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Research Paper

THE EFFECT OF THE SUM OF THE INVERSE-POWER OF ELEMENT ORDERS ON FINITE GROUPS

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Abstract. Let G be a finite group, and let $m(G) = \sum_{g\in G} 1/o(g)$, where $o(g)$ denotes the order of g. In this paper, we determine all groups such that $m(G) < 4$. Additionally, for a finite group G of odd order, we investigate the influence of $m(G)$ on supersolvability. Finally, we provide a criterion for p-solvability using the function $m(G)$, where $p \in \{7, 11\}$.

1. Introduction

Let G be a finite group, and let $m(G) = \sum_{g\in G} 1/o(g)$, where $o(g)$ denotes the order of $g \in G$. In [1], Asboei and Anabanti introduced the function

$$\mathcal{P}(G, x) := \sum_{g\in G} x^{o(g)}.$$

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Note that,

$$\int_0^1 \frac{1}{x}\mathcal{P}(G,x)\ dx = \sum_{g\in G}\frac{1}{o(g)} = m(G).$$

Let φ denotes Euler's totient function. In [9], Garonzi and Patassini introduced the function

$$R_G(r,s) = \sum_{g\in G}\frac{o(g)^s}{\varphi(o(g))^r},$$

for r, s real numbers. Note that, $R_G(0,-1) = \sum_{g\in G} 1/o(g) = m(G)$. They proved the following theorem.

Theorem 1.1. [9, Theorem 5] *If G is a non-cyclic group of order n, then $m(C_n) < m(G)$, where C_n denotes the cyclic group of order n.*

Recently, Archita proved the following theorem.

Theorem 1.2. [2, Theorem 1.2] *Let G be a non-cyclic group of order n, and C_n is the cyclic group of order n. Then*

$$m(G) \geqslant \frac{5}{4}m(C_n).$$

In [4], Baniasad Azad, Khosravi and Rashidi proved the following theorem:

Theorem 1.3. *Let G be a finite group. The following statements hold:*

1. *If $m(G) < m(A_5)$, then G is solvable* [4, Theorem 2.7].
2. *If $m(G) < m(A_4)$, then G is supersolvable* [4, Theorem 2.8].
3. *If $m(G) < m(S_3)$, then G is a cyclic group of order $n \in \{p, p^2, 2p, 8, 15, 16, 21, 27\}$, where p is prime or $G \cong C_2\times C_2$ or $G \cong Q_8$* [4, Theorem 2.9].

Recently, Kuang, Lu, Meng and Zhang, in [12], proved the following theorem.

Theorem 1.4. *Let G be a finite group. The following statements hold:*

1. *(*[12, Theorem 1.3]*) If $m(G) \leqslant |\pi(G)|$, then G is supersolvable.*
2. *(*[12, Theorem 1.4]*) If $m(A_5) \neq m(G) < m(SL(2,p))$, then G is solvable.*

Furthermore, many interesting results have been given, for example [3, 6, 11, 14].

In this paper, we determine all groups such that $m(G) < 4$. Also, for a finite group G of odd order we investigate the influence of $m(G)$ on supersolvability. Finally, we give a criterion for p-solvability by the function $m(G)$, where $p \in \{7, 11\}$. In fact, for a finite group of order n, we prove that the following:

- If $m(G) < 4$, then $G \cong C_2 \times C_2, S_3, Q_8, C_4 \times C_2, C_3 \times C_3$ or G is a cyclic group of order n such that
$$n \in \{p, p^2, p^3, pq, 4q, 16, 18, 32, 50, 81\},$$
where p and q are prime numbers.
- If $n = |G|$ is odd and $m(G) < 337/15$, then G is supersolvable.
- If G has no composition factor isomorphic to A_5 and $m(G) < m(\mathrm{PSL}(2,7))$, then G is a solvable group.
- If $m(G) < m(\mathrm{PSL}(2,7))$, then G is a 7-solvable group.
- If $m(G) < m(\mathrm{PSL}(2,11))$, then G is an 11-solvable group.

NOTATIONS

We use standard terminology and notation on groups that can be found in [10]. For a finite group G and $n \in \mathbb{N}$, we have the following notations.

