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Research Paper

SOME CONNECTIONS BETWEEN R-MODULES AND S-ACTS VIA THE k-REALIZATION FUNCTOR

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Abstract. For a commutative pointed monoid S and a commutative unital ring k, let $k[S]$ be the commutative ring consisting of finite k-linear sums of non-zero elements of S. In this paper, we investigate some properties of the k-realization functor from the category S-$\mathbf{Act}_0$ of all pointed S-acts to the category $k[S]$-$\mathbf{Mod}$ of all $k[S]$-modules, which is a left adjoint to the forgetful functor. Using this adjunction, we show that S-$\mathbf{Act}_0$ has enough injective objects. Finally, we prove that the functor $k[-]$ is faithful but not full.

1. Introduction

Throughout this paper, all monoids are assumed to be pointed and commutative. Let S be a monoid. An act over S is a pointed set X with an action $S \times X \to X$. These objects, called (pointed) S-acts, are the main focus of this paper. A morphism of S-acts is an S-equivariant pointed set map. A notable feature of S-acts is that the First Isomorphism Theorem does

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not generally hold. in other words, for an S-homomorphism $f : X \to Y$, it is not generally true that $X/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$. Moreover, $\operatorname{Ker}(f) = 0$ does not necessarily imply that f is injective. We address this by focusing on admissible morphisms, i.e., morphisms $f : X \to Y$ such that $f : X \to f(X)$ is a cokernel. Let k be a commutative unital ring, and let $k[S]$ be a commutative ring consisting of finite k-linear sums of non-zero elements of S. In this paper, we study some properties of the forgetful functor U and the k-realization functor $k[-]$. These functors were introduced in [4], and they form an adjunction. In this paper, it is shown that the functor U is not exact in the sense that it does not carry exact sequence to admissible exact sequence. Also, it is shown that while the functor U is not exact, the functor $k[-]$ is exact in the sense that it carries admissible exact sequences to exact sequences (in the usual sense of abelian categories). Additionally, it is proved that the functor U commutes with equalizers and the functor $k[-]$ commutes with coequalizers. Also, it is shown that the k-realization functor preserves and reflects monomorphisms. As an application, we prove that the category S-$\mathbf{Act}_0$ has enough injectives. We follow standard notation and terminology from [6]. Also, for further definitions in related algebraic structures, see [3] and [5].

2. MAIN RESULTS

Let S be a commutative pointed monoid, and if $st = 0_S$, then $s = 0_S$ or $t = 0_S$ for all $s, t \in S$. For a monoid S, a non-empty and pointed set X is called a *left S-act*, if there exists a mapping $S \times X \to X$ by $(s, x) \to sx$, satisfying the conditions $1x = x$, $0_S x = 0_X$, $s0_X = 0_X$, $(st)x = s(tx)$, and if $sx = 0_X$, then $s = 0_S$ or $x = 0_X$, for all $s, t \in S$ and $x \in X$. Throughout this paper, all monoids are assumed to be commutative having a zero element and an S-act is a left S-act. For S-acts X and Y, an *S-homomorphism* is a map $f : X \to Y$ such that $f(0_X) = 0_Y$, and $f(sx) = sf(x)$ for every $s \in S$ and $x \in X$. The category of all S-acts together with their S-morphisms will be denoted by S-$\mathbf{Act}_0$. For an S-map $f : X \to Y$, $\operatorname{Ker}(f)$ is the equalizer of f and 0, and $\operatorname{Coker}(f)$ is the coequalizer of f and 0, where 0 denotes the zero morphism. In S-$\mathbf{Act}_0$, kernels and cokernels always exist. However, unlike in some algebraic categories, in [4] it is shown that $\operatorname{Ker}(f) = 0$ does not imply injectivity of f in S-$\mathbf{Act}_0$. So, we restrict our attention to admissible morphisms. For such maps, the condition $\operatorname{Ker}(f) = 0$ implies that f is injective. Furthermore, a sequence of S-acts and S-maps

