some Sombor-type topological indices are defined in Table 1.

Much research has been done on this index [8, 10, 18, 32]. For more literature, on the topological indices and their applications, readers might refer to [5, 16, 19, 20, 21, 22, 23, 27, 28, 31].

Let R be a commutative ring and $Z(R)$ be the finite set of all zero-divisors of R. If $Z^*(R) = Z(R) \setminus \{0\}$, then the zero-divisor graph $\Gamma(R)$ is a simple graph with vertices $Z^*(R)$ and two vertices u and v are adjacent if and only if $uv = vu = 0$. Beck [4] first introduced the zero-divisor graph in 1988, and later Anderson and Livingstone [3] modified the definition of the zero-divisor graph. Studies in the field of commutative rings have various applications such as engineering, combinatorics, graph theory, advanced mechanics, communication theory, and cryptography. One of the studies done on these algebraic structures focuses on investigating the parameters of their related graphs.

Alikhani and Mirvakili [2] characterized the independent sets of the zero-divisor graph and the ideal-based zero-divisor graph of a commutative ring. In [25], all finite commutative rings with a planar or toroidal annihilator intersection graph were characterized. In [1], the topological polynomials based on distance and their related indices of the zero-divisor graph of a commutative ring are obtained. Mondal et al. [17] computed the topological indices of the total graph, the zero divisor graph, and the zero divisor graph using some algebraic polynomials. In [33] some topological indices for the zero-divisor graph related to commutative rings are given. Chokani et al. [6] obtained graph energy, Laplacian energy, signless Laplacian energy, edge energy and the minimum edge dominating energy of some zero-divisor graphs associated with the finite commutative rings. Movahedi and Akhbari in [24] investigated some of the topological indices such as graph energies, the Zagreb indices and the domination parameters of some zero-divisor graphs. In [7], the Laplacian eigenvalues of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ are studied.

In this paper, we compute the Sombor-type topological indices in Table 1 of the graph $\Gamma(R)$ for the commutative rings $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{2p}$ where $p > 2$ the prime number.

TABLE 1. The Sombor-type degree-based topological indices.

Topological indices	**Formulas**
Sombor index [9]	$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2}$
Reduced Sombor index [9]	$SO_{red}(G) = \sum_{uv \in E(G)} \sqrt{(d(u)-1)^2 + (d(v)-1)^2}$
Average Sombor index [9]	$SO_{avg}(G) = \sum_{uv \in E(G)} \sqrt{\left(d(u) - \frac{2\lvert E\rvert}{\lvert V\rvert}\right)^2 + \left(d(v) - \frac{2\lvert E\rvert}{\lvert V\rvert}\right)^2}$
Modified Sombor index [12]	${}^{m}SO(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)^2+d(v)^2}}$
Modified reduced Sombor index [12]	${}^{m}SO_{red}(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{(d(u)-1)^2+(d(v)-1)^2}}$
Reverse Sombor index [29]	$SO_{rev}(G) = \sum_{uv \in E(G)} \sqrt{(\Delta - d(u) + 1)^2 + (\Delta - d(v) + 1)^2}$
Increased Sombor index [26]	$SO^1(G) = \sum_{uv \in E(G)} \sqrt{(d(u)+1)^2 + (d(v)+1)^2}$
Revan Sombor index [13]	$RSO(G) = \sum_{uv \in E(G)} \sqrt{(\Delta + \delta - d(u))^2 + (\Delta + \delta - d(u))^2}$
First Banhatti-Sombor index [14]	$BSO_1(G) = \sum_{uv \in E(G)} \sqrt{\frac{1}{d(u)^2} + \frac{1}{d(v)^2}}$
First reduced Banhatti-Sombor index [14]	$RBSO_1(G) = \sum_{uv \in E(G), d(u) \neq 1, d(v) \neq 1} \sqrt{\frac{1}{(d(u)-1)^2} + \frac{1}{(d(v)-1)^2}}$
Second Banhatti-Sombor index [15]	$BSO_2(G) = \sum_{uv \in E(G)} \left(\frac{1}{d(u)^2} + \frac{1}{d(v)^2}\right)^{\frac{-1}{2}}$
Second reduced Banhatti-Sombor index [15]	$RBSO_2(G) = \sum_{uv \in E(G), d(u) \neq 1, d(v) \neq 1} \left(\frac{1}{(d(u)-1)^2} + \frac{1}{(d(v)-1)^2}\right)^{\frac{-1}{2}}$

2. MAIN RESULTS

In this section, we obtain the results of the topological indices defined in Table 1 on two zero-divisor graphs $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$.

First, we compute these topological indices of the zero-divisor graph $\Gamma(R)$ in which $R \simeq \mathbb{Z}_p \times \mathbb{Z}_{p^2}$. The graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ is a connected graph such that the vertices are partitioned as follows [30]

$$
\begin{aligned}
&V_1 = \{(u,0) : u \in \mathbb{Z}_p^*\}, \ |V_1| = p-1,\\
(1) \qquad &V_2 = \{(0,v) : v \in \mathbb{Z}_{p^2}^*, \ v \notin Z(\mathbb{Z}_{p^2}^*)\}, \ |V_2| = p^2 - p,\\
&V_3 = \{(0,w) : w \in \mathbb{Z}_{p^2}^*, \ w \in Z(\mathbb{Z}_{p^2}^*)\}, \ |V_3| = p-1,\\
&V_4 = \{(a,b) : a \in \mathbb{Z}_p^*, \ b \in \mathbb{Z}_{p^2}^* \ and \ b \in Z(\mathbb{Z}_{p^2}^*)\}, \ |V_4| = (p-1)^2.
\end{aligned}
$$

where $R^* = R \setminus \{0\}$. Therefore, the order of graph equals

$$\Big|V\left(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})\right)\Big| = \sum_{i=1}^{4} |V_i| = 2p^2 - p - 1.$$

Based on the above vertex-partitions, the degree of vertices is shown in Table 2 [30].

Table 2. The degree of vertex-partition sets (1) of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$.

$\mathbf{u} \in$	V_1	V_2	V_3	V_4
$\mathbf{d(u)}$	p^2-1	$p-1$	p^2-2	$p-1$

In [30], authors obtained the adjacency matrix of zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ as follows

$$
(2) \qquad A\left(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})\right) = \begin{bmatrix}
\mathbf{0}_{p-1} & \mathbf{1}_{(p-1)\times(p^2-p)} & \mathbf{1}_{p-1} & \mathbf{0}_{(p-1)\times(p-1)^2}\\
\mathbf{1}_{(p^2-p)\times(p-1)} & \mathbf{0}_{p^2-p} & \mathbf{0}_{(p^2-p)\times(p-1)} & \mathbf{0}_{(p^2-p)\times(p-1)^2}\\
\mathbf{1}_{p-1} & \mathbf{0}_{(p-1)\times(p^2-p)} & J_{p-1} & \mathbf{1}_{(p-1)\times(p-1)^2}\\
\mathbf{0}_{(p-1)^2\times(p-1)} & \mathbf{0}_{(p-1)^2\times(p^2-p)} & \mathbf{1}_{(p-1)^2\times(p-1)} & \mathbf{0}_{(p-1)^2}
\end{bmatrix},
$$

where $\mathbf{1}_i$, $\mathbf{0}_i$ and J_i denoting the all-ones matrix, the zero matrix, and a matrix with any elements on the main diagonal is equal to 0 and all other elements equal to 1, respectively, of order i.

In the following theorem, we compute the Sombor-type topological indices of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$.

