Algebraic Structures and Their Applications Vol. X No. X (20XX) pp XX-XX.

Research Paper

ON THE POWER GRAPH OF HAMILTONIAN GROUPS

FATEMEH SAMNEIA, HOSSEIN DOOSTIE* AND BAHRAM VAKILI

Abstract. A finite Hamiltonian group is a non-abelian group such that all of its subgroups are normal. This paper studies the power graphs of such groups. By computing the degrees of the vertices of the power graph we manage to give a necessary condition that a non-abelian group of even order to be a Hamiltonian group. Moreover, by presenting an infinite class of non-Hamiltonian groups we prove that the given condition is not sufficient. This study results in two infinite classes of non-isomorphic Eulerian graphs of the same vertex set. The proper power graphs of all Hamiltonian groups have been explicitly identified, as well.

1. Introduction

The study of associated graphs (directed or undirected) for algebraic structures is one of the broad areas of research because of its significant and valuable applications in mathematics and computer science. The well known *power graph* of a finite group is one of these graphs, one may consult [13, 14] as forerun sourcebook. This graph studied by many authors during the years

DOI: 10.22034/as.2026.23377.1807

MSC(2010): Primary: 05C25; Secondary: 20D60, 05C45.

Keywords: Enhanced power graph, Hamiltonian group, Power graph, Presentation of groups.

Received: 16 July 2025, Accepted: 27 June 2026.

∗Corresponding author

and many interesting properties of power graph have been achieved concerning abelian groups and their direct products, for an almost detailed list we may refer to [6, 7, 11, 3, 4, 15, 16, 17].

Our notation here is merely standard. Following [9] we recall the definition of an undirected *power graph* denoted by $P(S)$, for any algebraic structure S with a binary operation. The vertex set of $P(S)$ is S and two distinct vertices x and y are adjacent if and only if one of the equations $x = y^m$ or $y = x^m$ holds in S, for some integer $m \geq 2$. Also, the *enhanced power* graph of S denoted by $P_e(S)$ is defined to be an undirected connected graph where the vertex set of $P(S)$ is S and two distinct vertices x and y are adjacent if and only if for some element $z \in S$ the equations $x = z^m$ and $y = z^n$ hold for some positive integers m and n. Let us remind the definitions of $Eulerian$ and $semi-Eulerian$ graphs as well. A connected graph is called *Eulerian* or *semi-Eulerian* if every vertex is of even degree, either exactly two vertices are of odd degrees. The cyclic group of order n will be denoted by Z_n, the notation e will be used for the identity element of a group G and $o(g)$ denotes the order of an element g of a group. Finally, we will use the well-known notation $\pi = \langle X \mid R\rangle$ for a presentation of a group or semigroup where X is called the generating set and R is a set of relations (or relators) for the considered group (or semigroup). For a useful and prolific information on the presentation theory one may consult [8, 12, 19]. As usual, the notation $gp(\pi)$ will be used to denote a group presented by the presentation π. Note that as in the graph theory we use the notation "$\sim$" to denote the adjacency of two vertices of a graph.

On the study of power graphs of non-abelian groups one may see [10, 18], for examples. The power graphs of extra special p-groups have been identified in [18] and the article [10] proves that the power graph of a finite group of even order is not Eulerian. An interesting class of groups of even order is the class of Hamiltonian groups (groups where every subgroup is normal). The Hamiltonian groups have been classified as $Q_8 \times Z_2^k \times A$ where Q_8 is the quaternion group of order 8, k is any non-negative integer and A is an abelian group of odd order. Also We will consider the class of non-Hamiltonian groups $D_{2\times 4} \times Z_2^k \times A$ where, $D_{2\times 4}$ is the dihedral group of order 8. These two classes are non-isomorphic groups of the same order $2^{k+3}|A|$. By computing the degrees of all vertices of the power graphs of these classes of groups we will show that the complement of their power graphs are Eulerian. First of all consider the presentations,

