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Research Paper

THE CHARACTERIZATION OF BITONIC ALGEBRAS BY SHEFFER STROKE

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Abstract. The main objective of this study is to characterize bitonic algebras by means of the Sheffer stroke and to investigate some properties of related structures. It is shown that every Sheffer stroke bitonic algebra is a bitonic algebra, while the converse holds under special conditions. It is also illustrated that every Sheffer stroke bitonic algebra is a SUP-algebra; however, the converse is not true in general. By introducing certain filters of Sheffer stroke bitonic algebras, a congruence relation is defined on these algebraic structures. Moreover, quotient Sheffer stroke bitonic algebras are constructed via this congruence relation. Finally, homomorphisms on Sheffer stroke bitonic algebras are described, and fundamental homomorphism theorems are proved for these structures.

1. Introduction

In the late 19th century, H. M. Sheffer introduced the Sheffer stroke, also known as the Sheffer operation [21], which is a binary logical connective that is sufficient to define all other

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logical connectives. This remarkable property has made it an essential tool in mathematical logic and theoretical computer science.

Since the well-known examples are Boolean algebras, which are the algebraic counterparts of modern programming and digital computation, one of the most important applications of the Sheffer stroke arises in digital circuit design. In particular, logical circuits can be constructed using a single type of operation instead of multiple Boolean operations. This leads to a simplification in hardware design, where fewer logic gates (diodes) are required, making circuits more efficient, cost-effective, and easier to implement. From this perspective, the Sheffer stroke plays a fundamental role in reducing computational complexity at the hardware level.

Since algebraic structures serve as theoretical frameworks for computer science and logic systems, the Sheffer operation has provided new and fundamental axiom systems for many algebraic structures such as Boolean algebras, BL-algebras, BCK-algebra, UP-algebra, ortholattices, Hilbert algebras, and etc. (see [5], [14], [16]-[17]).

The concept of BCC-algebras, introduced by Komori [13] and Dudek [8], is a generalization of Iseki's BCK algebra [11]. A dual BCC-algebra is also a generalization of DBCK-algebras ([4], [12], [22]), Hilbert algebras ([7], [9], [10], [15]), Heyting algebras ([3], [6]), implication algebras [1], and lattice implication algebras ([24], [25]). These algebraic systems satisfy the following property (P):

$$x \leq y \;\Rightarrow\; z * x \leq z * y \text{ and } y * z \leq x * z.$$

Then bitonic algebras were introduced by Yong Ho Yon and Şule Ayar Özbal as a generalization of dual BCC-algebras [23], and the direct product of bitonic algebras has been studied in [2].

Motivated by these developments, and in order to further investigate algebraic structures equipped with Sheffer-type operations, we introduce the notion of s-Bitonic algebras. These structures aim to capture and generalize key algebraic behaviors arising from Sheffer stroke operations while preserving important order-theoretic and logical properties. In this sense, s-Bitonic algebras provide a unified framework for studying algebraic and logical systems with a Sheffer-based operation.

From an algebraic logic perspective, such structures may also contribute to the understanding of computational logic systems, formal reasoning, and the algebraic modeling of digital circuits.

The manuscript is organized as follows. In the first section, the historical background and recent developments of the related structures are presented. In the second section, the basic definitions and notions used throughout the study are given. In the third, fourth, and fifth sections, the main results are presented in detail and supported with illustrative examples.

Since these results are new in the literature, the manuscript contributes to pure mathematics in the areas of bitonic algebras, Sheffer operations, and abstract algebra.

2. Preliminaries

In this section, fundamental definitions and notions about Sheffer operation and Bitonic algebras are presented.

Definition 2.1. [5] Let $\mathcal{S} = (S, |)$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke* (or *Sheffer operation*) if it satisfies the following conditions:

$(S1)$ $a|b = b|a$,

$(S2)$ $(a|a)|(a|b) = a$,

$(S3)$ $a|((b|c)|(b|c)) = ((a|b)|(a|b))|c$,

$(S4)$ $(a|((a|a)|(b|b)))|(a|((a|a)|(b|b))) = a$,

for all $a, b, c \in \mathcal{S}$.

Lemma 2.2. [5] *Let $\mathcal{S} = (S, |)$ be a groupoid. Then the binary relation $\leq$ on S as defined by*

$$a \leq b \text{ if and only if } a|b = a|a$$

is an order on S.

Lemma 2.3. [5] *Let $|$ be a Sheffer stroke on S and $\leq$ the induced order of $\mathcal{S} = (S, |)$. Then*

(i) $a \leq b$ *if and only if* $b|b \leq a|a$,

(ii) $a|(b|(a|a)) = a|a$ *is the identity of* S,

(iii) $a \leq b$ *implies* $b|c \leq a|c$ *for all* $c \in S$,

(iv) $x \leq a$ *and* $x \leq b$ *imply* $a|b \leq x|x$.

Definition 2.4. [23] A bitonic algebra is an algebraic system $(A, *, 1)$, where A is a set, 1 an element in A and $*$ a binary operation on A, satisfying the following axioms. For every $a, b, c \in A$,

(B1) $a * 1 = 1$,

(B2) $1 * a = a$,

(B3) $a * b = 1$ and $b * a = 1$ implies $a = b$,

(B4) $a * b = 1$ implies $(c * a) * (c * b) = 1$ and $(b * c) * (a * c) = 1$.

Lemma 2.5. [23] *Let $(A, *, 1)$ be a bitonic algebra. Then*

(1) $a * a = 1$,

(2) $a * b = b * c = 1$ *implies* $a * c = 1$,

(3) $a * (b * a) = 1$,

for every $a, b, c \in A$.

Lemma 2.6. [23] *Let $(A, *, 1)$ be a bitonic algebra. If we define a binary relation "$\leq$" on A by*

$$a \leq b \iff a * b = 1,$$

for any $a, b \in A$, then $\leq$ is a partial order on A. Hence $(A, \leq)$ is a poset and 1 is the greatest element of A.

Lemma 2.7. [23] *Let $(A, *, 1)$ be a bitonic algebra. Then*

(1) *$a \leq b$ implies $c * a \leq c * b$ and $b * c \leq a * c$,*
(2) *$a \leq b * a$,*

for every $a, b, c \in A$.

Definition 2.8. [19] A Sheffer stroke UP-algebra (briefly, a SUP-algebra) is a structure $(A, |)$ of type (2) such that 0 is the fixed element in A and the following conditions are satisfied for all $a, b, c \in A$:

(SUP-1) $(((c|(a|a))|(c|(a|a)))|(((b|(a|a))|(c|(b|b)))|((b|(a|a))|(c|(b|b)))))|$
$(((c|(a|a))|(c|(a|a)))|(((b|(a|a))|(c|(b|b)))|((b|(a|a))|(c|(b|b))))) = 0$,
(SUP-2) $a|a = a|(0|0)$, and
(SUP-3) $(a|(b|b))|(a|(b|b)) = 0$ and $(b|(a|a))|(b|(a|a)) = 0$ imply $a = b$.

Lemma 2.9. [19] *In SUP-algebra $\langle A, |\rangle$, the following hold for all $a, b, c \in A$:*

(1) *$(a|(a|a))|(a|(a|a)) = 0$,*
(2) *$(b|(a|a))|(b|(a|a)) = 0$ and $(c|(b|b))|(c|(b|b)) = 0$ imply $(c|(a|a))|(c|(a|a)) = 0$,*
(3) *$(b|(a|a))|(b|(a|a)) = 0$ implies*
$(((b|(c|c))|(b|(c|c)))|(a|(c|c)))|(((b|(c|c))|(b|(c|c)))|(a|(c|c))) = 0$,
(4) *$(b|(a|a))|(b|(a|a)) = 0$ implies*
$(((c|(a|a))|(c|(a|a)))|(c|(b|b)))|(((c|(a|a))|(c|(a|a)))|(c|(b|b))) = 0$,
(5) *$(((a|(b|b))|(a|(b|b)))|(a|a))|(((a|(b|b))|(a|(b|b)))|(a|a)) = 0$,*
(6) *$a|(a|(b|b)) = 0$ if and only if $a|a = a|(b|b)$,*
(7) *$(((b|(b|b))|(b|(b|b)))|(a|a))|(((b|(b|b))|(b|(b|b)))|(x|x)) = 0$,*
(8) *$a|0 = 0|0$.*

3. s-Bitonic algebras

In this section, Sheffer stroke bitonic algebras are defined and studied in details.

Definition 3.1. A Sheffer stroke bitonic (for short, s-bitonic) algebra is an algebraic structure $(A, |, 1)$ where A is a set, 1 is a distinguished element of A, and $|$ is a Sheffer stroke on A satisfying the following axioms for all $a, b, c \in A$:

(sb1) $a|(1|1) = 1$,

(sb2) $a|(b|b) = b|(a|a) = 1$ implies $a = b$,

(sb3) $a|(b|b) = 1$ implies $(b|c)|((a|c)|(a|c)) = 1$, where $0 = 1|1$.

