

Research Paper

THE EFFECT OF THE SUM OF THE INVERSE-POWER OF ELEMENT ORDERS ON FINITE GROUPS

SARA POUYANDEH AND LALEH OFTADEH*

ABSTRACT. Let G be a finite group, and let $m(G) = \sum_{g \in G} 1/o(g)$, where $o(g)$ denotes the order of g . In this paper, we determine all groups such that $m(G) < 4$. Additionally, for a finite group G of odd order, we investigate the influence of $m(G)$ on supersolvability. Finally, we provide a criterion for p -solvability using the function $m(G)$, where $p \in \{7, 11\}$.

1. INTRODUCTION

Let G be a finite group, and let $m(G) = \sum_{g \in G} 1/o(g)$, where $o(g)$ denotes the order of $g \in G$. In [1], Asboei and Anabanti introduced the function

$$\mathcal{P}(G, x) := \sum_{g \in G} x^{o(g)}.$$

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*Corresponding author

Note that,

$$\int_0^1 \frac{1}{x} \mathcal{P}(G, x) \, dx = \sum_{g \in G} \frac{1}{o(g)} = m(G).$$

Let φ denotes Euler's totient function. In [9], Garonzi and Patassini introduced the function

$$R_G(r, s) = \sum_{g \in G} \frac{o(g)^s}{\varphi(o(g))^r},$$

for r, s real numbers. Note that, $R_G(0, -1) = \sum_{g \in G} 1/o(g) = m(G)$. They proved the following theorem.

Theorem 1.1. [9, Theorem 5] *If G is a non-cyclic group of order n , then $m(C_n) < m(G)$, where C_n denotes the cyclic group of order n .*

Recently, Archita proved the following theorem.

Theorem 1.2. [2, Theorem 1.2] *Let G be a non-cyclic group of order n , and C_n is the cyclic group of order n . Then*

$$m(G) \geq \frac{5}{4}m(C_n).$$

In [4], Baniasad Azad, Khosravi and Rashidi proved the following theorem:

Theorem 1.3. *Let G be a finite group. The following statements hold:*

- (1) *If $m(G) < m(A_5)$, then G is solvable [4, Theorem 2.7].*
- (2) *If $m(G) < m(A_4)$, then G is supersolvable [4, Theorem 2.8].*
- (3) *If $m(G) < m(S_3)$, then G is a cyclic group of order $n \in \{p, p^2, 2p, 8, 15, 16, 21, 27\}$, where p is prime or $G \cong C_2 \times C_2$ or $G \cong Q_8$ [4, Theorem 2.9].*

Recently, Kuang, Lu, Meng and Zhang, in [12], proved the following theorem.

Theorem 1.4. *Let G be a finite group. The following statements hold:*

- (1) ([12, Theorem 1.3]) *If $m(G) \leq |\pi(G)|$, then G is supersolvable.*
- (2) ([12, Theorem 1.4]) *If $m(A_5) \neq m(G) < m(SL(2, p))$, then G is solvable.*

Furthermore, many interesting results have been given, for example [3, 6, 11, 14].

In this paper, we determine all groups such that $m(G) < 4$. Also, for a finite group G of odd order we investigate the influence of $m(G)$ on supersolvability. Finally, we give a criterion for p -solvability by the function $m(G)$, where $p \in \{7, 11\}$. In fact, for a finite group of order n , we prove that the following:

- If $m(G) < 4$, then $G \cong C_2 \times C_2, S_3, Q_8, C_4 \times C_2, C_3 \times C_3$ or G is a cyclic group of order n such that

$$n \in \{p, p^2, p^3, pq, 4q, 16, 18, 32, 50, 81\},$$

where p and q are prime numbers.

- If $n = |G|$ is odd and $m(G) < 337/15$, then G is supersolvable.
- If G has no composition factor isomorphic to A_5 and $m(G) < m(\text{PSL}(2, 7))$, then G is a solvable group.
- If $m(G) < m(\text{PSL}(2, 7))$, then G is a 7-solvable group.
- If $m(G) < m(\text{PSL}(2, 11))$, then G is an 11-solvable group.

NOTATIONS

We use standard terminology and notation on groups that can be found in [10]. For a finite group G and $n \in \mathbb{N}$, we have the following notations.