Theorem 2.1. *Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ for prime number $p > 2$ be the zero-divisor graph. Then*

(i) $SO\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)\right) = \frac{p-1}{2}\Big[\sqrt{2}\beta(p-2) + 2(p-1)^2\sqrt{\alpha - 2\beta + 1} + 2p(p-1)\sqrt{\alpha}$
$+ 2(p-1)\sqrt{2(\alpha - 2\beta) + (4p-7)}\Big]$,

(ii) $SO_{red}\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)\right) = \frac{p-1}{2}\Big[\sqrt{2}(\beta-1)(p-2) + 2(p-1)^2\sqrt{\gamma - 2\beta + 1} + 2p(p-1)\sqrt{\gamma}$
$+ 2(p-1)\sqrt{2(\gamma - 2\beta) + (11-8p)}\Big]$,

(iii) $SO_{avg}\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)\right) = \frac{p-1}{2}\Big[2p\sqrt{(p^2+\lambda)^2+(p+\lambda)^2}$
$+\,2(p-1)\sqrt{(p^2+\lambda)^2+(p^2+\lambda-1)^2}$
$+\,2(p-1)^2\sqrt{(p^2+\lambda-1)^2+(p+\lambda)^2}+\sqrt{2}(p-2)(p^2+\lambda-1)\Big].$

(iv) ${}^{m}SO\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)\right) = \frac{(p-1)}{4}\Big[\frac{\sqrt{2}(p-2)}{\beta}+\frac{4(p-1)^2}{\sqrt{\alpha-2\beta-1}}+\frac{4p(p-1)}{\sqrt{\alpha}}+\frac{4(p-1)}{\sqrt{2(\alpha-2\beta)+(4p-7)}}\Big].$

(v) ${}^{m}SO_{red}\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)\right) = \frac{(p-1)}{4}\Big[\frac{\sqrt{2}(p-2)}{\beta-1}+\frac{4(p-1)^2}{\sqrt{\gamma-2\beta-1}}+\frac{4p(p-1)}{\sqrt{\gamma}}+\frac{4(p-1)}{\sqrt{2(\gamma-2\beta)-(8p-11)}}\Big].$

(vi) $SO_{rev}\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)\right) = (p-1)\Big[\sqrt{2}(p-2)+p(p-1)\sqrt{(p^2-p+1)^2+1}$
$+\sqrt{5}(p-1)+(p-1)^2\sqrt{(p^2-p+1)^2+4},$

(vii) $SO^{1}\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)\right) = \frac{(p-1)^2}{2}\Big[\sqrt{2}(p-2)(p+1)+2p^2(p-1)\sqrt{p^2+1}$
$+\,2(p-1)^2\sqrt{p^4-p^2+1}+2(p-1)\sqrt{2p^4-2p^2+1}\Big].$

(viii) $RSO\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)\right) = \frac{(p-1)}{2}\Big[\sqrt{2}p(p-2)+2p(p-1)^2\sqrt{p^2+2p+2}$
$+\,2(p-1)^2\sqrt{p^4-p^2+1}+2(p-1)\sqrt{2p^2-2p+1}\Big].$

where $\alpha = p^4-p^2-2p+2,\ \beta = p^2-2,\ \gamma = p^4-5p^2-4p+13$ *and* $\lambda = -1-\frac{(p-1)(4pq-3p-2)}{p^2+pq-p-1}$.

Proof. Assume that G is the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)$ with $2p^2-p-1$ vertices. Based on the adjacency matrix (2), the partition of edges is as follows. Table 3 shows the cardinality of edge subsets E_i for $i = 1, \ldots, 4$.

$$E_1 = \big\{uv \in E : u \in V_1,\ v \in V_2\big\},\ E_2 = \big\{uv \in E : u \in V_1,\ v \in V_3\big\},$$
$$E_3 = \big\{uv \in E : u, v \in V_3 \textit{ and } u \neq v\big\},\ E_4 = \big\{uv \in E : u \in V_3,\ v \in V_4\big\}.$$

Table 3. The cardinality of edge-partition sets of the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)$.

Edge-partition sets	E_1	E_2	E_3	E_4
Number of edges	$p(p-1)^2$	$(p-1)^2$	$\frac{(p-1)(p-2)}{2}$	$(p-1)^3$

Consequently, we have

$$\Big|E\left(\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\big)\right)\Big| = \sum_{i=1}^{4}|E_i| = \frac{1}{2}(p-1)(4p^2-3p-2).$$

Based on the definition of the Sombor indices in Table 1 and using Tables 2 and 3, we get

(i)

$$
\begin{aligned}
SO(G) &= \sum_{uv\in E(G)} \sqrt{d(u)^2+d(v)^2} \\
&= |E_1|\sqrt{(p^2-1)^2+(p-1)^2}+|E_2|\sqrt{(p^2-1)^2+(p^2-2)^2} \\
&\quad +|E_3|\sqrt{(p^2-2)^2+(p^2-2)^2}+|E_4|\sqrt{(p^2-2)^2+(p-1)^2} \\
&= \left(p(p-1)^2\right)\sqrt{(p^2-1)^2+(p-1)^2}+\left((p-1)^2\right)\sqrt{(p^2-1)^2+(p^2-2)^2} \\
&\quad +\left(\frac{(p-1)(p-2)}{2}\right)\sqrt{(p^2-2)^2+(p^2-2)^2}+\left((p-1)^3\right)\sqrt{(p^2-2)^2+(p-1)^2}.
\end{aligned}
$$

With putting $\alpha = p^4-p^2-2p+2$ and $\beta = p^2-2$. After simplifying the above relation, we get

$$
\begin{aligned}
SO(G) = \frac{p-1}{2}\Big[&\sqrt{2}\beta(p-2)+2(p-1)^2\sqrt{\alpha-2\beta+1} \\
&+2p(p-1)\sqrt{\alpha}+2(p-1)\sqrt{2(\alpha-2\beta)+(4p-7)}\Big].
\end{aligned}
$$

(ii)

$$
\begin{aligned}
SO_{red}(G) &= \sum_{uv\in E(G)} \sqrt{(d(u)-1)^2+(d(v)-1)^2} \\
&= |E_1|\sqrt{(p^2-2)^2+(p-2)^2}+|E_2|\sqrt{(p^2-2)^2+(p^2-3)^2} \\
&\quad +|E_3|\sqrt{(p^2-3)^2+(p^2-3)^2}+|E_4|\sqrt{(p^2-3)^2+(p-2)^2} \\
&= \left(p(p-1)^2\right)\sqrt{(p^2-2)^2+(p-2)^2}+\left((p-1)^2\right)\sqrt{(p^2-2)^2+(p^2-3)^2} \\
&\quad +\left(\frac{(p-1)(p-2)}{2}\right)\sqrt{(p^2-3)^2+(p^2-3)^2}+\left((p-1)^3\right)\sqrt{(p^2-3)^2+(p-2)^2}.
\end{aligned}
$$

After simplifying above relation and putting $\beta = p^2-2$ and $\gamma = p^4-3p^2-4p+8$, the result follows.