Example 3.2. Let $G = \{0, u, v, 1\}$ be a set and let a binary operation $|$ on G be defined by Table 1. Then $(G, |, 1)$ is an s-Bitonic algebra.

TABLE 1. Cayley table of the Sheffer stroke on G in Example 3.2.

$\mid$	0	u	v	1
0	1	1	1	1
u	1	v	1	v
v	1	1	u	u
1	1	v	u	0

Example 3.3. Given a set $H = \{0, a, b, c, d, e, f, 1\}$, and let a binary operation $|$ on H be defined by Table 2 as follows. Then $(H, |, 1)$ is an s-Bitonic algebra.

TABLE 2. Cayley table of the Sheffer stroke on H in Example 3.3.

$\mid$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	0

Lemma 3.4. *In an s-Bitonic algebra $(A, |, 1)$, the following properties hold for all $a, b, c \in A$:*

(1) $1|(a|a) = 1$,

(2) $a|a = a|1$,

(3) $a|0 = 1$,

(4) $(a|1)|(a|1) = a$,

(5) $a|(a|a) = 1$,

(6) $a|((a|b)|(a|b)) = a|b = ((a|b)|(a|b))|b$,

(7) $a|(b|b) = 1 = b|(c|c)$ *implies* $a|(c|c) = 1$,

(8) $a|((b|(a|a))|(b|(a|a))) = 1$.

Proof. The result is established by applying (sb1), (sb3), and (S1)–(S3). □

Lemma 3.5. *Let $(A, |, 1)$ be an s-Bitonic algebra and let a binary relation $\leq$ on A be defined by*

(1) $$a \leq b \Leftrightarrow a|(b|b) = 1,$$

for any $a, b \in A$. Then $\leq$ is a partial order on A. Also, 0 is the least element and 1 is the greatest element of A.

Proof. Let $(A, |, 1)$ be an s-Bitonic algebra and let a binary relation $\leq$ on A be defined as in (3.1). Then:

- Reflexivity follows from Lemma 3.4(5),
- By (sb2), antisymmetry holds,
- Transitivity follows immediately from Lemma 3.4(7).

Thus, $\leq$ is a partial order on A, and hence $(A, \leq)$ is a poset. Also, 0 is the least element and 1 is the greatest element of A, by Lemma 3.4(3), (S1), and (sb1). □

Lemma 3.6. *Let $(A, |, 1)$ be an s-Bitonic algebra and let $\leq$ be the induced order on A. Then the following properties hold for all $a, b, c \in A$:*

(1) $a \leq b|(a|a)$,
(2) $(a|b)|(a|b) \leq c \Leftrightarrow a \leq b|(c|c)$,
(3) $a \leq b \Rightarrow b|c \leq a|c$,
(4) $a \leq b \Leftrightarrow b|b \leq a|a$,
(5) $a \leq (a|(b|b))|(b|b)$,
(6) $a|(b|b) \leq (b|(c|c))|((a|(c|c))|(a|(c|c)))$,
(7) $a|(b|b) \leq (c|(a|a))|((c|(b|b))|(c|(b|b)))$.

Proof. (1) It is obtained from Lemma 3.4(8) and Lemma 3.5.

(2) By Lemma 3.5 and (S3),

$$\begin{aligned}(a|b)|(a|b) \leq c &\Leftrightarrow ((a|b)|(a|b))|(c|c) = 1 \quad \text{by Lemma 3.5}\\ &\Leftrightarrow a|((b|(c|c))|(b|(c|c))) = 1 \quad \text{by (S3)}\\ &\Leftrightarrow a \leq b|(c|c) \quad \text{by Lemma 3.5.}\end{aligned}$$

(3) It follows from Lemma 3.5 and (sb3).

(4)

$$\begin{aligned} a \leq b &\Leftrightarrow a|(b|b) = 1 \quad \text{by Lemma 3.5} \\ &\Leftrightarrow (b|b)|((a|a)|(a|a)) = 1 \quad \text{by (S1)–(S2)} \\ &\Leftrightarrow b|b \leq a|a. \end{aligned}$$

(5)

$$\begin{aligned} a|(((a|(b|b))|(b|b))|((a|(b|b))|(b|b))) &= a|(((b|b)|(a|(b|b)))|((b|b)|(a|(b|b)))) \\ &= ((a|(b|b))|(a|(b|b)))|(a|(b|b)) \\ &= (a|(b|b))|((a|(b|b))|(a|(b|b))) \\ &= 1, \end{aligned}$$

from (S1), (S3) and Lemma 3.4(5).

(6) Since $b \leq (b|(c|c))|(c|c)$, by (3), (4), (S1) and (S3),

$$\begin{aligned} a|(b|b) &\leq a|(((b|(c|c))|(c|c))|((b|(c|c))|(c|c))) \\ &= a|(((c|c)|(b|(c|c)))|((c|c)|(b|(c|c)))) \\ &= ((a|(c|c))|(a|(c|c)))|(b|(c|c)) \\ &= (b|(c|c))|((a|(c|c))|(a|(c|c))). \end{aligned}$$

(7) By (S1), (S2) and (6),

$$\begin{aligned} a|(a|b) &= (b|b)|((a|a)|(a|a)) \\ &\leq (c|(a|a))|((c|(b|b))|(c|(b|b))). \end{aligned}$$

□

Proposition 3.7. *Let $(A, |, 1)$ be an s-Bitonic algebra. Then $a|b = a|a$ if and only if $a|(b|b) = 1$ for all $a, b \in A$.*

Proof. Let $(A, |, 1)$ be an s-Bitonic algebra. Assume that $a|b = a|a$ for any $a, b \in A$. Then

$$\begin{aligned} a|(b|b) &= (b|b)|((a|a)|(a|a)) \\ &= (b|b)|((a|b)|(a|b)) \\ &= (b|b)|((b|a)|(b|a)) \\ &= a|((b|(b|b))|(b|(b|b))) \\ &= a|(1|1) \\ &= 1, \end{aligned}$$

from (S1)–(S3), Lemma 3.4(5) and (sb1).

Conversely, assume that $a|(b|b) = 1$ for any $a, b \in A$. Then $a \leq b$ by Lemma 3.5. Since

$$a|a \leq b|((a|a)|(a|a)) = a|b,$$

by Lemma 3.6(1), (S1)–(S2), and also

$$a|b \leq a|a,$$

by Lemma 3.6(3) and (S1), we obtain $a|b = a|a$ for all $a, b \in A$. □

Proposition 3.8. *Let $(A, |, 1)$ be a s-Bitonic algebra. If $k \leq a$ and $k \leq b$, then $a|b \leq k|k$, for any $a, b, k \in A$.*

Proof. Let $(A, |, 1)$ be a s-Bitonic algebra. Suppose that $k \leq a$ and $k \leq b$ for any $a, b, k \in A$. Since $k|(a|a) = 1$ and $k|(b|b) = 1$ from Lemma 3.5, we have from Proposition 3.7 that $k|a = k|k$ and $k|b = k|k$. Then $a|b \leq k|b = k|k$ and $a|b \leq k|a = k|k$ from Lemma 3.6 (3) and (S1). □

Proposition 3.9. *Let $(A, |, 1)$ be an s-Bitonic algebra. Then, for all $a, b \in A$, the following identities hold:*

(1) $(a|(b|b))|(b|b) = (a|a)|(b|b)$,
(2) $a|(a|(b|b)) = a|b$.

Proof. Let $(A, |, 1)$ be an s-Bitonic algebra.

(1) From Lemma 3.6(1), together with axioms (S1) and (S2), we obtain

$$a|a \leq (b|b)|((a|a)|(a|a)) = a|(b|b).$$

Then, by Lemma 3.6(5), it follows that

$$(a|(b|b))|(b|b) \leq (a|a)|(b|b),$$

for all $a, b \in A$.

On the other hand, by Lemma 3.6(1) and Lemma 3.6(5), we have

$$a, b \leq (a|(b|b))|(b|b).$$

Hence, using Lemma 3.6(4), Proposition 3.8, and axiom (S2), we obtain

$$(a|a)|(b|b) \leq (a|(b|b))|(b|b).$$

Therefore,

$$(a|(b|b))|(b|b) = (a|a)|(b|b),$$

for all $a, b \in A$.