- $o(g)$: order of element g .
- $meo(G)$: maximum order of an element of G .
- $\pi(G)$: set of all prime divisors of $|G|$.
- C_n : cyclic group of order n .
- A_n : alternating group of degree n .
- S_n : symmetric group of degree n .
- $\text{PSL}(n, q)$: projective special linear group (of degree n) over field of size q .

2. PRELIMINARIES

For the proof of our results, we need the following lemmas.

Lemma 2.1. [15, Lemma 1] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\text{Out}(N/M)|$.*

Lemma 2.2. [13] *Let A be a proper cyclic subgroup of a finite group G , and let $K = \text{core}_G(A)$. Then $|A : K| < |G : A|$, and in particular, if $|A| > |G : A|$, then $K > 1$.*

Lemma 2.3. [4, Lemma 2.1] *If $m(G) < t$, then G has an element x such that $|G : \langle x \rangle| < t$.*

Lemma 2.4. [4, Lemma 2.2] *If $H \leq G$, then $m(H) \leq m(G)$, with equality if and only if $H = G$.*

Lemma 2.5. [4, Lemma 2.3] *If $N \trianglelefteq G$, then $m(G/N) \leq m(G)$, with equality if and only if $N = 1$.*

Lemma 2.6. [4, Lemma 2.6] *If G and H are finite groups, then $m(G)\psi(H) \leq m(G \times H)$. Furthermore, $m(G \times H) = m(G)m(H)$ if and only if $\gcd(|G|, |H|) = 1$.*

Lemma 2.7. [4, Lemma 2.5] *Let G be a finite group satisfying $G = P \rtimes F$, where P is a cyclic p -group for some prime p , $|F| > 1$ and $(p, |F|) = 1$. Then*

$$m(G) = m(P)m(C_F(P)) + |P|m(F \setminus C_F(P)).$$

3. MAIN RESULTS

In this section, we prove the main results.

Remark 3.1. By [4, Example 1.2], we have

$$(1) \quad m(C_{p^n}) = 1 + n - n/p; \quad m(D_{2p}) = m(C_p) + p/2 = 2 - 1/p + p/2.$$

Remark 3.2. Using Lemma 2.4, we see that

$$(2) \quad m(C_p) < m(C_{p^2}) < m(C_{p^3}) < m(C_{p^4}) < \cdots < m(C_{p^n}).$$

Lemma 3.3. *Let p and q be prime numbers. Then $p < q$ if and only if $m(C_p) < m(C_q)$.*

Proof. We have

$$\begin{aligned} m(C_p) < m(C_q) &\Leftrightarrow 2 - \frac{1}{p} < 2 - \frac{1}{q} \Leftrightarrow \frac{2p-1}{p} < \frac{2q-1}{q} \\ &\Leftrightarrow 2pq - q < 2pq - p \Leftrightarrow -q < -p \Leftrightarrow p < q, \end{aligned}$$

as required. $\square$

Theorem 3.4. *Suppose that $n = |G|$ and also $m(G) < 4$. Then*

$$G \cong C_2 \times C_2, S_3, Q_8, C_4 \times C_2, C_3 \times C_3,$$

or G is a cyclic group of order n such that

$$n \in \{p, p^2, p^3, pq, 4q, 16, 18, 32, 50, 81\},$$

where p and q are prime numbers.

Proof. Let $|\pi(G)| \geq 3$ and $\{p, q, r\} \subseteq \pi(G)$. Then $|G| = n = tp^{\alpha_1}q^{\alpha_2}r^{\alpha_3}$, where $(t, pqr) = 1$ and $\alpha_i \in \mathbb{N}$. Thus

$$(3) \quad 9/2 \leq \frac{2p-1}{p} \cdot \frac{2q-1}{q} \cdot \frac{2r-1}{r} = m(C_p)m(C_q)m(C_r).$$

By Lemma 2.4, we have $m(C_p) \leq m(C_{p^{\alpha_1}})$, $m(C_q) \leq m(C_{q^{\alpha_2}})$ and $m(C_r) \leq m(C_{r^{\alpha_3}})$. Using Lemma 2.6 and Theorem 1.1, we have

$$(4) \quad m(C_p)m(C_q)m(C_r) \leq m(C_t)m(C_{p^{\alpha_1}})m(C_{q^{\alpha_2}})m(C_{r^{\alpha_3}}) = m(C_n) \leq m(G) < 4.$$

Using (3) and (4), we have $9/2 < 4$, which is a contradiction.