$$\begin{aligned}
\pi_1 = \;& \langle a, b, c_1, \ldots, c_k \mid a^5 = a, b^2 = a^2, ba = a^3b, c_i^3 = c_i, ac_i = c_ia, bc_i = c_ib, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \ldots, k)\rangle, \\
\pi_2 = \;& \langle a, b, c_1, \ldots, c_k \mid a^5 = a, b^3 = b, ba = a^3b, c_i^3 = c_i, ac_i = c_ia, \\
& \qquad\qquad\qquad\qquad\qquad\qquad bc_i = c_ib, (i = 1, 2, \ldots, k)\rangle,
\end{aligned}$$

and let $G_1 = gp(\pi_1)$, $G_2 = gp(\pi_2)$ where k is a non-negative integer. Our main results are:

Theorem 1.1. *Suppose that A is an abelian group of odd order and $G = G_1$ or G_2. For an element $z = (g, a)$ of the group $G \times A$ the degree of z as a vertex of $P(G \times A)$, equals:*

$$deg(z) = \begin{cases} (s-1) \times deg(g) + s + r - 2, & \text{if } g \neq e_1 \text{ and } a \neq e_2, \\ deg(g), & \text{if } g \neq e_1 \text{ and } a = e_2, \\ s, & \text{if } g = e_1 \text{ and } a \neq e_2, \\ 2^{k+3} \times |A| - 1, & \text{if } z = (e_1, e_2), \end{cases}$$

where, $deg(g)$ is the degree of vertex g in $P(G)$, $\mid A \mid$ denotes the order of the group A and e_i, $(i = 1, 2)$ denote the identity elements of G and A, respectively.

Theorem 1.2. *If a finite non-abelian group G is a Hamiltonian group then $\overline{P(G)}$, the complement of the power graph of G is an Eulerian graph.*

Theorem 1.3. *The Eulerianity of the graph $\overline{P(G)}$ is not sufficient for a group G to be Hamiltonian.*

Corollary 1.4. *For every positive integer $n \equiv 0 \pmod 8$ there exist two non-isomorphic Eulerian graphs of the same vertex set of cardinality n.*

The power graphs of the groups G_1 and G_2 will be studied in Section 2. The proofs of the main results will be given in Section 3. Finally, the last section is devoted to study of the *enhanced power graphs* and the *proper power graphs* of the studied groups.

2. The power graphs of G_1 and G_2

Throughout this section suppose that k is a positive integer. The simplified presentations for G_1 and G_2 are as follows:

$$\begin{aligned} G_1 = \; & \langle a, b, c_1, \ldots, c_k \mid a^4 = e, b^2 = a^2, ba = a^3 b, c_i^2 = e, ac_i = c_i a, bc_i = c_i b, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \ldots, k) \rangle, \\ G_2 = \; & \langle a, b, c_1, \ldots, c_k \mid a^4 = e, b^2 = e, ba = a^3 b, c_i^2 = e, ac_i = c_i a, bc_i = c_i b, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \ldots, k) \rangle, \end{aligned}$$

where, e represents the identity element. Evidently, both of these groups are of exponent 4 and any element may be represented as $y = a^i b^j c_1^{\varepsilon_1} c_2^{\varepsilon_2} \ldots c_k^{\varepsilon_k}$ where, $i \in \{0, 1, 2, 3\}$, $j \in \{0, 1\}$ and $\varepsilon_1, \ \varepsilon_2, \ldots \varepsilon_k \in \{0, 1\}$. The study of the power graphs of these groups needs some preliminary results concerning the new relations in these groups. The following lemma is one of these preliminaries.

Lemma 2.1. *For every values of* $\varepsilon_1,\ \varepsilon_2, \dots \varepsilon_k \in \{0,1\}$, *the relations*

$$\begin{aligned}
&(i).\ (a^i c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = a^2, && i = 1, 3,\\
&(ii).\ (a^i c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = e, && i = 2,\\
&(iii).\ (a^i b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = a^2, && i = 1, 2, 3,\\
&(iv).\ (b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = a^2,
\end{aligned}$$

hold in G_1.