(2) By applying part (1) with the substitutions

$$a := b|b \quad \text{and} \quad b := a|a,$$

and using axioms (S1) and (S2), we obtain

$$\begin{aligned} a|(a|(b|b)) &= ((b|b)|((a|a)|(a|a)))|((a|a)|(a|a)) \\ &= ((b|b)|(b|b))|((a|a)|(a|a)) \\ &= a|b. \end{aligned}$$

This completes the proof. □

Proposition 3.10. *Let $(A,|,1)$ be an s-Bitonic algebra. Then, for all $a,b,c \in A$,*

$$a|(b|c) = \big((a|(b|b))|(a|(c|c))\big)\big|\big((a|(b|b))|(a|(c|c))\big).$$

Proof. First, we show that

$$a|(b|c) \le \big((a|(b|b))|(a|(c|c))\big)\big|\big((a|(b|b))|(a|(c|c))\big).$$

By Lemma 3.6(1), together with axioms (S1) and (S2), we have

$$b|b \le c|((b|b)|(b|b)) = b|c,$$

and

$$c|c \le b|((c|c)|(c|c)) = b|c.$$

Hence, by Lemma 3.6(3),

$$a|(b|c) \le a|(b|b) \quad \text{and} \quad a|(b|c) \le a|(c|c),$$

for all $a,b,c \in A$. Therefore, applying Lemma 3.6(4), we obtain

$$a|(b|c) \le \big((a|(b|b))|(a|(c|c))\big)\big|\big((a|(b|b))|(a|(c|c))\big).$$

Next, we prove the converse inequality. Consider

$$\begin{aligned}
&\Big(((a|(b|b))|(a|(c|c)))|((a|(b|b))|(a|(c|c)))\Big)|((a|(b|c))|(a|(b|c)))\\
&= \Big(\Big(((a|(b|b))|(a|(c|c)))|((a|(b|b))|(a|(c|c)))\Big)|a\Big)\\
&\quad |\Big(\Big(((a|(b|b))|(a|(c|c)))|((a|(b|b))|(a|(c|c)))\Big)|a\Big)|(b|c)\\
&= \Big(((a|(a|(b|b)))|(a|(a|(b|b))))|(a|(c|c))\Big)\\
&\quad |\Big(((a|(a|(b|b)))|(a|(a|(b|b))))|(a|(c|c))\Big)|(b|c)\\
&= \Big(((a|b)|(a|b))|(a|(c|c))\Big)\\
&\quad |\Big(((a|b)|(a|b))|(a|(c|c))\Big)|(b|c)\\
&= \Big(b|((a|(a|(c|c)))|(a|(a|(c|c))))\Big)\\
&\quad |\Big(b|((a|(a|(c|c)))|(a|(a|(c|c))))\Big)|(b|c)\\
&= ((b|((a|c)|(a|c)))|(b|((a|c)|(a|c))))|(b|c)\\
&= ((a|((b|c)|(b|c)))|(a|((b|c)|(b|c))))|(b|c)\\
&= a|\Big(((b|c)|((b|c)|(b|c)))|((b|c)|((b|c)|(b|c)))\Big)\\
&= a|(1|1)\\
&= 1.
\end{aligned}$$

By axioms (S1) and (S3), Proposition 3.9(2), Lemma 3.4(5), and condition (sb1), it follows from Lemma 3.5 that

$$((a|(b|b))|(a|(c|c)))|((a|(b|b))|(a|(c|c))) \leq a|(b|c),$$

for all $a, b, c \in A$.

Consequently,

$$a|(b|c) = ((a|(b|b))|(a|(c|c)))|((a|(b|b))|(a|(c|c))),$$

for all $a, b, c \in A$. □

Theorem 3.11. *Let $(A, |, 1)$ be an s-Bitonic algebra, and let $\leq$ be the induced order on A. Then $(A, \leq)$ (equivalently, $(A, \vee, \wedge, 0, 1)$) forms a bounded distributive lattice, where*

$$a \vee b = (a|a)|(b|b),$$

and

$$a \wedge b = (a|b)|(a|b),$$

for all $a, b \in A$.

Proof. Let $(A, |, 1)$ be an s-Bitonic algebra, and let $\leq$ be the induced order on A.

By Lemma 3.6(1) and axiom (S1), we have

$$a \leq (b|b)|(a|a) = (a|a)|(b|b),$$

and

$$b \leq (a|a)|(b|b).$$

Hence, $(a|a)|(b|b)$ is an upper bound of a and b.

Now, let $c \in A$ be any upper bound of a and b, that is,

$$a \leq c \quad \text{and} \quad b \leq c.$$

Then

$$a|(c|c) = 1 \quad \text{and} \quad b|(c|c) = 1.$$

By Proposition 3.7, it follows that

$$a|c = a|a \quad \text{and} \quad b|c = b|b.$$

Using Lemma 3.6(3), we obtain

$$c|c \leq a|c = a|a,$$

and

$$c|c \leq b|c = b|b.$$

Therefore, by Proposition 3.8 and axiom (S2),

$$(a|a)|(b|b) \leq (c|c)|(c|c) = c.$$

Consequently,

$$a, b \leq (a|a)|(b|b) \leq c,$$

which shows that $(a|a)|(b|b)$ is the least upper bound of a and b. Hence,

$$a \vee b = (a|a)|(b|b),$$

for all $a, b \in A$.

Similarly,

$$a \wedge b = (a|b)|(a|b),$$

is the greatest lower bound of a and b for all $a, b \in A$.

Since 0 and 1 are respectively the least and greatest elements of A, it follows from Lemma 3.5 that

$$(A, \vee, \wedge, 0, 1),$$

is a bounded lattice.

Finally, we prove distributivity. Using Proposition 3.10 and axiom (S2), we obtain

$$\begin{aligned} a \vee (b \wedge c) &= a \vee ((b|c)|(b|c)) \\ &= (a|a)|((b|c)|(b|c)) \\ &= (a|(b|c))|(a|(b|c)) \\ &= \big(((a|a)|(b|b))|((a|a)|(c|c))\big) \\ &\quad \big|\big(((a|a)|(b|b))|((a|a)|(c|c))\big) \\ &= (a \vee b) \wedge (a \vee c). \end{aligned}$$

Similarly,

$$\begin{aligned} a \wedge (b \vee c) &= (a|((b|b)|(c|c)))|(a|((b|b)|(c|c))) \\ &= (a|(b \vee c))|(a|(b \vee c)) \\ &= ((a|b)|(a|c))|((a|b)|(a|c)) \\ &= \big(((a|b)|(a|b))|((a|c)|(a|c))\big) \\ &\quad \big|\big(((a|b)|(a|b))|((a|c)|(a|c))\big) \\ &= (a \wedge b) \vee (a \wedge c). \end{aligned}$$

Therefore, the lattice $(A, \vee, \wedge, 0, 1)$ is distributive. □

Definition 3.12. Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be two s-Bitonic algebras. Define the binary operation $|_{A\times B}$ on the Cartesian product $A \times B$ by

$$(a_1, b_1) \,|_{A\times B}\, (a_2, b_2) = (a_1|_A a_2,\; b_1|_B b_2),$$

for all $(a_1, b_1), (a_2, b_2) \in A \times B$.

Moreover, define $1_{A\times B} = (1_A, 1_B)$ and $0_{A\times B} = (0_A, 0_B)$.

Theorem 3.13. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be two s-Bitonic algebras. Then $(A \times B,\; |_{A\times B},\; 1_{A\times B})$ is an s-Bitonic algebra.*

Proof. The proof follows from Definition 3.12. □

Lemma 3.14. *Let $(A, |, 1)$ be an s-Bitonic algebra. Define a binary operation $*$ on A by*

$$a * b := a|(b|b),$$

for all $a, b \in A$. Then $(A, *, 1)$ is a bitonic algebra.

Proof. Let $(A, |, 1)$ be an s-Bitonic algebra, and define the binary operation $*$ on A by

$$a * b = a|(b|b),$$

for all $a, b \in A$.

We verify the axioms of a bitonic algebra.

(B1)

$$a * 1 = a|(1|1) = 1,$$

by condition (sb1).

(B2)

$$1 * a = 1|(a|a) = a,$$

by Lemma 3.4(1).

(B3) Suppose that

$$a * b = 1 \quad \text{and} \quad b * a = 1.$$

Then

$$a|(b|b) = 1 \quad \text{and} \quad b|(a|a) = 1.$$

Hence, by condition (sb2), we conclude that

$$a = b.$$

(B4) Suppose that

$$a * b = 1.$$

Then

$$a|(b|b) = 1.$$

By condition (sb3), we obtain

$$(c * a) * (c * b) = (c|(a|a))|((c|(b|b))|(c|(b|b))) = 1,$$

and

$$(b * c) * (a * c) = (b|(c|c))|((a|(c|c))|(a|(c|c))) = 1.$$

Therefore, $(A, *, 1)$ is a bitonic algebra. □

Example 3.15. Consider the s-Bitonic algebra $(G, |, 1)$ given in Example 3.2. Then the bitonic algebra $(G, *, 1)$ induced by this s-Bitonic algebra is given by Table 3 below.

TABLE 3. Cayley table of the binary operation $*$ on G.