(iii) Since $|V| = 2p^2 - p - 1$ and $|E| = \frac{1}{2}(p-1)(4p^2 - 3p - 2)$, we have $\frac{2|E|}{|V|} = \frac{4p^2-3p-2}{2p+1}$. Thus,

$$\begin{aligned}
SO_{avg}(G) &= \sum_{uv \in E(G)} \sqrt{\left(d(u) - \frac{2|E|}{|V|}\right)^2 + \left(d(v) - \frac{2|E|}{|V|}\right)^2} \\
&= |E_1| \sqrt{\left(p^2 - 1 - \frac{4p^2 - 3p - 2}{2p+1}\right)^2 + \left(p - 1 - \frac{4p^2 - 3p - 2}{2p+1}\right)^2} \\
&\quad + |E_2| \sqrt{\left(p^2 - 1 - \frac{4p^2 - 3p - 2}{2p+1}\right)^2 + \left(p^2 - 2 - \frac{4p^2 - 3p - 2}{2p+1}\right)^2} \\
&\quad + |E_3| \sqrt{\left(p^2 - 2 - \frac{4p^2 - 3p - 2}{2p+1}\right)^2 + \left(p^2 - 2 - \frac{4p^2 - 3p - 2}{2p+1}\right)^2} \\
&\quad + |E_4| \sqrt{\left(p^2 - 2 - \frac{4p^2 - 3p - 2}{2p+1}\right)^2 + \left(p - 1 - \frac{4p^2 - 3p - 2}{2p+1}\right)^2}.
\end{aligned}$$

After simplifying the above relation and putting $\lambda = -1 - \frac{4p^2-3p-2}{2p+1}$, we have

$$\begin{aligned}
SO_{avg}(G) = \frac{p-1}{2} \Big[& 2p\sqrt{(p^2+\lambda)^2 + (p+\lambda)^2} + 2(p-1)\sqrt{(p^2+\lambda)^2 + (p^2+\lambda-1)^2} \\
& + 2(p-1)^2\sqrt{(p^2+\lambda-1)^2 + (p+\lambda)^2} + \sqrt{2}(p-2)(p^2+\lambda-1) \Big].
\end{aligned}$$

(iv)

$$\begin{aligned}
{}^{m}SO(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)^2 + d(v)^2}} \\
&= |E_1| \left(\frac{1}{\sqrt{(p^2-1)^2 + (p-1)^2}} \right) + |E_2| \left(\frac{1}{\sqrt{(p^2-1)^2 + (p^2-2)^2}} \right) \\
&\quad + |E_3| \left(\frac{1}{\sqrt{(p^2-2)^2 + (p^2-2)^2}} \right) + |E_4| \left(\frac{1}{\sqrt{(p^2-2)^2 + (p-1)^2}} \right) \\
&= \frac{p(p-1)^2}{\sqrt{(p^2-1)^2 + (p-1)^2}} + \frac{(p-1)^2}{\sqrt{(p^2-1)^2 + (p^2-2)^2}} \\
&\quad + \frac{(p-1)(p-2)}{2\sqrt{(p^2-2)^2 + (p^2-2)^2}} + \frac{(p-1)^3}{\sqrt{(p^2-2)^2 + (p-1)^2}}.
\end{aligned}$$

After simplifying the above relation and putting $\alpha = p^4 - p^2 - 2p + 2$ and $\beta = p^2 - 2$, we have

$${}^{m}SO(G) = \frac{(p-1)}{4} \Big[\frac{\sqrt{2}(p-2)}{(p^2-2)} + \frac{4(p-1)^2}{\sqrt{\alpha - 2\beta^2 - 1}} + \frac{4p(p-1)}{\sqrt{\alpha}} + \frac{4(p-1)}{\sqrt{2(\alpha - 2\beta) + (4p-7)}} \Big].$$

(v)

$$
\begin{aligned}
{}^{m}SO_{red}(G) &= \sum_{uv\in E(G)} \frac{1}{\sqrt{(d(u)-1)^2+(d(v)-1)^2}}\\
&= |E_1|\left(\frac{1}{\sqrt{(p^2-2)^2+(p-2)^2}}\right)+|E_2|\left(\frac{1}{\sqrt{(p^2-2)^2+(p^2-3)^2}}\right)\\
&\quad +|E_3|\left(\frac{1}{\sqrt{(p^2-3)^2+(p^2-3)^2}}\right)+|E_4|\left(\frac{1}{\sqrt{(p^2-3)^2+(p-2)^2}}\right)\\
&= \frac{p(p-1)^2}{\sqrt{(p^2-2)^2+(p-2)^2}}+\frac{(p-1)^2}{\sqrt{(p^2-2)^2+(p^2-3)^2}}\\
&\quad +\frac{(p-1)(p-2)}{2\sqrt{(p^2-3)^2+(p^2-3)^2}}+\frac{(p-1)^3}{\sqrt{(p^2-3)^2+(p-2)^2}}.
\end{aligned}
$$

After simplifying the above relation and putting $\beta = p^2-2$ and $\gamma = p^4-5p^2-4p+13$, we have

$$
{}^{m}SO_{red}(G) = \frac{(p-1)}{4}\Big[\frac{\sqrt{2}(p-2)}{\beta-1}+\frac{4(p-1)^2}{\sqrt{\gamma-2\beta-1}}+\frac{4p(p-1)}{\sqrt{\gamma}}+\frac{4(p-1)}{\sqrt{2(\gamma-2\beta)-(8p-11)}}\Big].
$$

(vi) Since the maximum degree of G is $\Delta = p^2-1$, from the definition of the Reverse Sombor index in Table 1 and using Tables 2 and 3, we have

$$
\begin{aligned}
SO_{rev}(G) &= \sum_{uv\in E(G)} \sqrt{(\Delta-d(u)+1)^2+(\Delta-d(v)+1)^2}\\
&= |E_1|\sqrt{(p^2-1-(p^2-1)+1)^2+(p^2-1-(p-1)+1)^2}\\
&\quad +|E_2|\sqrt{(p^2-1-(p^2-1)+1)^2+(p^2-1-(p^2-2)+1)^2}\\
&\quad +|E_3|\sqrt{(p^2-1-(p^2-2)+1)^2+(p^2-1-(p^2-2)+1)^2}\\
&\quad +|E_4|\sqrt{(p^2-1-(p^2-2)+1)^2+(p^2-1-(p-1)+1)^2}\\
&= \left(p(p-1)^2\right)\sqrt{1+(p^2-p+1)}+\left((p-1)^2\right)\sqrt{2}\\
&\quad +\left(\frac{(p-1)(p-2)}{2}\right)\sqrt{8}+\left((p-1)^3\right)\sqrt{4+(p^2-p+1)^2}.
\end{aligned}
$$

After simplifying the above relation, the result is completed.

(vii)

$$\begin{aligned}
SO^1(G) &= \sum_{uv\in E(G)} \sqrt{(d(u)+1)^2+(d(v)+1)^2} \\
&= |E_1|\sqrt{(p^2)^2+(p)^2}+|E_2|\sqrt{(p^2)^2+(p^2-1)^2} \\
&\quad +|E_3|\sqrt{(p^2-1)^2+(p^2-1)^2}+|E_4|\sqrt{(p^2-1)^2+(p)^2} \\
&= \left(p(p-1)^2\right)\sqrt{p^4+p^2}+\left((p-1)^2\right)\sqrt{p^4+(p^2-1)^2} \\
&\quad +\left(\frac{(p-1)(p-2)}{2}\right)\sqrt{(p^2-1)^2+(p^2-1)^2}+\left((p-1)^3\right)\sqrt{(p^2-1)^2+p^4}.
\end{aligned}$$

After simplifying above relation, the result is completed.