Now we consider the following cases:

Case (I). Let $|\pi(G)| = 1$. Then $|G| = p^k$. Thus

$$k + 1 - \frac{k}{p} = \frac{pk + p - k}{p} = m(C_{p^k}) \leq m(G) < 4.$$

If $k \geq 6$, then $7 - 6/p < 4$, which is impossible. Therefore $k \leq 5$. So we consider the following cases:

- Let $p \geq 5$.
 If $k \in \{4, 5\}$, then $5 - 4/p < k + 1 - k/p = m(C_{p^k}) \leq m(G) < 4$, which is a contradiction. Therefore $k \in \{1, 2, 3\}$.
 If $|G| = p$, then $G \cong C_p$.
 If $|G| = p^2$, then $G \cong C_{p^2}$ or $C_p \times C_p$. Since $m(C_p \times C_p) = 1 + p - 1/p \geq 4$, we get that $G \cong C_{p^2}$.
 If $|G| = p^3$ and G is not cyclic, then $o(g) \leq p^2$, for any $g \in G$. Therefore $1/o(g) \geq 1/p^2$, for any $g \in G$. Thus $m(G) \geq p^3/p^2 = p$, which is a contradiction. Therefore $G \cong C_{p^3}$.
- Let $p = 3$.
 If $|G| = 3^5$, then $13/3 = m(C_{243}) \leq m(G) < 4$, which is a contradiction.
 If $|G| = 3^4 = 81$ and G is not cyclic, then G has a subgroup of order 3. Therefore $m(C_3) + 78/27 = 5/3 + 78/27 = 123/27 \leq m(G) < 4$, which is a contradiction. Thus $G \cong C_{81}$.
 If $|G| \in \{3, 3^2, 3^3\}$, then easily we can see that $G \in \{C_3, C_9, C_3 \times C_3, C_{27}\}$.
- Let $p = 2$.
 Let $|G| = 2^5$. If $meo(G) = 32$, then $G \cong C_{32}$ and so we get the result. If $meo(G) = 16$, then $m(C_{16}) + 16/16 = 4 \leq m(G) < 4$, which is a contradiction. If $meo(G) \leq 8$, then $32/8 = 4 \leq m(G) < 4$, which is a contradiction. If $|G| \in \{2, 4, 8, 16\}$, then

$$G \in \{C_2, C_4, C_2 \times C_2, C_8, C_4 \times C_2, Q_8, C_{16}\}.$$

Case (II). Let $|\pi(G)| = 2$. Then $|G| = p^m q^n$, where $p < q$.

If $m, n \geq 2$, then

$$14/3 \leq (3 - 2/p)(3 - 2/q) \leq m(C_{p^2})m(C_{q^2}) \leq m(C_n) \leq m(G) < 4,$$

which is a contradiction. If $m \geq 3$ and $n = 1$, then

$$25/5 \leq (4 - 3/p)(2 - 1/q) \leq m(C_{p^3})m(C_q) \leq m(C_n) \leq m(G) < 4,$$

which is a contradiction. If $m = 1$ and $n \geq 3$, then

$$9/2 \leq (2 - 1/p)(4 - 3/q) \leq m(C_p)m(C_{q^3}) \leq m(C_n) \leq m(G) < 4.$$

So we consider the following cases:

- Let $|G| = pq$, where $p < q$.

If G is a non-cyclic group, then $G \cong C_q \rtimes C_p$. By Lemma 2.7, we get that

$$m(G) = m(C_q \rtimes C_p) = (2 - 1/q) + q(1 - 1/p) < 4,$$

so $p = 2$ and $q = 3$. Thus $G \cong S_3$.

If G is a cyclic group, then $G \cong C_{pq}$, and we get the result.