$*$	0	u	v	1
0	1	1	1	1
u	v	1	v	1
v	u	u	1	1
1	0	u	v	1

Proposition 3.16. *Let $(A, *, 1)$ be a bitonic algebra with the least element 0. Define a unary operation "$\circ$" on A by $a^\circ = a * 0$ for all $a \in A$. Then the following statements hold for all $a, b \in A$:*

(1) $0^\circ = 1$ *and* $1^\circ = 0$,
(2) $0 * a = 1$,
(3) *If* $a \leq b$, *then* $b^\circ \leq a^\circ$.

Proof. Let $(A, *, 1)$ be a bitonic algebra with the least element 0, and define the unary operation "$\circ$" on A by

$$a^\circ = a * 0,$$

for all $a \in A$.

(1) By Lemma 2.5(1), we have

$$0^\circ = 0 * 0 = 1.$$

Furthermore, by axiom (B2),

$$1^\circ = 1 * 0 = 0.$$

(2) Since 0 is the least element of A, it follows from Lemma 2.6 that

$$0 * a = 1,$$

for all $a \in A$.

(3) Assume that $a \leq b$. Then, by Lemma 2.7,

$$b^\circ = b * 0 \leq a * 0 = a^\circ.$$

□

Proposition 3.17. *Let $(A, *, 1)$ be a bitonic algebra with the least element 0 satisfying $a * (b * c) = b * (a * c)$ and $(a * b) * a = a$ for all $a, b, c \in A$. Define $a^\circ := a * 0$ and $a|b := a * b^\circ$ for all $a, b \in A$. Then $a|a = a^\circ$, $(a^\circ)^\circ = a$, and $(A, |, 1)$ is an s-Bitonic algebra.*

Proof. We verify the axioms of an s-Bitonic algebra.

(S1)

$$\begin{aligned} a|b &= a*b^\circ \\ &= a*(b*0) \\ &= b*(a*0) \\ &= b*a^\circ \\ &= b|a. \end{aligned}$$

(S2)

$$\begin{aligned} (a|a)|(a|b) &= a^\circ*(a*b^\circ)^\circ \\ &= a^\circ*((a*b^\circ)*0) \\ &= (a*b^\circ)*(a^\circ*0) \\ &= (a*b^\circ)*(a^\circ)^\circ \\ &= (a*b^\circ)*a \\ &= a. \end{aligned}$$

(S3)

$$\begin{aligned} a|((b|c)|(b|c)) &= a*\big((b*c^\circ)^\circ\big)^\circ \\ &= a*(b*c^\circ) \\ &= a*(b*(c*0)) \\ &= c*(a*(b*0)) \\ &= c*(a*b^\circ) \\ &= c*\big((a*b^\circ)^\circ\big)^\circ \\ &= c*((a*b^\circ)^\circ*0) \\ &= (a*b^\circ)^\circ*(c*0) \\ &= (a*b^\circ)^\circ*c^\circ \\ &= ((a|b)|(a|b))|c. \end{aligned}$$

(S4)

$$\begin{aligned} (a|((a|a)|(b|b)))|(a|((a|a)|(b|b))) &= (a*(a^\circ*b)^\circ)^\circ \\ &= (a*((a^\circ*b)*0))^\circ \\ &= ((a^\circ*b)*(a*0))^\circ \\ &= ((a^\circ*b)*a^\circ)^\circ \\ &= (a^\circ)^\circ \\ &= a. \end{aligned}$$

(sb1)

$$a|(1|1) = a*(1^\circ)^\circ = a*1 = 1,$$

by axiom (B1).

(sb2) Let

$$a|(b|b) = 1 \quad \text{and} \quad b|(a|a) = 1.$$

By Lemma 2.6 and axiom (S1), we obtain

$$a * b = a * (b^\circ)^\circ = 1,$$

and

$$b * a = b * (a^\circ)^\circ = 1.$$

Hence, by axiom (B3), we conclude that

$$a = b.$$

(sb3) Let

$$a|(b|b) = 1.$$

Using Lemma 2.6 and axiom (S1), we obtain

$$a * b = a * (b^\circ)^\circ = 1.$$

Then, by axiom (B4),

$$(b * c^\circ) * (a * c^\circ) = 1.$$

Moreover,

$$\begin{aligned} (b|c)|((a|c)|(a|c)) &= (b * c^\circ) * ((a * c^\circ)^\circ)^\circ \\ &= (b * c^\circ) * (a * c^\circ) \\ &= 1. \end{aligned}$$

Therefore, $(A, |, 1)$ is an s-Bitonic algebra. □

Example 3.18. Consider the bitonic algebra $(H, *, 1)$ given by Table 4.

TABLE 4. Cayley table of the binary operation $*$ on H.

$*$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	f	1	f	f	1	1	f	1
b	e	e	1	e	1	e	1	1
c	d	d	d	1	d	1	1	1
d	c	e	f	c	1	e	f	1
e	b	d	b	f	d	1	f	1
f	a	a	d	e	d	e	1	1
1	0	a	b	c	d	e	f	1

Then the s-Bitonic algebra $(H, |, 1)$ induced by this bitonic algebra coincides with the s-Bitonic algebra presented in Example 3.3.

Lemma 3.19. *Every s-Bitonic algebra is a SUP-algebra.*

Proof. Let $(A, |, 1)$ be an s-Bitonic algebra. We verify the axioms of a SUP-algebra.

(SUP-1)

$$\begin{aligned}
&\Big(((c|(a|a))|(c|(a|a)))|(((b|(a|a))|(c|(b|b)))|((b|(a|a))|(c|(b|b))))\Big)\\
&\quad|\Big(((c|(a|a))|(c|(a|a)))|(((b|(a|a))|(c|(b|b)))|((b|(a|a))|(c|(b|b))))\Big)\\
&= (c|(b|b))|\Big(((b|(a|a))|((c|(a|a))|(c|(a|a))))\\
&\qquad |((b|(a|a))|((c|(a|a))|(c|(a|a))))\Big)\\
&= 1|1\\
&= 0.
\end{aligned}$$

This follows from axiom (S3), axiom (S1), Lemma 3.6(6), and Lemma 3.5.

(SUP-2) By Lemma 3.4(2), we have

$$a|a = a|1 = a|(0|0).$$

(SUP-3) Suppose that

$$(a|(b|b))|(a|(b|b)) = 0,$$

and

$$(b|(a|a))|(b|(a|a)) = 0.$$

Then, by axiom (S2),

$$a|(b|b) = 0|0 = 1$$

and

$$b|(a|a) = 0|0 = 1.$$

Hence, by condition (sb2), we conclude that

$$a = b.$$

□

Example 3.20. The s-Bitonic algebra $(G, |, 1)$ presented in Example 3.2 is a SUP-algebra.

Lemma 3.21. *Let $(A, |)$ be a SUP-algebra. If 1 is an element of A defined by $1 = 0|0$, then $(A, |, 1)$ is an s-Bitonic algebra.*

Proof. Let $(A, |)$ be a SUP-algebra, and assume that$1 = 0|0$. We verify conditions (sb1)–(sb3).

(sb1) By axiom (S2) and Lemma 2.9(8), we obtain

$$a|(1|1) = a|0 = 0|0 = 1.$$

(sb2) Suppose that

$$a|(b|b) = 1 \quad \text{and} \quad b|(a|a) = 1.$$

Then, by axiom (S2),

$$(a|(b|b))|(a|(b|b)) = 1|1 = 0,$$

and

$$(b|(a|a))|(b|(a|a)) = 1|1 = 0.$$

Hence, by axiom (SUP-3), we conclude that $a = b$.

(sb3) Suppose that $a|(b|b) = 1$. Then, by axiom (S2),

$$(a|(b|b))|(a|(b|b)) = 1|1 = 0.$$

Therefore, using Lemma 2.9(3) together with axioms (S1) and (S2), we obtain

$$(b|c)|((a|c)|(a|c)) = 0|0 = 1.$$

Therefore, $(A, |, 1)$ is an s-Bitonic algebra. □

Example 3.22. Consider a SUP-algebra $(A, |)$ where $A = \{0, u, v, w, t, x, y, 1\}$, and the Sheffer operation $|$ on A is defined by Table 5. Then $(A, |, 1)$ is a s-Bitonic algebra.

Table 5. Cayley table of the Sheffer stroke on A in Example 3.22

$\vert$	0	1	u	v	w	t	x	y
0	1	1	1	1	1	1	1	1
1	1	0	y	x	t	w	v	u
u	1	y	y	1	1	y	y	1
v	1	x	1	x	1	x	1	x
w	1	t	1	1	t	1	t	t
t	1	w	y	x	1	w	y	x
x	1	v	y	1	t	y	v	t
y	1	u	1	x	t	x	t	u

4. Filters

In this section, we introduce several types of filters on s-Bitonic algebras and investigate their properties. Unless otherwise stated, A denotes an s-Bitonic algebra.