- $o(g)$: order of element g.
- $meo(G)$: maximum order of an element of G.
- $\pi(G)$: set of all prime divisors of $|G|$.
- C_n : cyclic group of order n.
- A_n : alternating group of degree n.
- S_n : symmetric group of degree n
- $\mathrm{PSL}(n,q)$: projective special linear group (of degree n) over field of size q.

2. PRELIMINARIES

For the proof of our results, we need the following lemmas.

Lemma 2.1. [15, Lemma 1] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(N/M)|$.*

Lemma 2.2. [13] *Let A be a proper cyclic subgroup of a finite group G, and let $K = \mathrm{core}_G(A)$. Then $|A : K| < |G : A|$, and in particular, if $|A| > |G : A|$, then $K > 1$.*

Lemma 2.3. [4, Lemma 2.1] *If $m(G) < t$, then G has an element x such that $|G : \langle x\rangle| < t$.*

Lemma 2.4. [4, Lemma 2.2] *If $H \leq G$, then $m(H) \leq m(G)$, with equality if and only if $H = G$.*

Lemma 2.5. [4, Lemma 2.3] *If $N \trianglelefteq G$, then $m(G/N) \leq m(G)$, with equality if and only if $N = 1$.*

Lemma 2.6. [4, Lemma 2.6] *If G and H are finite groups, then $m(G)\psi(H) \leq m(G \times H)$. Furthermore, $m(G \times H) = m(G)m(H)$ if and only if* $\gcd(|G|, |H|) = 1$.

Lemma 2.7. [4, Lemma 2.5] *Let G be a finite group satisfying $G = P \rtimes F$, where P is a cyclic p-group for some prime p, $|F| > 1$ and $(p, |F|) = 1$. Then*

$$m(G) = m(P)m(C_F(P)) + |P|m(F \setminus C_F(P)).$$

3. MAIN RESULTS

In this section, we prove the main results.

Remark 3.1. By [4, Example 1.2], we have

$$m(C_{p^n}) = 1 + n - n/p; \qquad m(D_{2p}) = m(C_p) + p/2 = 2 - 1/p + p/2. \tag{1}$$

Remark 3.2. Using Lemma 2.4, we see that

$$m(C_p) < m(C_{p^2}) < m(C_{p^3}) < m(C_{p^4}) < \cdots < m(C_{p^n}). \tag{2}$$

Lemma 3.3. *Let p and q be prime numbers. Then $p < q$ if and only if $m(C_p) < m(C_q)$.*

Proof. We have

$$m(C_p) < m(C_q) \Leftrightarrow 2 - \frac{1}{p} < 2 - \frac{1}{q} \Leftrightarrow \frac{2p-1}{p} < \frac{2q-1}{q}$$
$$\Leftrightarrow 2pq - q < 2pq - p \Leftrightarrow -q < -p \Leftrightarrow p < q,$$

as required. □

Theorem 3.4. *Suppose that $n = |G|$ and also $m(G) < 4$. Then*

$$G \cong C_2 \times C_2, S_3, Q_8, C_4 \times C_2, C_3 \times C_3,$$

or G is a cyclic group of order n such that

$$n \in \{p, p^2, p^3, pq, 4q, 16, 18, 32, 50, 81\},$$

where p and q are prime numbers.

Proof. Let $|\pi(G)| \geqslant 3$ and $\{p, q, r\} \subseteq \pi(G)$. Then $|G| = n = tp^{\alpha_1}q^{\alpha_2}r^{\alpha_3}$, where $(t, pqr) = 1$ and $\alpha_i \in \mathbb{N}$. Thus

$$9/2 \leqslant \frac{2p-1}{p} \cdot \frac{2q-1}{q} \cdot \frac{2r-1}{r} = m(C_p)m(C_q)m(C_r). \tag{3}$$

By Lemma 2.4, we have $m(C_p) \leqslant m(C_{p^{\alpha_1}})$, $m(C_q) \leqslant m(C_{q^{\alpha_2}})$ and $m(C_r) \leqslant m(C_{r^{\alpha_3}})$. Using Lemma 2.6 and Theorem 1.1, we have

$$m(C_p)m(C_q)m(C_r) \leqslant m(C_t)m(C_{p^{\alpha_1}})m(C_{q^{\alpha_2}})m(C_{r^{\alpha_3}}) = m(C_n) \leqslant m(G) < 4. \tag{4}$$

Using (3) and (4), we have $9/2 < 4$, which is a contradiction.