Proof. Proofs are easy by considering the relations of the group G_1. □

Lemma 2.2. *For every values of* $\varepsilon_1,\ \varepsilon_2, \dots \varepsilon_k \in \{0,1\}$, *the relations:*

$$(a^i b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^3 = \begin{cases} a^3 b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k}, & i = 1,\\ b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k}, & i = 2,\\ a b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k}, & i = 3,\end{cases}$$

hold in G_1.

Proof. We use the relations of G_1 and get:

$$\begin{aligned}
(abc_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^3 &= (ab)^3 c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k}\\
&= a(baba) b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k}\\
&= a(a^3 b a^3 b) b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k}\\
&= a^2 (bab^2) c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k}\\
&= a^2 (a^3 b^3) c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} = a^3 b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k},
\end{aligned}$$

when $i = 1$. There are similar proofs when $i = 2$ or $i = 3$. □

Our method of calculation based on decomposition of the set of elements of groups. The set $G_1 \setminus \{e\}$ may be decomposed into the union of disjoint sets:

$$\begin{aligned}
A_1 &= \{a c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2 \dots, \varepsilon_k \in \{0,1\}\},\\
A_2 &= \{a^2\},\\
A_3 &= \{a^2 c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}, \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k \neq 0\},\\
A_4 &= \{a^3 c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}\},\\
A_5 &= \{c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}, \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k \neq 0\},\\
A_6 &= \{a^i b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid i = 0, 1, 2, 3, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}\}.
\end{aligned}$$

To facilitate the proof of the next key lemma we summaries the results of previous lemmas in Table 2.

	$y \in G_1$	y^2	y^3
1	$ac_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$	a^2	$a^3c_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$
2	a^2	e	a^2
3	$a^2c_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}, (\varepsilon_1+\varepsilon_2+\dots+\varepsilon_k \neq 0)$	e	$a^2c_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$
4	$a^3c_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$	a^2	$ac_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$
5	$c_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$	e	$c_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$
6	$bc_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$	a^2	$a^2bc_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$
7	$abc_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$	a^2	$a^3bc_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$
8	$a^2bc_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$	a^2	$bc_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$
9	$a^3bc_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$	a^2	$abc_1^{\varepsilon_1}c_2^{\varepsilon_2}\dots c_k^{\varepsilon_k}$

Lemma 2.3. *For every positive integer k and all values of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0, 1\}$ if $y \in G_1$ then degree of y, as a vertex of $P(G_1)$ is equal to:*

$$deg(y) = \begin{cases} 1, & if\ y \in A_3 \cup A_5, \\ 3, & if\ y \in A_1 \cup A_4 \cup A_6, \\ 1 + 3 \times 2^{k+1}, & if\ y \in A_2, \\ 2^{k+3} - 1, & if\ y = e. \end{cases}$$

Moreover, $\overline{P(G_1)}$ is an Eulerian graph.

Proof. The vertex corresponds to e is adjacent to all of the other vertices in $P(G_1)$. Hence, $P(G_1)$ is connected and $deg(e) = 2^{k+3} - 1$. For the non-identity elements of G_1 we consider three cases. As the results of the last two lemmas which have been gathered in table 2, we simply refer to this table in the proof.

case 1: $y \in A_3 \cup A_5$. As a quick result of the third and fifth rows of the table we get $deg(y) = 1$.

case 2: $y \in A_1 \cup A_4 \cup A_6$. In this case y is adjacent to e, a^2 and a^2y. Indeed, the second and the third columns of the table together with the appropriate and possible rows give us the result by considering the relation $a^4 = 1$. For $y \in A_1$ consider the first row, for $y \in A_4$, the fourth row and in the case when $y \in A_6$ it is sufficient to observe the rows 6, 7, 8 and 9. So, $deg(y) = 3$.