(viii) From Table 1, $\Delta = p^2-1$ and $\delta = p-1$. Therefore, we get

$$\begin{aligned}
RSO(G) &= \sum_{uv\in E(G)} \sqrt{(\Delta+\delta-d(u))^2+(\Delta+\delta-d(u))^2} \\
&= |E_1|\sqrt{(p^2+p-2-(p^2-1))^2+(p^2+p-2-(p-1))^2} \\
&\quad +|E_2|\sqrt{(p^2+p-2-(p^2-1))^2+(p^2+p-2-(p^2-2))^2} \\
&\quad +|E_3|\sqrt{(p^2+p-2-(p^2-2))^2+(p^2+p-2-(p^2-2))^2} \\
&\quad +|E_4|\sqrt{(p^2+p-2-(p^2-2))^2+(p^2+p-2-(p-1))^2} \\
&= \left(p(p-1)^2\right)\sqrt{(p-1)^2+(p^2-1)^2}+\left((p-1)^2\right)\sqrt{(p-1)^2+p^2} \\
&\quad +\left(\frac{(p-1)(p-2)}{2}\right)\sqrt{2p^2}+\left((p-1)^3\right)\sqrt{p^2+(p^2-1)^2}.
\end{aligned}$$

After simplifying the above relation, the result is completed. □

In the following theorem, we obtain the Banhatti-Sombor topological indices defined in Table 1 of the zero-divisor graph $\Gamma\left(\mathbb{Z}_p\times\mathbb{Z}_{p^2}\right)$.

Theorem 2.2. *Let $\Gamma\left(\mathbb{Z}_p\times\mathbb{Z}_{p^2}\right)$ for prime number $p>2$ be the zero-divisor graph. Then*

(i) $BSO_1\left(\Gamma\left(\mathbb{Z}_p\times\mathbb{Z}_{p^2}\right)\right) = \frac{(p-1)}{2\beta}\Big[\sqrt{2}(p-2)+2(p-1)\sqrt{\alpha-2\beta-1} + \left(\frac{2p\beta}{p+1}\right)\sqrt{\beta+2(p+2)}+\left(\frac{2}{p+1}\right)\sqrt{2(\alpha-2\beta)+(4p-7)}\Big].$

(ii) $RBSO_1\left(\Gamma\left(\mathbb{Z}_p\times\mathbb{Z}_{p^2}\right)\right) = \frac{(p-1)}{2(\beta-1)}\Big[\sqrt{2}(p-2)+\frac{2(p-1)^2}{p-2}\sqrt{\gamma-2\beta-1} + \left(\frac{2p(p-1)}{\beta(p-2)}\right)\sqrt{\gamma}(\beta-1)+\frac{2(p-1)}{\beta-1}\sqrt{2(\gamma-2\beta)-(8p-11)}\Big].$

(iii) $BSO_2\left(\Gamma\left(\mathbb{Z}_p\times\mathbb{Z}_{p^2}\right)\right) = \frac{(p-1)\beta}{4}\Big[\sqrt{2}(p-2)+\frac{4(p-1)^2}{\sqrt{\alpha-2\beta-1}}+\frac{4p(p-1)(p+1)^2}{\beta\sqrt{\beta+2(p+2)}}+\frac{4(p-1)^2(p+1)}{\sqrt{2(\alpha-2\beta)+(4p-7)}}\Big].$

(iv) $RBSO_2\left(\Gamma\left(\mathbb{Z}_p\times\mathbb{Z}_{p^2}\right)\right) = \frac{(p-1)(\beta-1)}{4}\Big[\sqrt{2}(p-2)+\frac{4(p-1)^2(p-2)}{\sqrt{\gamma-2\beta-1}}+\frac{4p(p-1)(p-2)\beta}{(\beta-1)\sqrt{\gamma}} + \frac{4\beta(p-1)}{\sqrt{2(\gamma-2\beta)+(11-8p)}}\Big].$

where $\alpha = p^4-p^2-2p+2$, $\beta = p^2-2$ and $\gamma = p^4-5p^2-4p+13$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$. Based on the definition of the Banhatti-Sombor indices in Table 1 and using the degree of vertices and the edge-partition shown in Tables 2 and 3, respectively, we have

(i)

$$\begin{aligned}
BSO_1(G) =& \sum_{uv\in E(G)} \sqrt{\frac{1}{d(u)^2} + \frac{1}{d(v)^2}} \\
=& |E_1|\sqrt{\frac{1}{(p^2-1)^2} + \frac{1}{(p-1)^2}} + |E_2|\sqrt{\frac{1}{(p^2-1)^2} + \frac{1}{(p^2-2)^2}} \\
&+|E_3|\sqrt{\frac{1}{(p^2-2)^2} + \frac{1}{(p^2-2)^2}} + |E_4|\sqrt{\frac{1}{(p^2-2)^2} + \frac{1}{(p-1)^2}}.
\end{aligned}$$

We put $\alpha = p^4 - p^2 - 2p + 2$ and $\beta = p^2 - 2$. After simplifying the above relation, we have

$$\begin{aligned}
BSO_1(G) =& \frac{(p-1)}{2\beta}\Big[\sqrt{2}(p-2) + 2(p-1)\sqrt{\alpha - 2\beta - 1} \\
&+ \frac{2p(p^2-2)}{p+1}\sqrt{\beta + 2(p+2)} + \frac{2}{p+1}\sqrt{2(\alpha - 2\beta) + (4p-7)}\Big].
\end{aligned}$$

(ii)

$$\begin{aligned}
RBSO_1(G) =& \sum_{uv\in E(G)} \sqrt{\frac{1}{(d(u)-1)^2} + \frac{1}{(d(v)-1)^2}} \\
=& |E_1|\sqrt{\frac{1}{(p^2-2)^2} + \frac{1}{(p-2)^2}} + |E_2|\sqrt{\frac{1}{(p^2-2)^2} + \frac{1}{(p^2-3)^2}} \\
&+|E_3|\sqrt{\frac{1}{(p^2-3)^2} + \frac{1}{(p^2-3)^2}} + |E_4|\sqrt{\frac{1}{(p^2-3)^2} + \frac{1}{(p-2)^2}}
\end{aligned}$$

With putting $\beta = p^2 - 2$ and $\gamma = p^4 - 5p^2 - 4p + 13$ and simplifications of the above relation, we have

$$\begin{aligned}
RBSO_1(G) =& \frac{(p-1)}{2(\beta-1)}\Big[\sqrt{2}(p-2) + \frac{2(p-1)^2}{p-2}\sqrt{\gamma - 2\beta - 1} + \left(\frac{2p(p-1)}{\beta(p-2)}\right)\sqrt{\gamma}(\beta - 1) \\
&+ \frac{2(p-1)}{\beta - 1}\sqrt{2(\gamma - 2\beta) - (8p-11)}\Big].
\end{aligned}$$

(iii)

$$
\begin{aligned}
BSO_2(G) &= \sum_{uv\in E(G)} \left(\frac{1}{d(u)^2}+\frac{1}{d(v)^2}\right)^{\frac{-1}{2}} \\
&= |E_1|\left(\frac{1}{(p^2-1)^2}+\frac{1}{(p-1)^2}\right)^{\frac{-1}{2}} + |E_2|\left(\frac{1}{(p^2-1)^2}+\frac{1}{(p^2-2)^2}\right)^{\frac{-1}{2}} \\
&\quad + |E_3|\left(\frac{1}{(p^2-2)^2}+\frac{1}{(p^2-2)^2}\right)^{\frac{-1}{2}} + |E_4|\left(\frac{1}{(p^2-2)^2}+\frac{1}{(p-1)^2}\right)^{\frac{-1}{2}}
\end{aligned}
$$

With $\alpha = p^4 - p^2 - 2p + 2$ and $\beta = p^2 - 2$ and simplifications of the above relation, the result is completed.