$$\cdots \to X_{n+1} \overset{f_{n+1}}{\to} X_n \overset{f_n}{\to} X_{n+1} \to \cdots$$

is called *admissible* whenever f_i is admissible for all i. An admissible sequence is *exact* provided that $\operatorname{Im}(f_{i-1}) = \operatorname{Ker}(f_i)$ holds for every i. If a sequence

$$0 \to X_1 \overset{f_1}{\to} X_2 \overset{f_2}{\to} X_3 \to 0,$$

of S-acts and S-maps is admissible exact, then it is called an *admissible short exact sequence.* In this case, one has f_1 is an injection, f_2 is an epimorphism and $X_2/X_1 \cong X_3$.

Let $\mathcal{R}$ denote the category of commutative unital rings and let $\mathcal{M}$ be the category of commutative pointed monoids. Let $R \in \mathcal{R}$ and let $U(R)$ denote the underlying monoid of R obtained by forgetting addition. Then $(U(R), \cdot) \in \mathcal{M}$ is a monoid with unit 1 and basepoint 0. This construction induces a functor $U : \mathcal{R} \to \mathcal{M}$, called the *forgetful functor.* This functor induces the forgetful functor $U : R\text{-}\mathbf{Mod} \to U(R)\text{-}\mathbf{Act}_0$, where R-**Mod** is the category of R-modules. To every R-module M, the $U(R)$-act $U(M)$ has no addition and retains its R-action. The following example shows that the functor $U : R\text{-}\mathbf{Mod} \to U(R)\text{-}\mathbf{Act}_0$ is not exact in the sense that it does not carry exact sequences to admissible exact sequences.

Example 2.1. If $R = \mathbb{Z}$, the usual ring of integers, the short exact sequence of $\mathbb{Z}$-modules (i.e. of abelian groups) $0 \longrightarrow C_3 \xrightarrow{\iota_1} C_3 \times C_4 \xrightarrow{\pi_2} C_4 \longrightarrow 0$, where C_3 and C_4 are the cyclic groups of order 3 and 4, respectively, and ι_1 and π_2 are the canonical inclusion and projection, is sent by U to a sequence in which π_2 is not admissible, because its restriction to $(C_3 \times C_4) \setminus \mathrm{Ker}(\pi_2)$ is not an injection. So, the sequence is not admissible.

Let $k \in \mathcal{R}$ and let $S \in \mathcal{M}$ with a pointed element θ. For every $s \in S$, we set

$$e_s = (\ldots, 0, 0, \ldots, 0, 1, 0, \ldots, 0, \ldots),$$

where 1 is situated in the s-th component, and $1 = 1_k$, $0 = 0_k$. Then $\coprod_{s\in S} k$ is the free k-module with basis $\{e_s\}_{s\in S}$. We define

$$\begin{cases} \varphi : \coprod\limits_{s\in S} k \to \coprod\limits_{\theta\neq s\in S} k, \\ \sum\limits_{\substack{i=1 \\ s_j=\theta}}^{n} k_i e_{s_i} \mapsto \sum\limits_{\substack{i=1 \\ s_i\neq\theta \\ i\neq j}}^{n} k_i e_{s_i}. \end{cases}$$

It is easy to check that φ is a surjective k-homomorphism and $\mathrm{Ker}\,\varphi =< e_\theta >$. Hence, we have the k-isomorphism $\frac{\coprod_{s\in S} k}{<e_\theta>} \cong \coprod\limits_{\theta\neq s\in S} k$. Set $k[S] := \coprod\limits_{\theta\neq s\in S} k$. Therefore $k[S]$ is a free k-module and its basis is the non-zero elements of S. Hence, every arbitrary element of $k[S]$ is a finite k-linear sum of non-zero elements of S. Suppose that $\sum\limits_{\substack{i=1 \\ s_i\neq\theta}}^{n} k_i e_{s_i}$ and $\sum\limits_{\substack{j=1 \\ s'_j\neq\theta}}^{n} k'_j e_{s'_j}$ are two elements of $k[S]$. We define