- Let $|G| = p^2q$, where $p < q$. If $p \geq 3$, then

$$21/5 \leq m(C_{p^2})m(C_q) = m(C_n) \leq m(G) < 4,$$

which is a contradiction.

If $p = 2$, then $|G| = 4q$. Assume that G is not cyclic. By Theorem 1.2, we have

$$\frac{50}{12} \leq \frac{5}{4} \cdot 2 \cdot (2 - \frac{1}{q}) = \frac{5}{4}m(C_4)m(C_q) = \frac{5}{4}m(C_n) \leq m(G) < 4,$$

which is a contradiction. Therefore $G \cong C_{4q}$.

- Let $|G| = pq^2$, where $p < q$. Assume that G is not cyclic. By Theorem 1.2,

$$\frac{35}{8} \leq \frac{5}{4} \cdot (2 - \frac{1}{p}) \cdot (3 - \frac{2}{q}) = \frac{5}{4}m(C_p)m(C_{q^2}) = \frac{5}{4}m(C_n) \leq m(G) < 4,$$

which is a contradiction. Therefore $G \cong C_{pq^2}$.

If $p = 2$, then

$$\begin{aligned} m(G) < 4 &\Leftrightarrow m(C_{2q^2}) < 4 \Leftrightarrow \frac{3}{2}(3 - \frac{2}{q}) < 4 \Leftrightarrow \frac{9q - 6}{2q} < 4 \\ &\Leftrightarrow 9q - 6 < 8q \Leftrightarrow q < 6. \end{aligned}$$

Thus $G \cong C_{18}$ or $G \cong C_{50}$.

If $p \geq 3$, then

$$13/3 \leq (2 - 1/3)(3 - 2/5) \leq m(C_p)m(C_{q^2}) \leq m(C_n) \leq m(G) < 4,$$

which is a contradiction.

Thus, the proof is now completed. $\square$

Corollary 3.5. *The only non-nilpotent group G such that $m(G) < 4$ is $G \cong S_3$.*

Proof. Using Theorem 3.4,

$$G \cong C_2 \times C_2, S_3, Q_8, C_4 \times C_2, C_3 \times C_3,$$

or G is a cyclic group of order n such that

$$n \in \{p, p^2, p^3, pq, 4q, 16, 18, 32, 50, 81\},$$

where p and q are prime numbers. We know p -groups and cyclic groups are nilpotent. Therefore $G \cong S_3$. $\square$

Theorem 3.6. *Let G be a finite group such that $|G|$ is odd. If $m(G) < 337/15$, then G is supersolvable.*

Proof. We prove the result by induction on $|G|$. If $|G| = 1$, then the result is trivial. Assume $|G| \geq 1$ and that G contains a normal cyclic subgroup $N \neq 1$. Using Lemma 2.5, we have $m(G/N) < m(G)$. Therefore, $m(G/N) < 337/15$, and by the inductive hypothesis, G/N is supersolvable. Consequently, G is supersolvable. Now suppose G has no non-trivial normal cyclic subgroup. By Lemma 2.3, there exists $x \in G$ such that $|G : \langle x \rangle| < 337/15 < 23$. Moreover,

$$|G : \text{core}_G(\langle x \rangle)| = |G : \langle x \rangle| \cdot |\langle x \rangle : \text{core}_G(\langle x \rangle)| \leq 22 \cdot 21 = 462.$$

Since $\text{core}_G(\langle x \rangle) = 1$, it follows that $|G| \leq 462$. Let G be a non-supersolvable group of odd order. By GAP [8], we obtain the following Table.

TABLE 1. The value of $m(G)$ for some non-supersolvable groups.

G	$ G $	$m(G)$
$(C_5 \times C_5) : C_3 = \text{SmallGroup}(75, 2)$	75	$337/15 \simeq 22.46$
$(C_5 \times C_5) : C_9 = \text{SmallGroup}(225, 3)$	225	$79/3 \simeq 26.33$
$C_3 \times ((C_5 \times C_5) : C_3) = \text{SmallGroup}(225, 5)$	225	$179/3 \simeq 59.66$
$(C_3 \times C_3 \times C_3) : C_{13} = \text{SmallGroup}(351, 12)$	351	$1349/39 \simeq 34.58$
$(C_{11} \times C_{11}) : C_3 = \text{SmallGroup}(363, 2)$	363	$3055/33 \simeq 92.57$
$((C_5 \times C_5) : C_5) : C_3 = \text{SmallGroup}(375, 2)$	375	$279/5 = 55.80$
$C_5 \times ((C_5 \times C_5) : C_3) = \text{SmallGroup}(375, 6)$	375	$279/5 = 55.80$
$(C_3 \times C_3 \times C_3 \times C_3) : C_5 = \text{SmallGroup}(405, 15)$	405	$1387/15 \simeq 92.46$