(iv)

$$
\begin{aligned}
RBSO_2(G) &= \sum_{uv\in E(G)} \left(\frac{1}{(d(u)-1)^2}+\frac{1}{(d(v)-1)^2}\right)^{\frac{-1}{2}} \\
&= |E_1|\left(\frac{1}{(p^2-2)^2}+\frac{1}{(p-2)^2}\right)^{\frac{-1}{2}} + |E_2|\left(\frac{1}{(p^2-2)^2}+\frac{1}{(p^2-3)^2}\right)^{\frac{-1}{2}} \\
&\quad + |E_3|\left(\frac{1}{(p^2-3)^2}+\frac{1}{(p^2-3)^2}\right)^{\frac{-1}{2}} + |E_4|\left(\frac{1}{(p^2-3)^2}+\frac{1}{(p-2)^2}\right)^{\frac{-1}{2}}
\end{aligned}
$$

With putting $\beta = p^2 - 2$ and $\gamma = p^4 - 5p^2 - 4p + 13$ and simplifications of the above relation, the result is completed.

□

Now, we consider the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ where $p > 2$ a prime. The vertex set of this graph is partitioned as follows [30].

$$
\begin{aligned}
&W_1 = \{(u,0) : u \in \mathbb{Z}_p^*\},\ |W_1| = p-1, \\
&W_2 = \{(0,v) : v \in \mathbb{Z}_{2p}^*,\ v \notin Z(\mathbb{Z}_{2p}^*)\},\ |W_2| = p-1, \\
&W_3 = \{(0,w) : w \in \mathbb{Z}_{2p}^*,\ w \in Z(\mathbb{Z}_{2p}^*)\},\ |W_3| = p, \\
&W_4 = \{(a,b) : a \in \mathbb{Z}_p^*,\ b \in \mathbb{Z}_{2p}^* \ and\ b \in Z(\mathbb{Z}_{2p}^*)\},\ |W_4| = p(p-1),
\end{aligned} \tag{3}
$$

where $\mathbb{Z}_n^*$ is denoted $\mathbb{Z}_n \setminus \{0\}$ for $n = p, 2p$. Therefore, the order of graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$

$$\Big|V\left(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})\right)\Big| = \sum_{i=1}^{4}|W_i| = p^2 + 2p - 2.$$

Based on the partition of vertices in (3), the degree of vertices is shown in Table 4 [30].

TABLE 4. The degree of vertex-partition sets (3) of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$.

Vertex	$u \in W_1$	$u \in W_2$	$u=(0,2m) \in W_3$	$u=(0,p) \in W_3$	$u=(a,2m) \in W_4$	$u=(a,p) \in W_4$
$\mathbf{d(u)}$	$2p-1$	$p-1$	$2p-1$	p^2-1	1	$p-1$

Based on the structure of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ in [30], the adjacency matrix of graph G is as follows

$$A(G)=\begin{bmatrix} \mathbf{0}_{p-1} & \mathbf{1}_{p-1} & \mathbf{1}_{(p-1)\times p} & \mathbf{0}_{(p-1)\times(p^2-p)} \\ \mathbf{1}_{p-1} & \mathbf{0}_{p-1} & \mathbf{0}_{(p-1)\times p} & \mathbf{0}_{(p-1)\times(p^2-p)} \\ \mathbf{1}_{p\times(p-1)} & \mathbf{0}_{p\times(p-1)} & M_1 & M_2 \\ \mathbf{0}_{(p^2-p)\times(p-1)} & \mathbf{0}_{(p^2-p)\times(p-1)} & M_2^T & \mathbf{0}_{p^2-p} \end{bmatrix}, \tag{4}$$

in which $\mathbf{1}_i$ and $\mathbf{0}_i$ denote the all-ones matrix and the zero matrix of order i, respectively. Also, the matrix M_2^T is the transpose of the matrix M_2 and the matrices M_1 of order $p \times p$ and M_2 of order $p \times (p^2-p)$ are as following.

$$M_1=\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix},$$

and

$$M_2=\begin{bmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}.$$

Based on the matrix (4), the edge set of graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{2p}\big)$ can be partitioned $E(G) = \bigcup_{i=1}^{6} E_i$ such that

$$E_1 = \big\{uv \in E : u \in W_1,\ v \in W_2\big\},$$

$$E_2 = \big\{uv \in E : u \in W_1,\ v = (0,p) \in W_3\big\},$$

$$E_3 = \big\{uv \in E : u \in W_1,\ v = (0,2m) \in W_3\big\},$$

$$E_4 = \big\{uv \in E : u, v \in W_3,\ u = (0,p)\ and\ v = (0,2m)\big\},$$

$$E_5 = \big\{uv \in E : u = (0,p) \in W_3,\ v = (a,2m) \in W_4\big\},$$

$$E_6 = \big\{uv \in E : u = (0,2m) \in W_3,\ v = (a,p) \in W_4\big\},$$

in which $1 \leq m \leq p-1$ and $a \in Z(\mathbb{Z}_p^*)$.

Table 5. The cardinality of edge-partition sets of the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{2p}\big)$.

Edge-partition sets	E_1	E_2	E_3	E_4	E_5	E_6
Number of edges	$(p-1)^2$	$p-1$	$(p-1)^2$	$p-1$	$(p-1)^2$	$(p-1)^2$

Consequently, the number of edges of the graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{2p}\big)$ is equal to

$$|E(\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{2p}\big))| = \sum_{i=1}^{6} |E_i| = 4p^2 - 6p + 2.$$

Theorem 2.3. *Let $\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)$ for prime number $p > 2$ be the zero-divisor graph. Then*

(i) $SO\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{2p}\right)\right) = (p-1)\Big[2(p-1)\sqrt{p(5p-6)+2} + 2\sqrt{p^4 + 2(p-1)^2} + \sqrt{2}(p-1)(2p-1) + (p-1)\sqrt{p^2(p^2-2)+2}\Big]$,

(ii) $SO_{red}\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{2p}\right)\right) = (p-1)\Big[2(p-1)\sqrt{p(5p-12)+8} + 2(p-1)\sqrt{p^4 - 8(p-1)} + 2\sqrt{2}(p-1)^2 + (p-1)(p^2-2)\Big]$,

(iii) $SO_{avg}\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{2p}\right)\right) = (p-1)\Big[\sqrt{2}(p-1)(\alpha-2p) + 2(p-1)\sqrt{(\alpha-2p)^2 + (\alpha-p)^2} + (p-1)\sqrt{(\alpha-2)^2 + (\alpha-p^2)^2} + 2\sqrt{(\alpha-2p)^2 + (\alpha-p^2)^2}\Big]$,

(iv) ${}^{m}SO\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{2p}\right)\right) = \frac{(p-1)}{2}\Big[\frac{4(p-1)}{\sqrt{p(5p-6)+2}} + \frac{4}{\sqrt{p^4+2(p-1)^2}} + \frac{\sqrt{2}(p-1)}{(2p-1)} + \frac{2(p-1)}{\sqrt{p^2(p^2-2)+2}}\Big]$,

(v) ${}^{m}SO_{red}\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{2p}\right)\right) = \frac{(p-1)}{4}\Big[\sqrt{2} + \frac{4(p-1)}{(p^2-2)} + \frac{8(p-1)}{\sqrt{5p^2-12p+8}} + \frac{8}{\sqrt{p^4-8p+8}}\Big]$,

(vi) $SO_{rev}\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{2p}\right)\right) = (p-1)\Big[\sqrt{2}(p-1)^3 + (p-1)\sqrt{(p^2-1)^2+1} + 2\sqrt{(p-1)^4+1} + 2(p-1)\sqrt{(p-1)^4 + (p^2-p+1)}\Big]$,

(vii) $SO^1\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{2p}\right)\right) = (p-1)\Big[2\sqrt{5}p(p-1) + 2p\sqrt{p^2+4} + 2\sqrt{2}p(p-1) + (p-1)\sqrt{p^4+4}\Big]$.