$$(\sum_{i=1}^{n} k_i e_{s_i})(\sum_{j=1}^{n} k'_j e_{s'_j}) := \sum_{i=1}^{n}\sum_{j=1}^{n} k_i k'_j e_{s_i} e_{s'_j},$$

which turns $k[S]$ into a commutative ring. Note that $s_i \neq \theta$ and $s'_j \neq \theta$ imply that $s_i s'_j \neq \theta$, since S is a pointed monoid. Now, we are ready to recall the definition of the k-realization functor. Let us first recall that a ring R is a k-*algebra*, where k is a commutative ring, if R is a k-module satisfying $k^{'}(rr^{'}) = (k^{'}r)r^{'} = r(k^{'}r^{'})$ for all $r, r^{'} \in R$ and $k^{'} \in k$.

Definition 2.2. [4, Page 46] Let $k \in \mathcal{R}$ and $S \in \mathcal{M}$. The k-*realization* functor $k[-] : \mathcal{M} \to k$-algebra assigns to every monoid S the free k-module $k[S]$ with basis the non-zero elements of S, endowed with multiplication induced by that of S. This can be extended to the k-realization of S-acts denoted as $k[-] : S - \mathbf{Act}_0 \to k[S]$-**Mod**. If X is an S-act, then $k[X]$ is the free $k[S]$-module with a basis as the set of non-zero elements of X and the $k[S]$-action given by S-action on X.

Remark 2.3. Let $f : X \to Y$ be an S-homomorphism. Then $k[f] : k[X] \to k[Y]$ is a $k[S]$-homomorphism which is defined by

$$k[f]\Big(\sum_{\substack{i=1 \\ 0\neq x_i \in X}}^{n} k_i e_{x_i}\Big) = \sum_{\substack{i=1 \\ 0\neq f(x_i)\in Y}}^{n} k_i e_{f(x_i)},$$

for every $\sum\limits_{\substack{i=1 \\ 0\neq x_i \in X}}^{n} k_i e_{x_i} \in k[X]$. Therefore, $k[-] : S - \mathbf{Act}_0 \to k[S]$-**Mod** is a covariant functor.

In the sequel, we investigate some properties of the functor $k[-]$.

Proposition 2.4. [4, Proposition 3.2.6] *The functor $k[-]$ is exact in the sense that it carries admissible exact sequences to exact sequences* (*in the usual sense of abelian categories*).

Proposition 2.5. *Let $a \in S$, and let $f : S \to S$ be a monoid homomorphism, such that $f(s) = as$, for every $s \in S$. Then $k[S/aS] \cong k[S]/ak[S]$.*

Proof. The sequence $0 \to aS \xrightarrow{\iota} S \xrightarrow{\pi} S/aS \to 0$ is an admissible exact sequence of S-acts by [8, Proposition 3.2]. Then the sequence $0 \to k[aS] \xrightarrow{k[\iota]} k[S] \xrightarrow{k[\pi]} k[S/aS] \to 0$ is an exact sequence of $k[S]$-modules, by Proposition 2.4. Then $k[S/aS] \cong k[S]/k[aS]$. Note that, $k[aS] = ak[S]$, and so we get the assertion. □

It is well-known that the functors $k[-]$ and U form an adjunction with $k[-]$ a left adjoint to U by [4]. According to this fact and [7, Lemma 7.5.1], we have the following result.