Now we consider the following cases:

Case (I). Let $|\pi(G)| = 1$. Then $|G| = p^k$. Thus

$$k + 1 - \frac{k}{p} = \frac{pk + p - k}{p} = m(C_{p^k}) \leqslant m(G) < 4.$$

If $k \geqslant 6$, then $7 - 6/p < 4$, which is impossible. Therefore $k \leqslant 5$. So we consider the following cases:

- Let $p \geqslant 5$.
  If $k \in \{4, 5\}$, then $5-4/p < k+1-k/p = m(C_{p^k}) \leqslant m(G) < 4$, which is a contradiction. Therefore $k \in \{1, 2, 3\}$.
  If $|G| = p$, then $G \cong C_p$.
  If $|G| = p^2$, then $G \cong C_{p^2}$ or $C_p \times C_p$. Since $m(C_p \times C_p) = 1 + p - 1/p \geqslant 4$, we get that $G \cong C_{p^2}$.
  If $|G| = p^3$ and G is not cyclic, then $o(g) \leqslant p^2$, for any $g \in G$. Therefore $1/o(g) \geqslant 1/p^2$, for any $g \in G$. Thus $m(G) \geqslant p^3/p^2 = p$, which is a contradiction. Therefore $G \cong C_{p^3}$.
- Let $p = 3$.
  If $|G| = 3^5$, then $13/3 = m(C_{243}) \leqslant m(G) < 4$, which is a contradiction.
  If $|G| = 3^4 = 81$ and G is not cyclic, then G has a subgroup of order 3. Therefore $m(C_3) + 78/27 = 5/3 + 78/27 = 123/27 \leqslant m(G) < 4$, which is a contradiction. Thus $G \cong C_{81}$.
  If $|G| \in \{3, 3^2, 3^3\}$, then easily we can see that $G \in \{C_3, C_9, C_3 \times C_3, C_{27}\}$.
- Let $p = 2$.
  Let $|G| = 2^5$. If $meo(G) = 32$, then $G \cong C_{32}$ and so we get the result. If $meo(G) = 16$, then $m(C_{16}) + 16/16 = 4 \leqslant m(G) < 4$, which is a contradiction. If $meo(G) \leqslant 8$, then $32/8 = 4 \leqslant m(G) < 4$, which is a contradiction. If $|G| \in \{2, 4, 8, 16\}$, then

$$G \in \{C_2, C_4, C_2 \times C_2, C_8, C_4 \times C_2, Q_8, C_{16}\}.$$

Case (II). Let $|\pi(G)| = 2$. Then $|G| = p^m q^n$, where $p < q$.

If $m, n \geqslant 2$, then

$$14/3 \leqslant (3 - 2/p)(3 - 2/q) \leqslant m(C_{p^2})m(C_{q^2}) \leqslant m(C_n) \leqslant m(G) < 4,$$

which is a contradiction. If $m \geqslant 3$ and $n = 1$, then

$$25/5 \leqslant (4 - 3/p)(2 - 1/q) \leqslant m(C_{p^3})m(C_q) \leqslant m(C_n) \leqslant m(G) < 4,$$

which is a contradiction. If $m = 1$ and $n \geqslant 3$, then

$$9/2 \leqslant (2 - 1/p)(4 - 3/q) \leqslant m(C_p)m(C_{q^3}) \leqslant m(C_n) \leqslant m(G) < 4.$$

So we consider the following cases:

- Let $|G| = pq$, where $p < q$.

  If G is a non-cyclic group, then $G \cong C_q \rtimes C_p$. By Lemma 2.7, we get that

  $$m(G) = m(C_q \rtimes C_p) = (2 - 1/q) + q(1 - 1/p) < 4,$$

  so $p = 2$ and $q = 3$, Thus $G \cong S_3$.

  If G is a cyclic group, then $G \cong C_{pq}$, and we get the result.
- Let $|G| = p^2q$, where $p < q$. If $p \geqslant 3$, then

  $$21/5 \leqslant m(C_{p^2})m(C_q) = m(C_n) \leqslant m(G) < 4,$$

  which is a contradiction.