(viii) $$RSO\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{2p}\right)\right) = (p-1)\Big[\sqrt{2}(p-1)^3 + (p-1)\sqrt{(p^2-1)^2+1} + 2\sqrt{(p-1)^4+1} + 2(p-1)\sqrt{(p-1)^4+(p^2-p+1)}\Big],$$

where $\alpha = \frac{8p^2-12p+4}{p^2+2p-2} + 1$.

Proof. Let G be the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_{2p}\big)$ with p^2+2p-2 vertices and $4p^2-6p+2$ edges. Based on Table 4, the maximum and minimum degrees are $\Delta = p^2-1$ and $\delta = 1$, respectively. Using the definition of the Sombor indices in Table 1, Tables 4 and 5, we get

(i)

$$\begin{aligned}
SO(G) &= \sum_{uv \in E(G)} \sqrt{d(u)^2+d(v)^2}\\
&= |E_1|\sqrt{(2p-1)^2+(p-1)^2} + |E_2|\sqrt{(2p-1)^2+(p^2-1)^2}\\
&\quad + |E_3|\sqrt{(2p-1)^2+(2p-1)^2} + |E_4|\sqrt{(p^2-1)^2+(2p-1)^2}\\
&\quad + |E_5|\sqrt{(p^2-1)^2+(1)^2} + |E_6|\sqrt{(p-1)^2+(2p-1)^2}\\
&= (p-1)^2\sqrt{(2p-1)^2+(p-1)^2} + (p-1)\sqrt{(2p-1)^2+(p^2-1)^2}\\
&\quad + (p-1)^2\sqrt{2(2p-1)^2} + (p-1)\sqrt{(p^2-1)^2+(2p-1)^2}\\
&\quad + (p-1)^2\sqrt{(p^2-1)^2+1} + (p-1)^2\sqrt{(p-1)^2+(2p-1)^2}.
\end{aligned}$$

After simplifying the above relation, the result is completed.

(ii)

$$\begin{aligned}
SO_{red}(G) &= \sum_{uv \in E(G)} \sqrt{(d(u)-1)^2+(d(v)-1)^2}\\
&= |E_1|\sqrt{(2p-2)^2+(p-2)^2} + |E_2|\sqrt{(2p-2)^2+(p^2-2)^2}\\
&\quad + |E_3|\sqrt{(2p-2)^2+(2p-2)^2} + |E_4|\sqrt{(p^2-2)^2+(2p-2)^2}\\
&\quad + |E_5|\sqrt{(p^2-2)^2+(1-1)^2} + |E_6|\sqrt{(p-2)^2+(2p-2)^2}\\
&= (p-1)^2\sqrt{4(p-1)^2+(p-2)^2} + (p-1)\sqrt{4(p-1)^2+(p^2-2)^2}\\
&\quad + (p-1)^2\sqrt{8(p-1)^2} + (p-1)\sqrt{(p^2-2)^2+4(p-1)^2}\\
&\quad + (p-1)^2(p^2-2) + (p-1)^2\sqrt{(p-2)^2+4(p-1)^2}.
\end{aligned}$$

After simplifying the above relation, the result is completed.

(iii) Since $|V| = p^2 + 2p - 2$ and $|E| = 4p^2 - 6p + 2$, we have $\frac{2|E|}{|V|} = \frac{8p^2-12p+4}{p^2+2p-2}$. Thus,

$$\begin{aligned}
SO_{avg}(G) =& \sum_{uv \in E(G)} \sqrt{\left(d(u) - \frac{2|E|}{|V|}\right)^2 + \left(d(v) - \frac{2|E|}{|V|}\right)^2} \\
=& |E_1| \sqrt{\left(2p - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2 + \left(p - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2} \\
&+ |E_2| \sqrt{\left(2p - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2 + \left(p^2 - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2} \\
&+ |E_3| \sqrt{\left(2p - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2 + \left(2p - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2} \\
&+ |E_4| \sqrt{\left(p^2 - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2 + \left(2p - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2} \\
&+ |E_5| \sqrt{\left(p^2 - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2 + \left(1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2} \\
&+ |E_6| \sqrt{\left(2p - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2 + \left(p - 1 - \frac{8p^2 - 12p + 4}{p^2 + 2p - 2}\right)^2}
\end{aligned}$$

After simplifying the above relation and putting $\alpha = \frac{8p^2-12p+4}{p^2+2p-2} + 1$ the result follows.

(iv)

$$\begin{aligned}
{}^{m}SO(G) =& \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)^2 + d(v)^2}} \\
=& |E_1| \left(\frac{1}{\sqrt{(2p-1)^2 + (p-1)^2}}\right) + |E_2| \left(\frac{1}{\sqrt{(2p-1)^2 + (p^2-1)^2}}\right) \\
&+ |E_3| \left(\frac{1}{\sqrt{(2p-1)^2 + (2p-1)^2}}\right) + |E_4| \left(\frac{1}{\sqrt{(p^2-1)^2 + (2p-1)^2}}\right) \\
&+ |E_5| \left(\frac{1}{\sqrt{(p^2-1)^2 + (1)^2}}\right) + |E_6| \left(\frac{1}{\sqrt{(p-1)^2 + (2p-1)^2}}\right).
\end{aligned}$$

After simplifying above relation and using Table 5, the result is completed.

(v)

$$\begin{aligned}
{}^{m}SO_{red}(G) =& \sum_{uv \in E(G)} \frac{1}{\sqrt{(d(u)-1)^2 + (d(v)-1)^2}} \\
=& |E_1| \left(\frac{1}{\sqrt{(2p-2)^2 + (p-2)^2}}\right) + |E_2| \left(\frac{1}{\sqrt{(2p-2)^2 + (p^2-2)^2}}\right) \\
&+ |E_3| \left(\frac{1}{\sqrt{(2p-2)^2 + (2p-2)^2}}\right) + |E_4| \left(\frac{1}{\sqrt{(p^2-2)^2 + (2p-2)^2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+|E_5|\left(\frac{1}{\sqrt{(p^2-2)^2+(1-1)^2}}\right)+|E_6|\left(\frac{1}{\sqrt{(p-2)^2+(2p-2)^2}}\right)\\
&=\frac{(p-1)^2}{\sqrt{4(p-1)^2+(p-2)^2}}+\frac{p-1}{\sqrt{4(p-1)^2+(p^2-2)^2}}+\frac{(p-1)^2}{\sqrt{8(p-1)^2}}\\
&+\frac{p-1}{\sqrt{(p^2-2)^2+4(p-1)^2}}+\frac{(p-1)^2}{(p^2-2)}+\frac{(p-1)^2}{\sqrt{(p-2)^2+4(p-1)^2}}.
\end{aligned}
$$

After simplifying the above relation, the result is completed.

(vi) Since $\Delta = p^2-1$, we get

$$
\begin{aligned}
SO_{rev}(G) &= \sum_{uv\in E(G)}\sqrt{(\Delta-d(u)+1)^2+(\Delta-d(v)+1)^2}\\
&=|E_1|\sqrt{(p^2-1-(2p-1)+1)^2+(p^2-1-(p-1)+1)^2}\\
&\quad+|E_2|\sqrt{(p^2-1-(2p-1)+1)^2+(p^2-1-(p^2-1)+1)^2}\\
&\quad+|E_3|\sqrt{(p^2-1-(2p-1)+1)^2+(p^2-1-(2p-1)+1)^2}\\
&\quad+|E_4|\sqrt{(p^2-1-(p^2-1)+1)^2+(p^2-1-(2p-1)+1)^2}\\
&\quad+|E_5|\sqrt{(p^2-1-(p^2-1)+1)^2+(p^2-1-(1)+1)^2}\\
&\quad+|E_6|\sqrt{(p^2-1-(2p-1)+1)^2+(p^2-1-(p-1)+1)^2}\\
&=(p-1)^2\sqrt{(p-1)^4+(p^2-p+1)^2}+(p-1)\sqrt{(p-1)^4+1}\\
&\quad+(p-1)^2\sqrt{2(p-1)^4}+(p-1)\sqrt{1+(p-1)^4}\\
&\quad+(p-1)^2\sqrt{1+(p^2-1)^2}+(p-1)^2\sqrt{(p-1)^4+(p^2-p+1)^2}.
\end{aligned}
$$

After simplifying the above relation, the result is completed.