case 3: $y \in A_2$. Considering all possible values of ε_i,s in the rows numbered 1, 4, 6, 7, 8 and 9 yields that a^2 is adjacent to $2^k + 2^k + 2^k + 2^k + 2^k + 2^k$ different vertices. Hence, $deg(a^2) = 1 + 3 \times 2^{k+1}$. Consequently, $\{1, 3, 1 + 3 \times 2^{k+1}\}$ is the set of degrees of $P(G_1)$. Since G_1 is of even degree then $\overline{P(G_1)}$ is a graph with every vertex of even degree. □

The set of elements of $G_2 \setminus \{e\}$ may be decomposed into the following eight disjoint sets:

$$\begin{aligned}
B_1 &= \{c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}, \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k \neq 0\},\\
B_2 &= \{b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}\},\\
B_3 &= \{a^2 c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}, \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k \neq 0\},\\
B_4 &= \{a^2 b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}\},\\
B_5 &= \{a c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2 \dots, \varepsilon_k \in \{0,1\}\},\\
B_6 &= \{a^3 c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}\},\\
B_7 &= \{a^i b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k} \mid (i = 1,3), \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}\},\\
B_8 &= \{a^2\}.
\end{aligned}$$

By these comments we get:

Lemma 2.4. *Let $G = G_2$. Then, the set of degrees of the vertices of $P(G)$ equals $\{1, 3, 1 + 2^{k+1}, 2^{k+3} - 1\}$, for every positive integer k. Moreover, $\overline{P(G_2)}$ is an Eulerian graph.*

Proof. Evidently, $deg(y) = 2^{k+3} - 1$ if $y = e$, Hence, $P(G_2)$ is connected. Suppose that y is a non-identity element of G_2. Then, the degree of y, as a vertex of $P(G_2)$, satisfies,

$$deg(y) = \begin{cases} 1, & \text{if } y \in B_1 \cup B_2 \cup B_3 \cup B_4,\\ 3, & \text{if } y \in B_5 \cup B_6 \cup B_7,\\ 1 + 2^{k+1}, & \text{if } y \in B_8. \end{cases}$$

This may be checked by considering the relations of G_2 together with the new relations:

$$\begin{aligned}
&(c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = e, \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k \neq 0,\\
&(b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = e,\\
&(a^2 b^j c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = e, \ (j = 0,1),\\
&(a c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^4 = (a^3 c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^4 = e,\\
&(a b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = (a^3 b c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = e,\\
&(a c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^3 = a^3 c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k},\\
&(a^3 c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^3 = a c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k},\\
&(a c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = (a^3 c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots c_k^{\varepsilon_k})^2 = a^2,\\
&(a^2)^2 = e,
\end{aligned}$$

which hold for every integer $k \geq 1$ and all values of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}$.

The last two lines give us $deg(a^2) = 1 + 2^{k+1}$ because of the 2×2^k possible values of ε_i,s and the adjacency of the vertices e and a^2. The first three lines yield that $deg(y) = 1$ for every $y \in B_1 \cup B_2 \cup B_3 \cup B_4$. Finally, by the lines four, five, six and seven we conclude that $deg(y) = 3$, for every $y \in B_5 \cup B_6 \cup B_7$. Since G_2 is of even order and every vertex of $P(G_2)$ is of odd degree then $\overline{P(G_2)}$ is an Eulerian graph. This completes the proof. □

3. Proofs of main results

As well as in the last section suppose that k is a positive integer and A is an abelian group of odd order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ where, p_i,s are distinct odd primes and α_i,s are non-negative integers.

Proof of Theorem 1.1**.** Consider the vertex $z = (g, a_1)$ of $P(G \times A)$ where $g \neq e_1$ and $a_1 \neq e_2$. We show that there are exactly $(s-1) \times deg(g) + s + r - 2$ vertices adjacent to z where, $deg(g)$ is the degree of the vertex g in $P(G)$, $r = o(g)$ and $s = o(a_1)$.