which is a contradiction. $\square$

Lemma 3.7. *If G is non-solvable, then G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is a direct product of isomorphic non-abelian simple groups and*

$$(5) \quad m(N/M) \leq m(N) \leq m(G).$$

Proof. By Lemma 2.1, Lemma 2.4 and Lemma 2.5, we get the result. $\square$

Theorem 3.8. *(a) Let G has no composition factor isomorphic to A_5 . If $m(G) < m(\text{PSL}(2, 7))$, then G is solvable.*

(b) Let G be a finite group such that $m(G) < m(\text{PSL}(2, 7))$, then G is 7-solvable.

Proof. **(a)** We establish solvability of G through induction on its order $|G|$. If $|G| \leq 59$, then G is a solvable group.

Suppose G possesses a non-trivial normal solvable subgroup K . By Lemma 2.5,

$$m(G/K) \leq m(G) < m(\text{PSL}(2, 7)).$$

Hence, the inductive hypothesis ensures G/K is solvable. Consequently, G is a solvable group. Therefore suppose that G has no non-trivial normal solvable subgroup. Since $m(G) < m(\text{PSL}(2, 7)) = 998/21 \simeq 47.524$, using Lemma 2.3 we conclude that there exists an element $x \in G$ such that

$$(6) \quad |G : \langle x \rangle| \leq 47.$$

Using Lemma 2.2, $|\langle x \rangle : \text{core}_G(\langle x \rangle)| \leq 46$. Therefore

$$|G : \text{core}_G(\langle x \rangle)| = |G : \langle x \rangle| \cdot |\langle x \rangle : \text{core}_G(\langle x \rangle)| \leq 47 \cdot 46 = 2162.$$

Since $\text{core}_G(\langle x \rangle) = 1$, therefore $|G| \leq 2162$. Let G be a non-solvable group. By Lemma 2.1, G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is a direct product of isomorphic non-abelian simple groups and $|G/N| \mid |\text{Out}(N/M)|$.

If M is non-solvable, then $|N| = |N/M| \cdot |M|$ divides G . Therefore $3600 \leq |G|$, which is a contradiction. Thus M is solvable and so $M = 1$. Therefore N is a simple group such that $|N| \leq 2162$. Since G has no composition factor isomorphic to A_5 , using [7], we have

$$N \in \{\text{PSL}(2, 7), A_6, \text{PSL}(2, 8), \text{PSL}(2, 11), \text{PSL}(2, 13)\}.$$

By GAP [8], we have the following table.

TABLE 2. The value of $m(N)$ for some simple groups.

N	$\text{PSL}(2, 7)$	A_6	$\text{PSL}(2, 8)$	$\text{PSL}(2, 11)$	$\text{PSL}(2, 13)$
$m(N)$	$\frac{998}{21}$	$\frac{1522}{15} \simeq 101.4$	$\frac{4229}{42} \simeq 100.6$	$\frac{16193}{110} \simeq 147.2$	$\frac{39545}{182} \simeq 217.2$

We see that $m(G) \geq m(\text{PSL}(2, 7))$, which is a contradiction.

(b) Similarly to the above we get that $|G| \leq 2162$. Now suppose that G is not a 7-solvable group. Therefore G is nonsolvable. Thus by Lemma 2.1, G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is a direct product of isomorphic non-abelian simple groups and $|G/N| \mid |\text{Out}(N/M)|$. As we mentioned above M is a solvable group.

Let $N/M \cong A_5$. If M is 7-solvable, then G is 7-solvable. If M is not a 7-solvable group, then $7 \mid |M|$. If $|M| < 168$, then M is a solvable group and so M is a 7-solvable group. Therefore $|M| \geq 168$ and we have $|G| \geq 60 \cdot 168$, which is a contradiction.