Corollary 2.6. *Let $S \in \mathcal{M}$. Then the following statements hold.*

(*i*) *For a family $\{X_i\}_{i\in I}$ of S-acts, $k[\coprod\limits_{i\in I} X_i] \cong \coprod\limits_{i\in I} k[X_i]$.*

(*ii*) *For a family $\{M_i\}_{i\in I}$ of $k[S]$-modules, $U(\prod\limits_{i\in I} M_i) \cong \prod\limits_{i\in I} U(M_i)$.*

(*iii*) *For an S-homomorphism $f, g : X \to Y$,*

$$k[\mathrm{coeq}(f, g)] \cong \mathrm{coeq}(k[f], k[g]),$$

where $\operatorname{coeq}(f,g)$ *is the coequalizer of* f *and* g*.*

(iv) *For an* $k[S]$*-homomorphism* $f, g : M \to N$*,*

$$U(\operatorname{eq}(f,g)) \cong \operatorname{eq}(U(f), U(g)),$$

where $\operatorname{eq}(f,g)$ *is the equalizer of* f *and* g*.*

(v) *For an* S*-homomorphism* $f : X \to Y$*,*

$$k[\operatorname{Coker}(f)] \cong \operatorname{Coker}(k[f]).$$

(vi) *For an* $k[S]$*-homomorphism* $f : M \to N$*,*

$$U(\operatorname{Ker}(f)) \cong \operatorname{Ker}(U(f)).$$

Proposition 2.7. *Let* $f, g : X \to Y$ *be* S*-homomorphisms. Then* $k[\operatorname{eq}(f,g)] \subseteq \operatorname{eq}(k[f], k[g])$*.*

Proof. Let $E = \operatorname{eq}(f,g)$ and $E' = \operatorname{eq}(k[f], k[g])$. If $\sum\limits_{\substack{i=1 \\ 0\neq x_i \in E}}^{n} k_i e_{x_i} \in k[E]$, then $\sum\limits_{\substack{i=1 \\ 0\neq x_i \in E}}^{n} k_i e_{f(x_i)} = \sum\limits_{\substack{i=1 \\ 0\neq x_i \in E}}^{n} k_i e_{g(x_i)}$, and so $\sum\limits_{\substack{i=1 \\ 0\neq x_i \in E}}^{n} k_i e_{x_i} \in E'$, as desired. □

In the following, we show that the k-realization functor preserves and reflects monomorphisms.

Lemma 2.8. *Let* $f : X \to Y$ *be an* S*-homomorphism. Then* $k[f] : k[X] \to k[Y]$ *is a* $k[S]$*-monomorphism if and only if* $f : X \to Y$ *is an* S*-monomorphism.*

Proof. Necessity. Let $x_1, x_2 \in X$, $f(x_1) = f(x_2)$. Then $k[f](e_{x_1}) = e_{f(x_1)} = e_{f(x_2)} = k[f](e_{x_2})$ and since $k[f]$ is a $k[S]$-monomorphism, we have $e_{x_1} = e_{x_2}$. So, $x_1 = x_2$.
Sufficiency. Let $\sum\limits_{\substack{i=1 \\ 0\neq x_i \in X}}^{n} k_i e_{x_i} \in \operatorname{Ker}(k[f])$. Then $\sum\limits_{\substack{i=1 \\ 0\neq x_i \in X}}^{n} k_i e_{f(x_i)} = 0$. Since f is an S-monomorphism and $0 \neq x_i \in X$, $f(x_i) \neq 0$. Hence, $k_i = 0$, for each $i = 1, \ldots, n$, as desired. □

In what follows, we study some injectivity transferrings between modules and acts. Injective objects are defined in the categorical manner, see [8].

Lemma 2.9. *Let* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *be two functors such that* F *is left adjoint to* G*. Also, let* $\mathcal{M}$*,* $\mathcal{M}'$ *be certain subclasses of morphisms of* $\mathcal{C}$*,* $\mathcal{D}$*, respectively. If for all* $f \in \mathcal{M}$*,* $Ff \in \mathcal{M}'$*, then for any* $\mathcal{M}'$*-injective object* $D \in \mathcal{D}$*,* GD *is an* $\mathcal{M}$*-injective object of* $\mathcal{C}$*.*

Proof. See [1, Page 136]. □

Proposition 2.10. *Let M be an injective $k[S]$-module. Then M is an injective S-act.*

Proof. The functor $k[-]$ is a left adjoint to U. Also, the functor k preserves monomorphisms by Lemma 2.8. Therefore, Lemma 2.9 implies that $U({}_{k[S]}M) = {}_S M$ is an injective S-act, where $U({}_{k[S]}M)$ is $U(M)$ viewing M as a $k[S]$-module. $\square$