There are two types of vertices adjacent to z. The first type consists of the vertices of the set $\{z^m \mid m = 2, 3, \dots\}$. We show that this set is finite and is of cardinality $rs - 1$. Indeed, the existing distinct elements are:

$$
\begin{aligned}
&(g^i, a_1^i), \quad i = 2, 3, \dots, r, r+1, \\
&(g^i, a_1^i), \quad i = r+2, r+3, \dots, 2r, 2r+1, \\
&(g^i, a_1^i), \quad i = 2r+2, 2r+3, \dots, 3r, 3r+1, \\
&\vdots \\
&(g^i, a_1^i), \quad i = (s-1)r+2, (s-1)r+3, \dots, (s-1)r, (s-1)r+1.
\end{aligned}
$$

The second type of the vertices adjacent to z are vertices of the form $z' = (g', a_1')$ where $(z')^m = z$, for some integer $m \geq 2$. Hence, $(g')^m = g$ and $(a_1')^m = a_1$. So, $g' \sim g$ in $P(G)$ and $a_1' \sim a_1$ in $P(A)$. Since $o(a_1) = s$ then there are $s-1$ possible values for a_1'. However, the number of possible values for g' is $deg(g) - r + 1$ where $r = o(g)$ (indeed, by considering the vertices of the first type the powers of g are excluded.) Hence, the possible number of the vertices $z' = (g', a_1')$ of the second type is equal to $(deg(g) - r + 1)(s-1)$. Consequently, there are exactly,

$$deg(z) = rs - 1 + (s-1)(deg(g) - r + 1) = (s-1) \times deg(g) + s + r - 2,$$

adjacent vertices with $z = (g, a_1)$ where g and a_1 are non-identity. To complete the proof one may easily verify the assertion in the remained three cases. □

Proof of Theorem 1.2**.** By Lemma 2.3 and the Theorem 1.1 we conclude that every vertex of the graph $P(G_1 \times A)$ is of odd degree. Since the group $G_1 \times A$ is a group of even order then the degree of each vertex of the graph $\overline{P(G_1 \times A)}$ should be even. Consequently, $\overline{P(G_1 \times A)}$ is an Eulerian graph. □

Proof of Theorem 1.3**.** The results of Lemma 2.4 and the Theorem 1.1 yield that every vertex of the graph $\overline{P(G_2 \times A)}$ is of even degree. However, the group $G_2 \times A$ is not Hamiltonian.

□

Proof of Corollary 1.4**.** Every positive integer $n \equiv 0$ $(mod\ 8)$ may be decomposed in primes as $n = 2^{\alpha} p_1^{\beta_1} p_2^{\beta_2} \dots p_m^{\beta_m}$ where, $\alpha \geq 3$, $\beta_1, \beta_2, \dots, \beta_m \geq 0$ and p_i,s are distinct odd primes. We may rewrite n as $n = 2^3.2^k.p_1^{\beta_1} p_2^{\beta_2} \dots p_m^{\beta_m}$ where, $k = \alpha - 3 \geq 0$. There are exactly two non-isomorphic groups,

$$Q_8 \times Z_2^k \times Z_{p_1}^{\beta_1} \times Z_{p_2}^{\beta_2} \times \cdots \times Z_{p_m}^{\beta_m},$$
$$D_{2\times 4} \times Z_2^k \times Z_{p_1}^{\beta_1} \times Z_{p_2}^{\beta_2} \times \cdots \times Z_{p_m}^{\beta_m},$$

of order n, for, Q_8 and $D_{2\times 4}$ are two exclusive groups of order 8 which are non-isomorphic and non-abelian. The complements of the power graphs of these groups are Eulerian graphs as proved in the Theorems 1.2 and 1.3. The groups are non-isomorphic so the resulted graphs are non-isomorphic graphs. □