  If $p = 2$, then $|G| = 4q$. Assume that G is not cyclic. By Theorem 1.2, we have

  $$\frac{50}{12} \leqslant \frac{5}{4} \cdot 2 \cdot (2 - \frac{1}{q}) = \frac{5}{4}m(C_4)m(C_q) = \frac{5}{4}m(C_n) \leqslant m(G) < 4,$$

  which is a contradiction. Therefore $G \cong C_{4q}$.
- Let $|G| = pq^2$, where $p < q$. Assume that G is not cyclic. By Theorem 1.2,

  $$\frac{35}{8} \leqslant \frac{5}{4} \cdot (2 - \frac{1}{p}) \cdot (3 - \frac{2}{q}) = \frac{5}{4}m(C_p)m(C_{q^2}) = \frac{5}{4}m(C_n) \leqslant m(G) < 4,$$

  which is a contradiction. Therefore $G \cong C_{pq^2}$.

  If $p = 2$, then

  $$m(G) < 4 \Leftrightarrow m(C_{2q^2}) < 4 \Leftrightarrow \frac{3}{2}(3 - \frac{2}{q}) < 4 \Leftrightarrow \frac{9q - 6}{2q} < 4$$
  $$\Leftrightarrow 9q - 6 < 8q \Leftrightarrow q < 6.$$

  Thus $G \cong C_{18}$ or $G \cong C_{50}$.

  If $p \geqslant 3$, then

  $$13/3 \leqslant (2 - 1/3)(3 - 2/5) \leqslant m(C_p)m(C_{q^2}) \leqslant m(C_n) \leqslant m(G) < 4,$$

  which is a contradiction.

Thus, the proof is now completed. □

Corollary 3.5. *The only non-nilpotent group G such that $m(G) < 4$ is $G \cong S_3$.*

Proof. Using Theorem 3.4,

$$G \cong C_2 \times C_2, S_3, Q_8, C_4 \times C_2, C_3 \times C_3,$$

or G is a cyclic group of order n such that

$$n \in \{p, p^2, p^3, pq, 4q, 16, 18, 32, 50, 81\},$$

where p and q are prime numbers. We know p-groups and cyclic groups are nilpotent. Therefore $G \cong S_3$. □

Theorem 3.6. *Let G be a finite group such that $|G|$ is odd. If $m(G) < 337/15$, then G is supersolvable.*

Proof. We prove the result by induction on $|G|$. If $|G| = 1$, then the result is trivial. Assume $|G| \geqslant 1$ and that G contains a normal cyclic subgroup $N \neq 1$. Using Lemma 2.5, we have $m(G/N) < m(G)$. Therefore, $m(G/N) < 337/15$, and by the inductive hypothesis, G/N is supersolvable. Consequently, G is supersolvable. Now suppose has G has no non-trivial normal cyclic subgroup. By Lemma 2.3, there exists $x \in G$ such that $|G : \langle x \rangle| < 337/15 < 23$. Moreover,

$$|G : \mathrm{core}_G(\langle x \rangle)| = |G : \langle x \rangle| \cdot |\langle x \rangle : \mathrm{core}_G(\langle x \rangle)| \leqslant 22 \cdot 21 = 462.$$

Since $\mathrm{core}_G(\langle x \rangle) = 1$, it follows that $|G| \leqslant 462$. Let G be a non-supersolvable group of odd order. By GAP [8], we obtain the following Table.

Table 1. The value of $m(G)$ for some non-supersolvable groups.

G	$\lvert G\rvert$	$m(G)$
$(C_5 \times C_5) : C_3 = \mathrm{SmallGroup}(75, 2)$	75	$337/15 \simeq 22.46$
$(C_5 \times C_5) : C_9 = \mathrm{SmallGroup}(225, 3)$	225	$79/3 \simeq 26.33$
$C_3 \times ((C_5 \times C_5) : C_3) = \mathrm{SmallGroup}(225, 5)$	225	$179/3 \simeq 59.66$
$(C_3 \times C_3 \times C_3) : C_{13} = \mathrm{SmallGroup}(351, 12)$	351	$1349/39 \simeq 34.58$
$(C_{11} \times C_{11}) : C_3 = \mathrm{SmallGroup}(363, 2)$	363	$3055/33 \simeq 92.57$
$((C_5 \times C_5) : C_5) : C_3 = \mathrm{SmallGroup}(375, 2)$	375	$279/5 = 55.80$
$C_5 \times ((C_5 \times C_5) : C_3) = \mathrm{SmallGroup}(375, 6)$	375	$279/5 = 55.80$
$(C_3 \times C_3 \times C_3 \times C_3) : C_5 = \mathrm{SmallGroup}(405, 15)$	405	$1387/15 \simeq 92.46$