Definition 4.1. Let $(A,|)$ be an s-Bitonic algebra and let $\emptyset \neq F \subseteq A$. Then F is called a filter of A if it satisfies the following conditions:

(sF1) $1 \in F$,

(sF2) $a \in F$ and $a|(b|b) \in F$ imply $b \in F$,

for all $a, b \in A$.

Example 4.2. Consider the s-Bitonic algebra A given in Example 3.2. Then $\{u, 1\}$, $\{v, 1\}$, $\{1\}$, and A itself are filters of A.

Lemma 4.3. *Let $(A,|)$ be an s-Bitonic algebra and $\emptyset \neq F \subseteq A$. Then F is a filter of A if and only if the following conditions hold:*

(sF3) *$a, b \in F$ implies $a \wedge b \in F$,*

(sF4) *$a \in F$ and $a \leq b$ imply $b \in F$,*

for all $a, b \in A$.

Proof. Let $(A,|)$ be an s-Bitonic algebra and assume that $\emptyset \neq F \subseteq A$ is a filter of A.

(sF3) Let $a, b \in F$. Since $a|(((a|b)|b)|((a|b)|b)) = (a|b)|((a|b)|(a|b)) = 1 \in F$ from (S1), (S3), Lemma 3.4(5), and (sF1), it follows from (sF2) that $(a|b)|b \in F$. Moreover, since $b|(((a|b)|(a|b))|((a|b)|(a|b))) = (a|b)|b \in F$, using (sF2) and Theorem 3.11, we obtain $(a|b)|(a|b) \in F$, that is, $a \wedge b \in F$.

(sF4) Let $a \in F$ and $a \leq b$. By Theorem 3.13, we have $a|(b|b) = 1 \in F$. Hence, by (sF2), we conclude that $b \in F$.

Conversely, assume that F is a nonempty subset of A satisfying (sF3) and (sF4).

(sF1) Since $F \neq \emptyset$, take $a \in F$. As $a \leq 1$ for all $a \in A$, it follows from (sF4) that $1 \in F$.

(sF2) Let $a \in F$ and $a|(b|b) \in F$. Then, using Theorem 3.11 and Proposition 3.9(2), we obtain $a \wedge b = (a|b)|(a|b) = a \wedge (a|(b|b)) \in F$ by (sF3). Since $a \wedge b \leq b$, it follows from (sF4) that $b \in F$.

Therefore, F is a filter of A. □

Definition 4.4. Let $(A,|)$ be an s-Bitonic algebra. A nonempty subset B of A is called an s-subalgebra of A if $a|(b|b)$, $b|(a|a) \in B$ for all $a, b \in B$.

Example 4.5. Consider the s-Bitonic algebra $(H,|,1)$ given in Example 3.3. Then $\{0, c, d, 1\}$ is an s-subalgebra of H.

Lemma 4.6. *Every filter of an s-Bitonic algebra* $(A,|)$ *is a subalgebra of* A.

Proof. Let F be a filter of A, and let $a,b\in F$. Since

$$\begin{aligned}(a\wedge b)|((a|(b|b))|(a|(b|b))) &= ((a|b)|(a|b))|((a|(b|b))|(a|(b|b)))\\ &= ((a|(a|(b|b)))|(a|(a|(b|b))))|((a|(b|b))|(a|(b|b)))\\ &= a|\Big(((a|(b|b))|((a|(b|b))|(a|(b|b))))\\ &\qquad |((a|(b|b))|((a|(b|b))|(a|(b|b))))\Big)\\ &= a|(1|1)\\ &= 1\in F,\end{aligned}$$

and $a\wedge b\in F$ from Theorem 3.11, Proposition 3.9(2), (S3), Lemma 3.4(5), (sb1), (sF1), and (sF3), it follows from (sF2) that $a|(b|b)\in F$. Similarly, by interchanging a and b, we obtain $b|(a|a)\in F$. Hence F is a subalgebra of A. □

However, the converse of Lemma 4.6 does not hold in general.

Example 4.7. Consider the s-Bitonic algebra $(H,|,1)$ given in Example 3.3. Then $\{0,a,f,1\}$ is a subalgebra of H, but it is not a filter of H, since $a\in F$ and $a|(d|d)=1\in F$ while $d\notin F$.

Definition 4.8. Let F be a filter of an s-Bitonic algebra $(A,|,1)$. Then F is called an ultra filter of A if $a\vee b\in F$ implies $a\in F$ or $b\in F$, for any $a,b\in A$.

Example 4.9. Consider the s-Bitonic algebra $(H,|,1)$ given in Example 3.3. Then $\{c,e,f,1\}$ is an ultra filter of H, but $\{f,1\}$ is not an ultra filter of H, since $b,c\notin\{f,1\}$ while $b\vee c\in\{f,1\}$.

Lemma 4.10. *Let* F *be a filter of an s-Bitonic algebra* A. *Then* F *is an ultra filter of* A *if and only if* $a\in F$ *or* $a|a\in F$, *for all* $a\in A$.

Proof. Let F be an ultra filter of A. Since

$$a\vee(a|a)=(a|a)|((a|a)|(a|a))=a|(a|a)=1\in F,$$

by Theorem 3.11, (S1), (S2), Lemma 3.4(5), and (sF1), it follows from the definition of ultra filter that $a\in F$ or $a|a\in F$ for every $a\in A$.

Conversely, assume that F is a filter of A such that for every $a\in A$ we have $a\in F$ or $a|a\in F$. Let $a\vee b\in F$ and suppose that $a\notin F$. Then $a|a\in F$. Since

$$(a|a)|(b|b)=a\vee b\in F,$$

by Theorem 3.11, it follows from (sF2) that $b\in F$. Hence F is an ultra filter of A. □

Lemma 4.11. *Let F be a filter of an s-Bitonic algebra A. Then F is an ultra filter of A if and only if $a \notin F$ and $b \notin F$ imply $a|(b|b) \in F$ and $b|(a|a) \in F$, for all $a, b \in A$.*

Proof. Assume that F is an ultra filter of A and let $a \notin F$, $b \notin F$ for some $a, b \in A$. Then, by Lemma 4.10, we have $a|a \in F$ and $b|b \in F$. Since

$$a|a \leq (b|b)|((a|a)|(a|a)) = a|(b|b) \quad \text{and} \quad b|b \leq (a|a)|((b|b)|(b|b)) = b|(a|a),$$

by Lemma 3.6(1), (S1), and (S2), it follows from (sF4) that

$$a|(b|b) \in F \quad \text{and} \quad b|(a|a) \in F.$$

Conversely, assume that F is a filter of A satisfying that $a \notin F$ and $b \notin F$ imply $a|(b|b) \in F$ and $b|(a|a) \in F$ for all $a, b \in A$. Suppose that $a \notin F$ and $a|a \notin F$ for some $a \in A$. Then, by (S2), we obtain

$$a|a = a|((a|a)|(a|a)) \in F \quad \text{and} \quad a = (a|a)|(a|a) \in F,$$

which is a contradiction. Hence, for all $a \in A$, either $a \in F$ or $a|a \in F$. Therefore, by Lemma 4.10, F is an ultra filter of A. □

Lemma 4.12. *Let F be a filter of an s-Bitonic algebra A. Then F is an ultra filter of A if and only if $a|(b|b) \in F$ or $b|(a|a) \in F$, for all $a, b \in A$.*

Proof. Let F be an ultra filter of A. Since

$$\begin{aligned}
(a|(b|b)) \vee (b|(a|a)) &= ((a|(b|b))|(a|(b|b)))|((b|(a|a))|(b|(a|a))) \\
&= a\Big(((b|b)|((b|(a|a))|(b|(a|a)))) \\
&\qquad |((b|b)|((b|(a|a))|(b|(a|a))))\Big) \\
&= a\Big((((b|(b|b))|(b|(b|b)))|(a|a)) \\
&\qquad |(((b|(b|b))|(b|(b|b)))|(a|a))\Big) \\
&= a(((a|a)|(1|1))|((a|a)|(1|1))) \\
&= 1 \in F,
\end{aligned}$$

by Theorem 3.11, (S1), (S3), Lemma 3.4(5), (sb1), and (sF1). Hence, since F is an ultra filter, we obtain

$$a|(b|b) \in F \quad \text{or} \quad b|(a|a) \in F.$$

Conversely, assume that F is a filter of A such that for all $a, b \in A$,

$$a|(b|b) \in F \quad \text{or} \quad b|(a|a) \in F.$$

Suppose that $a \vee b \in F$. Then

$$(a \vee b)|(a|a) = ((a|a)|(b|b))|(a|a) = b|(a|a) \in F,$$

or

$$(a \vee b)|(b|b) = ((a|a)|(b|b))|(b|b) = a|(b|b) \in F,$$

by Theorem 3.11, (S1), (S2), and Proposition 3.9(1). Hence, by (sF2), we conclude that $a \in F$ or $b \in F$. Therefore, F is an ultra filter of A. □

5. Homomorphisms and quotient sets

In this section, homomorphisms between s-Bitonic algebras are introduced, and quotient structures are constructed via these homomorphisms. Furthermore, three fundamental homomorphism theorems are established for s-Bitonic algebras.