Corollary 2.11. *Let X be an S-act. If $k[X]$ is an injective $k[S]$-module, then $k[X]$ is an injective S-act.*

Corollary 2.12. *Let a $k[S]$-module M be an injective extension of a $k[S]$-module N. Then the S-act M is an injective extension of the S-act N.*

Proof. Let $f : N \to M$ be a $k[S]$-monomorphism, and let M be an injective $k[S]$-module. Then $U(f) = f : N \to M$ is an S-monomorphism. On the other hand, M is an injective S-act by Proposition 2.10. So, the result is obtained. $\square$

In the following, we show that the category $S - \mathbf{Act}_0$ has enough injective objects. It should be noted that this fact the ordinary category of S-acts has been proved in [6, Corollary 3.1.6]. Every object in the category $S - \mathbf{Act}_0$ has a unique zero element, but in [6, Theorem 3.1.5], it was shown that an S-act X is embedded in the cofree S-act X^S which is injective and has no unique zero element. Indeed, every constant map $g_{x_0} : S \to X$, $g_{x_0}(s) = x_0$ for all $s \in S$, is a zero element of X^S.

The following theorem has been proven in [2, Corollary 1], but it is proven here using methods of functor.

Theorem 2.13. [2] *The category $S - \mathbf{Act}_0$ has enough injective objects.*

Proof. Let X be an S-act. Consider the $k[S]$-module $k[X]$. Therefore, there exists an injective $k[S]$-module E and a $k[S]$-monomorphism $f : k[X] \to E$. By Corollary 2.12, the S-act E is an injective extension of the S-act $k[X]$, and $U(f) = f : k[X] \to E$ is an S-monomorphism. On the other hand, the morphism $g : X \to k[X]$ given by $g(x) = e_x$ for all $0 \neq x \in X$ and $g(0_X) = 0_{k[X]}$ is an S-monomorphism. Hence, $f \circ g : X \to E$ is an S-monomorphism, which completes the proof. $\square$

Recall from [7, Definition 4.1.2] that the functor $F : \mathcal{C} \to \mathcal{D}$ is called *full* if for every object $C_1, C_2 \in \mathcal{C}$ and every morphism $g : F(C_1) \to F(C_2)$ in $\mathcal{D}$, there exists a morphism $f : C_1 \to C_2$ in $\mathcal{C}$ such that $F(f) = g$. Moreover, F is called *faithful* if for every morphism $f, g \in \mathcal{C}$, the condition $F(f) = F(g)$ implies that $f = g$.

Proposition 2.14. *The functor $k[-]$ is faithful.*

Proof. Let $f, g : X \to Y$ be two S-homomorphisms such that $k[f] = k[g]$. Hence, $k[f](e_x) = k[g](e_x)$, for every $x \in X$. Therefore, $e_{f(x)} = e_{g(x)}$, for every $x \in X$, and so $f = g$. □

The following example shows that the faithful functor $k[-]$ is not full.

Example 2.15. Let $S = \{0, 1, r\}$ be a monoid with r being an idempotent element. Then $X = rS = \{r, 0\}$ and $Y = S$ are both S-acts. Hence, $k[X] = \{k^{'}e_r \mid k^{'} \in k\}$ and $k[Y] = \{k_1e_1 + k_re_r \mid k_1, k_r \in k\}$ are $k[S]$-module. Define a map $g : k[X] \to k[Y]$ by $g(k^{'}e_r) = k^{'}e_1 + k^{'}e_r$. It is routine to check that g is a $k[S]$-homomorphism. Notice that $\mid X \mid= 2$ and $\mid Y \mid= 3$. Then there are 9 maps from X to Y, only 6 of which are S-homomorphisms and for all of them $k[f] \neq g$, where, f is an S-homomorphism from X to Y. Therefore, the functor $k[-]$ is not full.

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