4. Conclusion

Remind the definition of enhanced power graph and recall the definition of *proper power graph* which is the graph $P^*(G) = P(G) - e$, for a finite or infinite group G. Coinciding the power graph and the enhanced power is of interest and studied in [2]. Recalling the Theorem 18 of this article yields us a necessary and sufficient condition on a finite group G as, the graphs $P(G)$ and $P_e(G)$ coincide if and only if every cyclic subgroup of G is of order a power of a prime. This gives us:

Remark 4.1. For an abelian group A of odd order and for every integer $k \geq 1$ consider the Hamiltonian group G where, $G = Q_8 \times Z_2^k \times A$ or $G = Q_8 \times A$. Then, $P_e(G) = P(G)$ if and only if A is the trivial group.

Proof. Let A be the trivial group. Since Q_8 and $Q_8 \times Z_2^k$ are 2-groups then every cyclic subgroup of each one is of order of a power of the prime $p = 2$. Moreover, if A is a non-trivial abelian group of odd order $n = {p_1}^{\beta_1} {p_2}^{\beta_2} \dots {p_m}^{\beta_m}$, the equation $P_e(G) = P(G)$ doesn't hold where $G = Q_8 \times A$ or $G = Q_8 \times Z_2^k \times A$. This is because of existing of cyclic subgroups of orders $2 \times {p_i}^{\beta_i}$ in G where, $1 \leq i \leq m$. □

The connectivity of the proper power graph and its complement studied in [1] to identify certain classes of finite groups (mainly abelian). In the next remark we identify the connectivity of proper power graph of Hamiltonian groups.

Remark 4.2. For a Hamiltonian group G,

(i). $P^*(G)$ is connected if $G = Q_8 \times A$ where, A is an abelian group of odd order,

(ii). $P^*(G)$ is connected if $G = Q_8 \times Z_2^k \times A$ where, k is a positive integer and A is a non-trivial abelian group of odd order,

(iii). $P^*(G)$ is disconnected if $G = Q_8 \times Z_2^k$ where, k is a positive integer.

Proof. To prove (i) we show that any vertex $z = (g, e_2)$ of $P^*(Q_8 \times A)$ is adjacent to every vertex $z' = (g, a_1)$. Since $g \neq e_1$ then $o(g) = 2$ or $o(g) = 4$. For a given $a_1 \in A$ let $o(a_1) = s$ where s is an odd integer. Then we get:

$$\begin{aligned} z^s &= (g^s, a_1^s) = (g^s, a_1) = (g, a_1), && \text{for, } o(g) = 2 \text{ and } s \equiv 1 (mod\ 2), \\ z^{s^2} &= (g^{s^2}, a_1^{s^2}) = (g^{s^2}, a_1) = (g, a_1), && \text{for, } o(g) = 4 \text{ and } s^2 \equiv 1 (mod\ 4). \end{aligned}$$

So, $P^*(Q_8 \times A)$ is connected.

There are almost similar proof for the part (ii). It is sufficient to consider the different cases for the non-identity element $g \in Q_8 \times Z_2^k$. Indeed, g is of order 2 or 4, in this case.

To prove (iii) we use the results of Section 2 where the degrees of vertices of $P(Q_8 \times Z_2^k)$ are identified. Following the notations of Lemma 2.3 the degree of vertices of $P(Q_8 \times Z_2^k)$ are:

$$deg(y) = \begin{cases} 1, & \text{if } y \in A_3 \cup A_5, \\ 3, & \text{if } y \in A_1 \cup A_4 \cup A_6, \\ 1 + 3 \times 2^{k+1}, & \text{if } y \in A_2, \\ 2^{k+3} - 1, & \text{if } y = e. \end{cases}$$

So, there are exactly $2(2^k - 1)$ vertices of degree 1 (each of the sets A_3 and A_5 are of cardinality $2^k - 1$). However, these vertices are of degree 0 in the proper power graph. Proving that $P^*(G)$ is not connected. □

References

[1] T. Anitha, R. Rajkumar and A. Gagarian, *The complement of proper power graph of finite groups*, Palestin J. Math, **7** No. 2 (2018) 579-597.