Definition 5.1. Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be two s-Bitonic algebras. A mapping $h : A \to B$ is called a homomorphism if

$$h(a_1|_A a_2) = h(a_1)|_B h(a_2),$$

for all $a_1, a_2 \in A$, and $h(1_A) = 1_B$.

Theorem 5.2. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be two s-Bitonic algebras, and let $h : A \to B$ be a homomorphism. Then the following properties hold:*

(1) *If F is a (ultra) filter of A, then $h(F)$ is a (ultra) filter of B.*
(2) *If G is a (ultra) filter of B and h is bijective, then $h^{-1}(G)$ is a (ultra) filter of A.*

Proof. Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be two s-Bitonic algebras, and let $h : A \to B$ be a homomorphism.

(1) Let F be a (ultra) filter of A. Since $h(1_A) = 1_B$, we have $1_B \in h(F)$. Let $h(a) \in h(F)$ and assume that

$$h(a|_A(b|_A b)) = h(a)|_B(h(b)|_B h(b)) \in h(F).$$

Then $a \in F$ and $a|_A(b|_A b) \in F$. Since F is a filter of A, it follows from (sF2) that $b \in F$, and hence $h(b) \in h(F)$. Therefore, $h(F)$ is a filter of B.

Now assume that F is an ultra filter of A and let $h(a) \vee_B h(b) \in h(F)$. Since

$$\begin{aligned} h(a \vee_A b) &= h((a|_A a)|_A(b|_A b)) \\ &= (h(a)|_B h(a))|_B(h(b)|_B h(b)) \\ &= h(a) \vee_B h(b), \end{aligned}$$

it follows that $a \vee_A b \in F$. Hence $a \in F$ or $b \in F$, and thus $h(a) \in h(F)$ or $h(b) \in h(F)$. Therefore, $h(F)$ is an ultra filter of B.

(2) Let G be a (ultra) filter of B and assume that h is bijective. Since $h(1_A) = 1_B \in G$, we obtain $1_A \in h^{-1}(G)$. Let $a \in h^{-1}(G)$ and assume that $a|_A(b|_A b) \in h^{-1}(G)$. Then $h(a) \in G$ and

$$h(a|_A(b|_A b)) = h(a)|_B(h(b)|_B h(b)) \in G.$$

Since G is a filter of B, we get $h(b) \in G$, and hence $b \in h^{-1}(G)$. Therefore, $h^{-1}(G)$ is a filter of A.

Now assume that G is an ultra filter of B and let $a \vee_A b \in h^{-1}(G)$. Then

$$\begin{aligned} h(a) \vee_B h(b) &= (h(a)|_B h(a))|_B(h(b)|_B h(b)) \\ &= h((a|_A a)|_A(b|_A b)) \\ &= h(a \vee_A b) \in G. \end{aligned}$$

Hence $h(a) \in G$ or $h(b) \in G$, which implies $a \in h^{-1}(G)$ or $b \in h^{-1}(G)$. Therefore, $h^{-1}(G)$ is an ultra filter of A.

□

Corollary 5.3. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be two s-Bitonic algebras, and let $h : A \to B$ be a homomorphism. Then $h(A)$ is a (ultra) filter of B. Moreover, $h(A)$ is a subalgebra of B.*

Definition 5.4. Let μ be an equivalence relation on an s-Bitonic algebra A. If $(a, b) \in \mu$ implies $(a|c, b|c) \in \mu$ for all $a, b, c \in A$, then μ is called a congruence relation on A.

Example 5.5. Consider the s-Bitonic algebra G given in Example 3.2. Then

$$\mu = \{(0, 0), (u, u), (v, v), (1, 1), (u, 1), (1, u), (v, 0), (0, v)\},$$

is a congruence relation on G.

Lemma 5.6. *An equivalence relation μ is a congruence relation on an s-Bitonic algebra A if and only if $(a, b) \in \mu$ and $(x, y) \in \mu$ imply $(a|x, b|y) \in \mu$, for all $a, b, x, y \in A$.*

Proof. Let μ be a congruence relation on A, and let $(a, b) \in \mu$ and $(x, y) \in \mu$ for any $a, b, x, y \in A$. Since $(a|x, b|x) \in \mu$ and $(b|x, b|y) \in \mu$ from (S1), and μ is transitive, it follows that $(a|x, b|y) \in \mu$ for all $a, b, x, y \in A$.

Conversely, assume that μ is an equivalence relation on A such that $(a, b) \in \mu$ and $(x, y) \in \mu$ imply $(a|x, b|y) \in \mu$ for all $a, b, x, y \in A$. Let $a, b, x \in A$ with $(a, b) \in \mu$. Since $(x, x) \in \mu$, we obtain $(a|x, b|x) \in \mu$. Hence, μ is a congruence relation on A. □

Theorem 5.7. *Let F be a filter of an s-Bitonic algebra A, and define a binary relation μ_F on A by*

$$(a,b) \in \mu_F \Longleftrightarrow a|(b|b) \in F \text{ and } b|(a|a) \in F, \tag{2}$$

for all $a, b \in A$. Then μ_F is a congruence relation on A.

Proof. Let F be a filter of A.

- *Reflexivity*: For any $a \in A$, we have $a|(a|a) = 1 \in F$ by Lemma 3.4(5) and (sF1). Hence $(a, a) \in \mu_F$.
- *Symmetry* The symmetry of μ_F follows directly from the definition.
- *Transitivity*: Let $(a, b) \in \mu_F$ and $(b, c) \in \mu_F$ for any $a, b, c \in A$. Then

$$a|(b|b),\ b|(a|a),\ b|(c|c),\ c|(b|b) \in F.$$

Since

$$a|(b|b) \leq (b|(c|c))|((a|(c|c))|(a|(c|c)))$$

by Lemma 3.6(6), it follows from (sF4) that

$$(b|(c|c))|((a|(c|c))|(a|(c|c))) \in F.$$

Hence, by (sF2), we obtain $a|(c|c) \in F$. Similarly, interchanging a and c yields $c|(a|a) \in F$. Thus $(a, c) \in \mu_F$, and so μ_F is an equivalence relation.

Now let $(a, b) \in \mu_F$ and $(x, y) \in \mu_F$. Then

$$a|(b|b),\ b|(a|a),\ x|(y|y),\ y|(x|x) \in F.$$

Using Lemma 3.6(7), (S1), and (S2), we obtain

$$a|(b|b) \leq (b|x)|((a|x)|(a|x)) \quad \text{and} \quad b|(a|a) \leq (a|x)|((b|x)|(b|x)).$$

Hence, by (sF4),

$$(b|x)|((a|x)|(a|x)) \in F \quad \text{and} \quad (a|x)|((b|x)|(b|x)) \in F,$$

which implies $(a|x, b|x) \in \mu_F$. Similarly, replacing x by b and b by y, we obtain $(b|x, b|y) \in \mu_F$. By transitivity of μ_F, we conclude $(a|x, b|y) \in \mu_F$.

Therefore, μ_F is a congruence relation on A. □

Theorem 5.8. *Let F be a filter of an s-Bitonic algebra A and let μ_F be the congruence relation on A defined as in 2. Then $(A/F, |_F, [1]_F)$ is an s-Bitonic algebra, where the quotient set*

$$A/F \equiv A/\mu_F = \{[a]_F : a \in A\},$$

the Sheffer operation $|_F$ *is defined by*

$$[a]_F|_F[b]_F = [a|b]_F,$$

and $[1]_F = F$.

Proof. Let F be a filter of an s-Bitonic algebra A and let μ_F be the congruence relation defined in (2).

We verify that the quotient structure satisfies axioms (S1)-(sb3).

(S1) follows directly from the definition of $|_F$ and (S1) in A:

$$[a]_F|_F[b]_F = [a|b]_F = [b|a]_F = [b]_F|_F[a]_F.$$

(S2):

$$([a]_F|_F[a]_F)|_F([a]_F|_F[b]_F) = [(a|a)|(a|b)]_F = [a]_F.$$

(S3):

$$\begin{aligned}[a]_F|_F(([b]_F|_F[c]_F)|_F([b]_F|_F[c]_F)) &= [a|((b|c)|(b|c))]_F \\ &= [((a|b)|(a|b))|c]_F \\ &= (([a]_F|_F[b]_F)|_F([a]_F|_F[b]_F))|_F[c]_F.\end{aligned}$$

(S4):

$$\begin{aligned}&([a]_F|_F(([a]_F|_F[a]_F)|_F([b]_F|_F[b]_F))) \\ &\qquad |_F([a]_F|_F(([a]_F|_F[a]_F)|_F([b]_F|_F[b]_F))) \\ &= [(a|((a|a)|(b|b)))|(a|((a|a)|(b|b)))]_F \\ &= [a]_F.\end{aligned}$$

(sb1):

$$[a]_F|_F([1]_F|_F[1]_F) = [a|(1|1)]_F = [1]_F.$$

(sb2): Assume that

$$[a]_F|_F([b]_F|_F[b]_F) = [1]_F = [b]_F|_F([a]_F|_F[a]_F).$$

Then

$$[a|(b|b)]_F = [1]_F = [b|(a|a)]_F.$$

Hence $a|(b|b)$, $b|(a|a) \in F$, and since μ_F is a congruence relation, we obtain $(a,b) \in \mu_F$, so $[a]_F = [b]_F$.