The proof is now completed. $\square$

Theorem 3.9. *Let G be a finite group such that $m(G) < m(\text{PSL}(2, 11))$. Then G is an 11-solvable group.*

Proof. We prove that G is solvable by induction on $|G|$. If $|G| < 660$ or $11 \nmid |G|$, then G is an 11-solvable group. If G has a non-trivial normal 11-solvable subgroup K , then by Lemma 2.5,

$$m(G/K) \leq m(G) < m(\text{PSL}(2, 11)),$$

and so by the inductive hypothesis, G/K is an 11-solvable group, and consequently, G is 11-solvable. Therefore suppose that G has no non-trivial normal 11-solvable subgroup.

Since $m(G) < m(\text{PSL}(2, 11)) = 16193/110 \simeq 147.2$, using Lemma 2.3 we get that there exists an element $x \in G$ such that

$$(7) \quad |G : \langle x \rangle| \leq 147.$$

Using Lemma 2.2, $|\langle x \rangle : \text{core}_G(\langle x \rangle)| \leq 146$. Therefore

$$|G : \text{core}_G(\langle x \rangle)| = |G : \langle x \rangle| \cdot |\langle x \rangle : \text{core}_G(\langle x \rangle)| \leq 147 \cdot 146 = 21462.$$

Since G has no non-trivial normal 11-solvable subgroup, $\text{core}_G(\langle x \rangle) = 1$, so

$$|G : \text{core}_G(\langle x \rangle)| = |G| \leq 21462.$$

If G is not an 11-solvable group, then G is not a solvable group. By Lemma 2.1, G has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N/M is isomorphic to a direct product of a non-abelian simple group S and $|G/N| \mid |\text{Out}(N/M)|$. If M is not an 11-solvable, then $|M| \geq 660$ and also we know $|N/M| \geq 60$. Therefore $|N| = |N/M| \cdot |M| \geq 60 \cdot 660 = 39600$. Since $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$, $|N| \leq |G|$. Therefore $39600 \leq |G| \leq 21462$, which is a contradiction. Thus M is 11-solvable. Since G has no non-trivial normal 11-solvable subgroup, $M = 1$.

By [7], we have

$$S \in \{\text{PSL}(2, q) | q = 5, 7, 8, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31\} \\ \cup \{A_6, A_7, A_8, \text{PSL}(3, 3), \text{PSL}(3, 4), \text{PSU}(3, 3), M_{11}\}.$$

Now, we consider two following cases:

- (1) Let $11 \nmid |S|$. Since G has no non-trivial normal 11-solvable subgroup, $M = 1$. We have normal series $1 \trianglelefteq N \trianglelefteq G$ such that N is isomorphic to a direct product of a non-abelian simple group S and $|G/N| \mid |\text{Out}(N)|$. We know $|G| = |G/N| \cdot |N|$ and also $|G/N| \mid |\text{Out}(N)|$. Then by [7, page 239], $11 \nmid |\text{Out}(N)|$ and also since $11 \nmid |S|$, we have $11 \nmid |N|$. Therefore $11 \nmid |G|$ and so G is 11-solvable.
- (2) Let $11 \mid |S|$. Then by [7], we have

$$S \in \{\text{PSL}(2, 11), \text{PSL}(2, 23), M_{11}\}.$$

TABLE 3. The value of $m(G)$ for some simple groups.

G	$PSL(2, 11)$	$PSL(2, 23)$	M_{11}
$m(G)$	$\frac{16193}{110}$	$\frac{656686}{759} \simeq 865.1$	$\frac{459649}{330} \simeq 1392.8$

We see that $m(G) \geq m(\text{PSL}(2, 11))$, which is a contradiction.

The proof is now completed. $\square$

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Sara Pouyandeh

Department of Mathematics,
Payamenoor University,
Tehran, Iran.

s.pouyandeh124@pnu.ac.ir

Laleh Oftadeh

Department of Mathematics,
Fir. C., Islamic Azad University,
Firoozabad, Iran.

lalehoftadeh@iaau.ac.ir, lalehoftadeh1359@gmail.com