[2] G. Aalipour, S. Akbari, P. J. Cameron, R. Nikandish and F. Shaveisi, *On the structure of the power graph and the enhanced power graph of a group*, Electron. J. Comb., **24** No. 3 (2017) 3-16.

[3] J. Abawajy, A. V. Kelarev and M. Chowdhury, *Power graphs: A survey*, Electron. J. Graph Theory Appl., **1** No. 2 (2013) 125-147.

[4] S. Bera, H. K. Dey, K. L. Patra and B. K. Sahoo, *On the domination number of proper power graphs of finite groups*, Discrete Math., **348** No. 10 (2025) 114557.

[5] A. K. Bhuniya and S. K. Mukherjee, *On the power graph of the direct product of two groups*, Electron. Notes Discrete. Math., **63** (2017) 197-202.

[6] P. J. Cameron, *The power graph of a finite group II*, J. Group Theory, **13**, (2010) 779-783.

[7] P. J. Cameron and S. Ghosh, *The power graph of a finite group*, Discrete Math., **311** (2011) 1220-1222.

[8] C. M. Campbell, E. F. Robertson, N. Ruškuc and R. M. Thomas, *On semigroups defined by Coxeter-type presentations*, Proc. Roy. Soc. Edinburgh Sect. A, **125** (1995) 1063-1075.

[9] I. Chakrabarty, S. Ghosh, M.K Sen, *Undirected power graphs of semigroups*, Semigr. Forum, **78** No. 3 (2009) 410-426.

[10] S. Chattopadhyay and P. Panigrahi, *Power graphs of finite groups of even oeder*, Internat. Conf. on Math. Modeling and Scientific Computations, (2012) 62-67.

[11] T. T. Chelvam and M. Sattanathan, *Power graphs of finite abelian groups*, Algebra and Discrete Math, **16** (2013) 33-41.

[12] D. L. Johnson, *Presentation of Groups*, Second Ed., Cambridge University Press, Cambridge, 1997.

[13] A. V. Kelarev and S. J. Quinn, *A combinatorial property and power graphs of groups*, Contrib. General Algebra, **12** (2000) 229-235.

[14] A. V. Kelarev, *Graph Algebras and Automata*, Marcel Dekker, New York, 2003.

[15] X. Ma, *Forbidden Subgraphs in Intersection Power Graphs of Finite Groups*, Algebra Colloq., **32** No. 1 (2025) 95-110.

[16] X. Ma, A. Doostabadi and K. Wang, *Notes on the diameter of the complement of the power graph of a finite group*, Ars Math. Contemp., **25** No. 1 (2025) P1.07.

[17] X. Ma, L. Li and G. Zhong, *Perfect codes in proper intersection power graphs of finite groups*, Appl. Algebra Eng. Commun. Comput. (AAECC), **36** No. 3 (2025) 543-555.

[18] M. Pourhasan and H. Doostie, *On the power graphs of elementary abelian and extra special p-groups*, Internat. J. Group Theory, **10** No. 2 (2021) 89-95.

[19] E. F. Robertson and Y. Ünlü, *On semigroup presentations*, Proc. Edinburgh Math. Soc, **36** (1993) 55-68.

[20] M. Tarnauceanu, *Some combinatorial aspects of finite Hamiltonian groups*, Bull. Iranian Math. Soc, **39** No. 5 (2013) 841-854.

Fatemeh Samneia

Department of Mathematics, Shab.C.,

Islamic Azad University, Shabestar, Iran.

fatemeh.samneia@iau.ac.ir

Hossein Doostie

Department of Mathematics, Kharazmi University,

49 Mofateh Ave., Tehran, Iran.

doostih@gmail.com

Bahram Vakili

Department of Mathematics, Shab.C.,

Islamic Azad University, Shabestar, Iran.

bvakil@iau.ac.ir and bvakil2011@gmail.com