(sb3): Assume

$$[a]_F|_F([b]_F|_F[b]_F) = [1]_F.$$

Then $a|(b|b) \in F$. Using Lemma 3.4(4)-(5), Lemma 3.6(1), and Proposition 3.10, we obtain

$$a|(b|b) \leq (b|c)|((a|c)|(a|c)).$$

By (sF4), this implies

$$(b|c)|((a|c)|(a|c)) \in F,$$

hence

$$([b]_F|_F[c]_F)|_F(([a]_F|_F[c]_F)|_F([a]_F|_F[c]_F)) = [1]_F.$$

Finally, we show $[1]_F = F$. Indeed, since $[1]_F = [1|1]_F = [1]_F|_F[1]_F$, we have:

$$a \in [1]_F \iff (a,1) \in \mu_F \iff a \in F.$$

Thus $[1]_F = F$. □

Example 5.9. Consider the s-Bitonic algebra $(H, |, 1)$ given in Example 3.3. For the filter $F = \{e, 1\}$ of H, the equivalence relation

$$\mu_F = \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(1,1),$$
$$(e,1),(1,e),(0,b),(b,0),(a,d),(d,a),(c,f),(f,c)\}.$$

on H is a congruence relation on H. Then $(H/F, |_F, [1]_F)$ is an s-Bitonic algebra, where

$$H/F \equiv H/\mu_F = \{[0]_F, [a]_F, [f]_F, [1]_F\},$$

and the Sheffer operation $|_F$ on H/F is given by Table 6.

Table 6. Cayley table of the Sheffer stroke $|_F$ on H/F.

$\|_F$	$[0]_F$	$[a]_F$	$[f]_F$	$[1]_F$
$[0]_F$	$[1]_F$	$[1]_F$	$[1]_F$	$[1]_F$
$[a]_F$	$[1]_F$	$[f]_F$	$[1]_F$	$[f]_F$
$[f]_F$	$[1]_F$	$[1]_F$	$[a]_F$	$[a]_F$
$[1]_F$	$[1]_F$	$[f]_F$	$[a]_F$	$[0]_F$

Also, $[1]_F = \{e, 1\} = F$.

Lemma 5.10. *Let F be a filter of a s-Bitonic algebra A and let μ_F be the congruence relation on A defined as in (2). Then the binary relation $\leq$ on A/F defined by*

(3) $$[a]_F \leq [b]_F \iff a|(b|b) \in F,$$

is a partial order on A/F.

Proof. Let F be a filter of a s-Bitonic algebra A, and let μ_F be the congruence relation defined in (2). By Theorem 5.8, $(A/F, |_F, [1]_F)$ is a s-Bitonic algebra.

- *Reflexivity*: Since $a|(a|a) = 1 \in F$ (by Lemma 3.4(5) and (sF1)), we obtain $[a]_F \leq [a]_F$ for all $a \in A$.
- *Antisymmetry*: Let $[a]_F \leq [b]_F$ and $[b]_F \leq [a]_F$. Then $a|(b|b) \in F$ and $b|(a|a) \in F$, which implies $(a, b) \in \mu_F$. Hence, $[a]_F = [b]_F$.
- *Transitivity*: Let $[a]_F \leq [b]_F$ and $[b]_F \leq [c]_F$. Then $a|(b|b) \in F$ and $b|(c|c) \in F$. Since $a|(b|b) \leq (b|(c|c))|((a|(c|c))|(a|(c|c)))$ by Lemma 3.6(6), it follows from (sF2) and (sF4) that $a|(c|c) \in F$. Hence, $[a]_F \leq [c]_F$.

Therefore, $\leq$ is a partial order on A/F. □

Theorem 5.11. *Let F be a filter of a s-Bitonic algebra $(A, |, 1)$. Then F is an ultra filter of A if and only if $(A/F, |_F, [1]_F)$ is a totally ordered s-Bitonic algebra.*

Proof. Let F be an ultra filter of a s-Bitonic algebra $(A, |, 1)$. By Theorem 5.8, $(A/F, |_F, [1]_F)$ is a s-Bitonic algebra. Since $a|(b|b) \in F$ or $b|(a|a) \in F$ (by Lemma 4.12), we obtain $[a]_F \leq [b]_F$ or $[b]_F \leq [a]_F$ for all $a, b \in A$. Hence, $(A/F, |_F, [1]_F)$ is totally ordered.

Conversely, assume that $(A/F, |_F, [1]_F)$ is totally ordered. Then for all $a, b \in A$, either $[a]_F \leq [b]_F$ or $[b]_F \leq [a]_F$. This implies that $a|(b|b) \in F$ or $b|(a|a) \in F$. By Lemma 4.12, it follows that F is an ultra filter of A. □

Corollary 5.12. *Let F be a filter of a s-Bitonic algebra $(A, |, 1)$. Then $(A/F, |_F, [1]_F)$ is a totally ordered s-Bitonic algebra if and only if $a \in F$ or $a|a \in F$, for all $a \in A$.*

Corollary 5.13. *Let F be a filter of a s-Bitonic algebra $(A, |, 1)$. Then $(A/F, |_F, [1]_F)$ is a totally ordered s-Bitonic algebra if and only if $a \notin F$ and $b \notin F$ imply $a|(b|b) \in F$ and $b|(a|a) \in F$, for all $a, b \in A$.*

Theorem 5.14. *Let F be a filter of a s-Bitonic algebra A. Then a subset G of A containing F is a filter of A if and only if the set*

$$G/F = \{[a]_F \in A/F : a \in G\},$$

is a filter of A/F.

Proof. Let F and G be filters of A such that $F \subseteq G$. By Theorem 5.8, $(A/F, |_F, [1]_F)$ is a s-Bitonic algebra and $[1]_F = F$. Hence, $[1]_F \in G/F$.

Assume that $[a]_F \in G/F$ and $[a]_F|_F([b]_F|_F[b]_F) \in G/F$. Then $a \in G$ and $a|(b|b) \in G$. Since G is a filter of A, it follows from (sF2) that $b \in G$, and thus $[b]_F \in G/F$. Hence, G/F is a filter of A/F.

Conversely, suppose that G/F is a filter of A/F. Since $[1]_F \in G/F$, we have $1 \in G$, and hence $F \subseteq G$. Therefore, G is a filter of A. □

Theorem 5.15. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be s-Bitonic algebras, and let $h : A \to B$ be a homomorphism. Then*

$$Ker\,(h) = \{a \in A : h(a) = 1_B\},$$

is a filter of A.

Proof. Let $h : A \to B$ be a homomorphism. Since $h(1_A) = 1_B$, we have $1_A \in \mathrm{Ker}\,(h)$.

Let $a \in \mathrm{Ker}\,(h)$ and assume that $a|_A(b|_A b) \in \mathrm{Ker}\,(h)$. Then $h(a) = 1_B$, and $h(a|_A(b|_A b)) = 1_B$. Since h is a homomorphism, we obtain

$$h(a)|_B(h(b)|_B h(b)) = 1_B.$$

Using Lemma 3.4(1), it follows that $h(b) = 1_B$, hence $b \in \mathrm{Ker}\,(h)$.

Therefore, $\mathrm{Ker}\,(h)$ is a filter of A. □

Theorem 5.16. *Let $(A, |, 1)$ be a s-Bitonic algebra and let F be a filter of A. Then the mapping $h : A \to A/F$ defined by $h(a) = [a]_F$ is a homomorphism, called the natural homomorphism.*

Proof. Let F be a filter of A. By Theorem 5.8, the quotient $(A/F, |_F, [1]_F)$ is a s-Bitonic algebra.

Define $h : A \to A/F$ by $h(a) = [a]_F$. Then, for all $a, b \in A$, we have

$$h(a|b) = [a|b]_F = [a]_F|_F[b]_F = h(a)|_F h(b),$$

and $h(1) = [1]_F = F = [1]_F$.