(vii)

$$
\begin{aligned}
SO^1(G) &= \sum_{uv\in E(G)}\sqrt{(d(u)+1)^2+(d(v)+1)^2}\\
&=|E_1|\sqrt{(2p)^2+(p)^2}+|E_2|\sqrt{(2p)^2+(p^2)^2}\\
&\quad+|E_3|\sqrt{(2p)^2+(2p)^2}+|E_4|\sqrt{(p^2)^2+(2p)^2}\\
&\quad+|E_5|\sqrt{(p^2)^2+(2)^2}+|E_6|\sqrt{(p)^2+(2p)^2}\\
&=(p-1)^2\sqrt{(2p)^2+(p)^2}+(p-1)\sqrt{(2p)^2+(p^2)^2}\\
&\quad+(p-1)^2\sqrt{2(2p)^2}+(p-1)\sqrt{(p^2)^2+(2p)^2}\\
&\quad+(p-1)^2\sqrt{(p^2)^2+4}+(p-1)^2\sqrt{(p)^2+(2p)^2}.
\end{aligned}
$$

After simplifying the above relation, the result is completed.

(viii) Since $\delta = 1$, we have

$$\begin{aligned} RSO(G) &= \sum_{uv\in E(G)} \sqrt{(\Delta+\delta-d(u))^2+(\Delta+\delta-d(u))^2} \\ &= \sum_{uv\in E(G)} \sqrt{(\Delta-d(u)+1)^2+(\Delta-d(v)+1)^2} \\ &= SO_{rev}(G). \end{aligned}$$

Therefore, $RSO(G) = SO_{rev}(G)$.

□

In the following theorem, we obtain the Banhatti-Sombor topological indices defined in Table 1 of the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$.

Theorem 2.4. *Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$ for prime number $p > 2$ be the zero-divisor graph. Then*

(i) $BSO_1(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})) = (p-1)\left(\frac{\sqrt{2}(p-1)}{(2p-1)} + \frac{2\sqrt{5p^2-6p+2}}{2p-1} + \frac{\sqrt{p^4-2p^2+2}}{p+1} + \frac{2\sqrt{p^4+2(p-1)^2}}{(2p-1)(p^2-1)}\right)$.

(ii) $RBSO_1(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})) = \frac{\sqrt{2}}{2}(p-1) + \frac{\sqrt{5p^2-12p+8}}{p-2} + \frac{\sqrt{p^4-8p+8}}{p^2-2}$.

(iii) $BSO_2(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})) = \frac{(p-1)^2}{2}$
$\left(\sqrt{2}(2p-1) + \frac{4(p-1)(2p-1)}{\sqrt{5p^2-6p+2}} + \frac{2(p^2-1)}{\sqrt{p^4-2p^2+2}} + \frac{2(p+1)(2p-1)}{\sqrt{p^4+2(p-1)^2}}\right)$.

(iv) $RBSO_2(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})) = (p-1)^2\left(\sqrt{2}(p-1) + \frac{4(p-2)(p-1)}{\sqrt{5p^2-12p+8}} + \frac{4(p^2-2)}{\sqrt{p^4-8p+8}}\right)$.

Proof. Let G be the zero-divisor graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{2p})$. Based on the definition of the Banhatti-Sombor indices in Table 1 and using the degree of vertices and the edge-partition shown in Tables 4 and 5, respectively, we have

(i)

$$\begin{aligned} BSO_1(G) &= \sum_{uv\in E(G)} \sqrt{\frac{1}{d(u)^2}+\frac{1}{d(v)^2}} \\ &= |E_1|\sqrt{\frac{1}{(2p-1)^2}+\frac{1}{(p-1)^2}} + |E_2|\sqrt{\frac{1}{(2p-1)^2}+\frac{1}{(p^2-1)^2}} \\ &\quad +|E_3|\sqrt{\frac{1}{(2p-1)^2}+\frac{1}{(2p-1)^2}} + |E_4|\sqrt{\frac{1}{(p^2-1)^2}+\frac{1}{(2p-1)^2}} \\ &\quad +|E_5|\sqrt{\frac{1}{(p^2-1)^2}+\frac{1}{(1)^2}} + |E_6|\sqrt{\frac{1}{(2p-1)^2}+\frac{1}{(p-1)^2}} \end{aligned}$$

After simplifying the above relation, the result is completed.

(ii) Based on the definition of the first Banhatti-Sombor index in Figure 1, for any $uv \in E(G)$ where $d(u) \neq 1$ and $d(v) \neq 1$, we get

$$
\begin{aligned}
RBSO_1(G) = & \sum_{uv \in E(G), d(u) \neq 1, d(v) \neq 1} \sqrt{\frac{1}{(d(u)-1)^2} + \frac{1}{(d(v)-1)^2}} \\
= & |E_1| \sqrt{\frac{1}{(2p-2)^2} + \frac{1}{(p-2)^2}} + |E_2| \sqrt{\frac{1}{(2p-2)^2} + \frac{1}{(p^2-2)^2}} \\
& + |E_3| \sqrt{\frac{1}{(2p-2)^2} + \frac{1}{(2p-2)^2}} + |E_4| \sqrt{\frac{1}{(p^2-2)^2} + \frac{1}{(2p-2)^2}} \\
& + |E_6| \sqrt{\frac{1}{(2p-2)^2} + \frac{1}{(p-2)^2}}
\end{aligned}
$$

After simplifying the above relation and using Table 5, the result is completed.

(iii)

$$
\begin{aligned}
BSO_2(G) = & \sum_{uv \in E(G)} \left(\frac{1}{d(u)^2} + \frac{1}{d(v)^2} \right)^{\frac{-1}{2}} \\
= & |E_1| \left(\frac{1}{(2p-1)^2} + \frac{1}{(p-1)^2} \right)^{\frac{-1}{2}} + |E_2| \left(\frac{1}{(2p-1)^2} + \frac{1}{(p^2-1)^2} \right)^{\frac{-1}{2}} \\
& + |E_3| \left(\frac{1}{(2p-1)^2} + \frac{1}{(2p-1)^2} \right)^{\frac{-1}{2}} + |E_4| \left(\frac{1}{(p^2-1)^2} + \frac{1}{(2p-1)^2} \right)^{\frac{-1}{2}} \\
& + |E_5| \left(\frac{1}{(p^2-1)^2} + \frac{1}{(1)^2} \right)^{\frac{-1}{2}} + |E_6| \left(\frac{1}{(2p-1)^2} + \frac{1}{(p-1)^2} \right)^{\frac{-1}{2}}.
\end{aligned}
$$

After simplifying the above relation, the result is completed.