which is a contradiction. □

Lemma 3.7. *If G is non-solvable, then G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is a direct product of isomorphic non-abelian simple groups and*

$$m(N/M) \leqslant m(N) \leqslant m(G). \tag{5}$$

Proof. By Lemma 2.1, Lemma 2.4 and Lemma 2.5, we get the result. □

Theorem 3.8. ***(a)*** *Let G has no composition factor isomorphic to A_5. If $m(G) < m(\mathrm{PSL}(2,7))$, then G is solvable.*
(b) *Let G be a finite group such that $m(G) < m(\mathrm{PSL}(2,7))$, then G is 7-solvable.*

Proof. **(a)** We establish solvability of G through induction on its order $|G|$. If $|G| \leq 59$, then G is a solvable group.

Suppose G possesses a non-trivial normal solvable subgroup K. By Lemma 2.5,

$$m(G/K) \leq m(G) < m(\mathrm{PSL}(2,7)).$$

Hence, the inductive hypothesis ensures G/K is solvable. Consequently, G is a solvable group. Therefore suppose that G has no non-trivial normal solvable subgroup. Since $m(G) < m(\mathrm{PSL}(2,7)) = 998/21 \simeq 47.524$, using Lemma 2.3 we conclude that there exists an element $x \in G$ such that

$$|G : \langle x\rangle| \leq 47. \tag{6}$$

Using Lemma 2.2, $|\langle x\rangle : \mathrm{core}_G(\langle x\rangle)| \leqslant 46$. Therefore

$$|G : \mathrm{core}_G(\langle x\rangle)| = |G : \langle x\rangle| \cdot |\langle x\rangle : \mathrm{core}_G(\langle x\rangle)| \leqslant 47 \cdot 46 = 2162.$$

Since $\mathrm{core}_G(\langle x\rangle) = 1$, therefore $|G| \leqslant 2162$. Let G be a non-solvable group. By Lemma 2.1, G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is a direct product of isomorphic non-abelian simple groups and $|G/N| \mid |\mathrm{Out}(N/M)|$.
If M is non-solvable, then $|N| = |N/M| \cdot |M|$ divides G. Therefore $3600 \leq |G|$, which is a contradiction. Thus M is solvable and so $M = 1$. Therefore N is a simple group such that $|N| \leqslant 2162$. Since G has no composition factor isomorphic to A_5, using [7], we have

$$N \in \{\mathrm{PSL}(2,7), A_6, \mathrm{PSL}(2,8), \mathrm{PSL}(2,11), \mathrm{PSL}(2,13)\}.$$

By GAP [8], we have the following table.

Table 2. The value of $m(N)$ for some simple groups.

N	$PSL(2,7)$	A_6	$PSL(2,8)$	$\mathrm{PSL}(2,11)$	$\mathrm{PSL}(2,13)$
$m(N)$	$\frac{998}{21}$	$\frac{1522}{15} \simeq 101.4$	$\frac{4229}{42} \simeq 100.6$	$\frac{16193}{110} \simeq 147.2$	$\frac{39545}{182} \simeq 217.2$

We see that $m(G) \geqslant m(\mathrm{PSL}(2,7))$, which is a contradiction.

(b) Similarly to the above we get that $|G| \leqslant 2162$. Now suppose that G is not a 7-solvable group. Therefore G is nonsolvable. Thus by Lemma 2.1, G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is a direct product of isomorphic non-abelian simple groups and $|G/N| \bigm| |\mathrm{Out}(N/M)|$. As we mentioned above M is a solvable group.