Hence, h is a homomorphism from A onto A/F. □

Theorem 5.17. *Let $(A, |, 1)$ be a s-Bitonic algebra and let F be a filter of A. Then there exists a canonical surjective homomorphism $h : A \to A/F$ defined by $h(a) = [a]_F$, and $Ker\,(h) = F$.*

Proof. Let $(A, |, 1)$ be a s-Bitonic algebra and let F be a filter of A. By Theorem 5.8, $(A/F, |_F, [1]_F)$ is a s-Bitonic algebra. Moreover, by Theorem 5.16, the mapping $h : A \to A/F$ defined by $h(a) = [a]_F$ is a surjective homomorphism.

Now we determine the kernel of h. For any $a \in A$, we have

$$a \in \mathrm{Ker}\,(h) \iff h(a) = [1]_F \iff [a]_F = [1]_F \iff (a, 1) \in \mu_F.$$

By the definition of μ_F, this is equivalent to

$$a|(1|1) \in F \quad \text{and} \quad 1|(a|a) \in F.$$

Using (sb1) and Lemma 3.4(1), we obtain $a \in F$. Hence, $\mathrm{Ker}\,(h) = F$. □

Theorem 5.18. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be s-Bitonic algebras, and let $h : A \longrightarrow B$ be a homomorphism. Then there exists a unique homomorphism $g : A/Kerh \longrightarrow B$ such that $h = g \circ f$, where f is the natural homomorphism $f : A \longrightarrow A/Kerh$. Moreover, h is surjective and g is injective.*

Proof. Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be s-Bitonic algebras and $h : A \longrightarrow B$ a homomorphism. Since $Kerh$ is a filter of A by Theorem 5.15, the quotient algebra $(A/Kerh, |_{Kerh}, [1]_{Kerh})$ is a s-Bitonic algebra by Theorem 5.8.

Define

$$g : A/Kerh \longrightarrow B, \qquad g([a]_{Kerh}) = h(a) \ \text{ for all } a \in A.$$

To show that g is well-defined, assume that $[a]_{Kerh} = [b]_{Kerh}$. Then $(a, b) \in \mu_{Kerh}$, hence $a|_A(b|_Ab) \in Kerh$ and $b|_A(a|_Aa) \in Kerh$. Applying h, we obtain

$$h(a)|_B(h(b)|_Bh(b)) = 1_B, \qquad h(b)|_B(h(a)|_Bh(a)) = 1_B,$$

which implies $h(a) \leq_B h(b)$ and $h(b) \leq_B h(a)$. Thus $h(a) = h(b)$, and so $g([a]_{Kerh}) = g([b]_{Kerh})$.

Now, for $[a]_{Kerh}, [b]_{Kerh} \in A/Kerh$,

$$g([a]_{Kerh}|_{Kerh}[b]_{Kerh}) = g([a|_Ab]_{Kerh}) = h(a|_Ab) = h(a)|_Bh(b) = g([a]_{Kerh})|_Bg([b]_{Kerh}),$$

and $g([1_A]_{Kerh}) = h(1_A) = 1_B$. Hence g is a homomorphism.

To show injectivity, let $g([a]_{Kerh}) = g([b]_{Kerh})$. Then $h(a) = h(b)$, so

$$h(a|_A(b|_Ab)) = 1_B, \qquad h(b|_A(a|_Aa)) = 1_B,$$

hence $(a, b) \in \mu_{Kerh}$ and therefore $[a]_{Kerh} = [b]_{Kerh}$. Thus g is injective.

For any $a \in A$, we have $(g \circ f)(a) = g([a]_{Kerh}) = h(a)$, so $h = g \circ f$. The surjectivity of f is clear.

Finally, uniqueness: if $\delta : A/Kerh \to B$ satisfies $h = \delta \circ f$, then for any $[a]_{Kerh}$,

$$\delta([a]_{Kerh}) = \delta(f(a)) = h(a) = g([a]_{Kerh}),$$

so $\delta = g$. □

Corollary 5.19. *(First Isomorphism Theorem) Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be s-Bitonic algebras, and let $h : A \longrightarrow B$ be a homomorphism. Then $A/Kerh \cong h(A)$. In particular, if h is surjective, then $A/Kerh \cong B$.*

Theorem 5.20. *(Second Isomorphism Theorem) Let* $(B,|,1)$ *be a subalgebra of a s-Bitonic algebra* $(A,|,1)$*, and let* F *be a filter of* A*. Then*

$$BF/F \cong B/(B\cap F),$$

where $BF = \bigcup_{a\in B}[a]_F$ *and* $BF/F = \{[a]_F \in A/F : a \in BF\}$*.*

Proof. Let $(B,|,1)$ be a subalgebra of a s-Bitonic algebra $(A,|,1)$ and let F be a filter of A. Since $1 \in B$, we have $1 \in BF$. Hence $(BF,|,1)$ is closed under $|$ and satisfies (sb1)–(sb3), so it is a s-Bitonic subalgebra of A.

From Theorem 5.8, $(A/F,|_F,[1]_F)$ is a s-Bitonic algebra, and consequently BF/F is a subalgebra of A/F.

Define

$$h : B \longrightarrow BF/F, \qquad h(a) = [a]_F.$$

This mapping is well-defined and a homomorphism.

Now,

$$Kerh = \{a \in B : h(a) = [1]_F\} = \{a \in B : (a,1) \in \mu_F\} = \{a \in B : a \in F\} = B \cap F.$$

Next, h is surjective since for any $[u]_F \in BF/F$, there exists $a \in B$ such that $u \in [a]_F$, hence $h(a) = [a]_F = [u]_F$.

Therefore, by the First Isomorphism Theorem (Corollary 5.19),

$$BF/F \cong B/(B\cap F).$$

□

Theorem 5.21. *(Third Isomorphism Theorem) Let* F *and* G *be filters of a s-Bitonic algebra* $(A,|,1)$ *such that* $F \subseteq G$*. Then*

$$(A/F)/(G/F) \cong A/G.$$

Proof. Let F and G be filters of $(A,|,1)$ with $F \subseteq G$. By Theorem 5.8, $(A/F,|_F,[1]_F)$ and $(A/G,|_G,[1]_G)$ are s-Bitonic algebras. Also, by Theorem 5.14, G/F is a filter of A/F, hence $(A/F)/(G/F)$ is well-defined.

Define

$$\eta : (A/F)/(G/F) \longrightarrow A/G, \qquad \eta([[a]_F]_{G/F}) = [a]_G.$$

Well-definedness. If $[[a]_F]_{G/F} = [[b]_F]_{G/F}$, then $a|(b|b) \in G$ and $b|(a|a) \in G$, hence $(a,b) \in \mu_G$ and so $[a]_G = [b]_G$.

Homomorphism. For any $[[a]_F]_{G/F}, [[b]_F]_{G/F} \in (A/F)/(G/F)$,

$$\eta([[a]_F]_{G/F}|_{G/F}[[b]_F]_{G/F}) = \eta([[a|b]_F]_{G/F}) = [a|b]_G = [a]_G|_G[b]_G = \eta([[a]_F]_{G/F})|_G\eta([[b]_F]_{G/F}).$$

Also, $\eta([[1]_F]_{G/F}) = [1]_G$, so η is a homomorphism.

Surjectivity. For any $[a]_G \in A/G$, there exists $[[a]_F]_{G/F}$ such that $\eta([[a]_F]_{G/F}) = [a]_G$.

Kernel. If $\eta([[a]_F]_{G/F}) = [1]_G$, then $a \in G$, hence $[[a]_F]_{G/F} = [[1]_F]_{G/F}$. Thus $Ker\eta = \{[[1]_F]_{G/F}\}$, and η is injective.

Therefore, η is an isomorphism and

$$(A/F)/(G/F) \cong A/G.$$

□

6. Conclusion

In this study, Sheffer stroke bitonic algebras (briefly, s-Bitonic algebras) are introduced. We define a partial order, subalgebras, filters, homomorphisms, and quotient structures in this setting, and investigate relationships between these notions and related algebraic structures in detail. It is shown that every s-Bitonic algebra forms a bounded distributive lattice, and that the Cartesian product of s-Bitonic algebras is again a s-Bitonic algebra.

A unary operation $^\circ$ is defined on bitonic algebras with a least element 0. It is shown that every s-Bitonic algebra is a bitonic algebra, while the converse holds only under certain special conditions. Moreover, every s-Bitonic algebra is shown to be a SUP-algebra; however, the converse does not hold in general.

Furthermore, we introduce subalgebras and (ultra)filters on s-Bitonic algebras. It is proved that every filter of a s-Bitonic algebra is a subalgebra, although the converse is not generally true. Homomorphisms are also defined, and it is shown that filters and subalgebras are preserved under homomorphisms.

By introducing a congruence relation induced by filters, quotient s-Bitonic algebras are constructed, and a partial order is defined on the quotient structure. It is shown that a s-Bitonic algebra is totally ordered if and only if its filter is an ultrafilter. Finally, the three fundamental isomorphism theorems are established for s-Bitonic algebras.

In future work, we plan to study fuzzy extensions of s-Bitonic algebras.

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