(iv) Based on the definition of the second Banhatti-Sombor index in Figure 1, for any $uv \in E(G)$ where $d(u) \neq 1$ and $d(v) \neq 1$, we get

$$
\begin{aligned}
RBSO_2(G) = & \sum_{uv \in E(G), d(u) \neq 1, d(v) \neq 1} \left(\frac{1}{(d(u)-1)^2} + \frac{1}{(d(v)-1)^2} \right)^{\frac{-1}{2}} \\
= & |E_1| \left(\frac{1}{(2p-2)^2} + \frac{1}{(p-2)^2} \right)^{\frac{-1}{2}} + |E_2| \left(\frac{1}{(2p-2)^2} + \frac{1}{(p^2-2)^2} \right)^{\frac{-1}{2}} \\
& + |E_3| \left(\frac{1}{(2p-2)^2} + \frac{1}{(2p-2)^2} \right)^{\frac{-1}{2}} + |E_4| \left(\frac{1}{(p^2-2)^2} + \frac{1}{(2p-2)^2} \right)^{\frac{-1}{2}} \\
& + |E_6| \left(\frac{1}{(2p-2)^2} + \frac{1}{(p-2)^2} \right)^{\frac{-1}{2}}.
\end{aligned}
$$

After simplifying the above relation and using Table 5, the result is completed.

□

3. Conclusion

In this paper, we investigated the Sombor-type topological indices of the zero-divisor graph $\Gamma(R)$ for the commutative rings $R = \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ and $R = \mathbb{Z}_p \times \mathbb{Z}_{2p}$, where $p > 2$ is a prime. By analyzing the structure of $\Gamma(R)$, we computed several topological indices, which provide insights into the properties of the graph and its connection to the algebraic structures of the rings.

Our results demonstrate that the topological indices of $\Gamma(R)$ are closely related to the prime p and the specific structure of the rings. The computations reveal patterns that highlight the interplay between ring theory and graph theory, particularly in the context of zero-divisor graphs.

Future work could extend this study to other classes of rings or explore additional topological indices to further understand the relationship between algebraic structures and their associated graphs.

References

[1] A. Ahmad, R. Hasni, N. Akhter and K. Elahi, *Analysis of distance-based topological polynomials associated with zero-divisor graphs*, Comput. Mater. Contin., **70** No. 2 (2022) 2895-2904.

[2] S. Alikhani and S. Mirvakili, *Independent sets of some graphs associated to commutative rings*, Alg. Struc. Appl., **1** No. 2 (2014) 85-103.

[3] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra., **167** (1999) 434-447.

[4] I. Beck, *Coloring of commutative rings*, J. Algebra., **116** (1988) 208-226.

[5] S. Bermudo, R. Hasni, F. Movahedi and J.E. Nápoles, *The Geometric-Arithmetic index of trees with a given total domination number*, Discret. Appl. Math., **345** (2024) 99-113.

[6] S. Chokani, F. Movahedi and S. M. Taheri, *Some of the graph energies of zero-divisor graphs of finite commutative rings*, Int. J. Nonlinear Anal. Appl., **14** No. 7 (2023) 207-216.

[7] S. Chattopadhyay, K. L. Patra and B. K. Sahoo, *Laplacian eigenvalues of the zero divisor graph of the ring* Z_n, Linear Algebra Appl., **584** (2020) 267-286.

[8] K. C. Das and I. Gutman, *On Sombor index of trees*, Appl. Math. Comput., **412** (2022) 126575.

[9] I. Gutman, *Geometric approach to degree-based topological indices: Sombor indices*, MATCH Commun. Math. Comput. Chem., **86** (2021) 11-16.

[10] I. Gutman, *Some basic properties of Sombor indices*, Open J. Discrete. Applied. Math., **4** No. 1 (2021) 1-3.

[11] F. Harary, *Graph Theory*, Addison-Wesley, Reading, 1972.

[12] V. R. Kulli and I. Gutman, *Computation of Sombor indices of certain networks*, SSRG Int. J. Appl. Chem., **8** (2021) 1-5.

[13] V. R. Kulli, I. Gutman, *Revan Sombor index*, J. math. informatics., **22** (2022) 23-27.

[14] V. R. Kulli, *On Banhatti-Sombor indices*, SSRG Int. J. Appl. Chem., **8** No. 1 (2021),21-25.

[15] V. R. Kulli, *On second Banhatti-Sombor indices*, Int. J. Math. Arch., **12** No. 5 (2021) 11-16.

[16] H. Liu, L. You and Y. Huang, *Ordering chemical graphs by Sombor indices and its applications*, MATCH Commun. Math. Comput. Chem., **87** (2022) 5-22.

[17] S. Mondal, M. Imran, N. De and A. Pal, *Topological indices of total graph and zero divisor graph of commutative ring: A polynomial approach*, Complex., **2023** No. 1 (2023) 6815657.

[18] F. Movahedi and M. H. Akhbari, *Entire Sombor index of graphs*, Iranian J. Math. Chem., **14**(1) (2023) 33-45.

[19] F. Movahedi and M. H. Akhbari, *Degree-based topological indices of the molecular structure of hyaluronic acid-methotrexate conjugates in cancer treatment*, Int. J. Quantum Chem., **123** (2023) e27106.

[20] F. Movahedi, M. H. Akhbari and R. Hasni, *Computing the Hosoya index of some nanostar dendrimers*, Mal. J. Fund. Appl. Sci, **20** No. 3 (2024), 588-596.

[21] F. Movahedi, M. H. Akhbari and R. Hasni, *Sharp lower bound for Randić index of trees with fixed Roman domination number*, Ars Combinatoria, **161** (2024) 175-185.

[22] F. Movahedi and M. H. Akhbari, *Investigating Sombor-type topological indices in Titania nanotube structures*, Nanoscale., **11**(3) (2024) 58-46.

[23] F. Movahedi, M. Zameni, M. H. Akhbari, M. H. Shahavi and S. Rajabnezhad, *Topological indices and data analysis techniques modeling to predict the physicochemical properties of tetracycline antibiotics*, J. Pharm. Sci., **114** (2025) 103871.

[24] F. Movahedi and M. H. Akhbari, *Some graph parameters of the Zero-divisor graphs of finite commutative rings*, J. Math. Ext., **17** No. 3 (2023) 1-23.

[25] M. Nazim, N. ur. Rehman and K. Selvakumar, *On the genus of annihilator intersection graph of commutative rings*, Alg. Struc. Appl., **11** No. 1 (2024) 25-36.

[26] W. Ning, Y. Song and K. Wang, *More on Sombor index of graphs*, Math., **10** No. 3 (2022) 301.

[27] Z. Samiei and F. Movahedi, *Investigating graph invariants for predicting properties of chemical structures of antiviral drugs*, Polycycl. Aromat. Compd., **44** No. 10 (2023) 6696-6713.

[28] Z. Samiei and F. Movahedi, *Some degree-based topological indices of the molecular structure of drugs in the treatment RNA-viruses*, J. Inf. Optim. Sci., **47** No. 2 (2026) 465-486.

[29] N. N. Swamy, T. Manohar, B. Sooryanarayana and I. Gutman, *Reverse Sombor index*, Bull. Int. Math. Virtual Inst., **12** No. 2 (2022) 267-272.

[30] K. Patra and P. P. Baruah, *On the adjacency matrix and neighborhood associated with zero-divisor graph for direct product of finite commutative rings*, Int. J. Comput. Appl., **2** No. 3 (2013) 315-323.

[31] I. Radžeporić, *Chemical applicability of Sombor indices*, J. Serb. Chem. Soc., **86** No. 5 (2021) 445-457.

[32] J. Rada, Juan, J. M. Rodríguez and J. M. Sigarreta, *General properties on Sombor indices*, Discret. Appl. Math., **299** (2021) 87-97.

[33] C. J. Rayer and R. S. Jeyaraj, *Applications on topological indices of zero-divisor graph associated with commutative rings*, Symmetry, **15** (2023) 335.

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