Let $N/M \cong A_5$. If M is 7-solvable, then G is 7-solvable. If M is not a 7-solvable group, then $7 \mid |M|$. If $|M| < 168$, then M is a solvable group and so M is a 7-solvable group. Therefore $|M| \geq 168$ and we have $|G| \geq 60 \cdot 168$, which is a contradiction.

The proof is now completed. $\square$

Theorem 3.9. *Let G be a finite group such that $m(G) < m(\mathrm{PSL}(2,11))$. Then G is an 11-solvable group.*

Proof. We prove that G is solvable by induction on $|G|$. If $|G| < 660$ or $11 \nmid |G|$, then G is an 11-solvable group. If G has a non-trivial normal 11-solvable subgroup K, then by Lemma 2.5,

$$m(G/K) \leq m(G) < m(\mathrm{PSL}(2,11)),$$

and so by the inductive hypothesis, G/K is an 11-solvable group, and consequently, G is 11-solvable. Therefore suppose that G has no non-trivial normal 11-solvable subgroup.

Since $m(G) < m(\mathrm{PSL}(2,11)) = 16193/110 \simeq 147.2$, using Lemma 2.3 we get that there exists an element $x \in G$ such that

$$|G : \langle x \rangle| \leq 147. \tag{7}$$

Using Lemma 2.2, $|\langle x \rangle : \mathrm{core}_G(\langle x \rangle)| \leqslant 146$. Therefore

$$|G : \mathrm{core}_G(\langle x \rangle)| = |G : \langle x \rangle| \cdot |\langle x \rangle : \mathrm{core}_G(\langle x \rangle)| \leqslant 147 \cdot 146 = 21462.$$

Since G has no non-trivial normal 11-solvable subgroup, $\mathrm{core}_G(\langle x \rangle) = 1$, so

$$|G : \mathrm{core}_G(\langle x \rangle)| = |G| \leqslant 21462.$$

If G is not an 11-solvable group, then G is not a solvable group. By Lemma 2.1, G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is isomorphic to a direct product of a non-abelian simple group S and $|G/N| \bigm| |\mathrm{Out}(N/M)|$. If M is not an 11-solvable, then $|M| \geqslant 660$ and also we know $|N/M| \geqslant 60$. Therefore $|N| = |N/M| \cdot |M| \geqslant 60 \cdot 660 = 39600$. Since $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$, $|N| \leq |G|$. Therefore $39600 \leq |G| \leq 21462$, which is a contradiction. Thus M is 11-solvable. Since G has no non-trivial normal 11-solvable subgroup, $M = 1$.

By [7], we have

$$\begin{aligned} S \in & \{\mathrm{PSL}(2,q) | q = 5,7,8,11,13,16,17,19,23,25,27,29,31\} \\ & \cup \{A_6, A_7, A_8, \mathrm{PSL}(3,3), \mathrm{PSL}(3,4), \mathrm{PSU}(3,3), M_{11}\}. \end{aligned}$$

Now, we consider two following cases:

(1) Let $11 \nmid |S|$. Since G has no non-trivial normal 11-solvable subgroup, $M = 1$. We have normal series $1 \trianglelefteq N \trianglelefteq G$ such that N is isomorphic to a direct product of a non-abelian simple group S and $|G/N| \mid |\mathrm{Out}(N)|$. We know $|G| = |G/N| \cdot |N|$ and also $|G/N| \mid |\mathrm{Out(N)}|$. Then by [7, page 239], $11 \nmid |\mathrm{Out}(N)|$ and also since $11 \nmid |S|$, we have $11 \nmid |N|$. Therefore $11 \nmid |G|$ and so G is 11-solvable.

(2) Let $11 \mid |S|$. Then by [7], we have

$$S \in \{\mathrm{PSL}(2,11), \mathrm{PSL}(2,23), M_{11}\}.$$

Table 3. The value of $m(G)$ for some simple groups.

G	$PSL(2,11)$	$\mathrm{PSL}(2,23)$	M_{11}
$m(G)$	$\frac{16193}{110}$	$\frac{656686}{759} \simeq 865.1$	$\frac{459649}{330} \simeq 1392.8$

We see that $m(G) \geqslant m(\mathrm{PSL}(2,11))$, which is a contradiction.

The proof is now completed. □

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