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THE GOLDIE EXTENDING PROPERTY APPLIED TO C-CLOSED SUBMODULES

ENAS MUSTAFA KAMIL*

Abstract. In this study, we define a module $\mathcal{H}$ to be Goldie CCLS if and only if there exists a direct summand $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{L} \cap \mathcal{B}$ is essential in each of $\mathcal{L}$ and $\mathcal{B}$ for each c-closed submodule $\mathcal{L}$ of $\mathcal{H}$. We examine the structural characteristics of Goldie CCLS modules and identify the connections with the other extending generalizations. We discuss the theory of decomposition. Using examples, we derive several essential features and characterizations of Goldie CCLS modules.

1. Introduction

Every module $\mathcal{H}$ in this work is a unitary right $\mathcal{R}$-module, and $\mathcal{R}$ stands for an arbitrary associative ring with identity. In this paper, the term "direct summand" is abbreviated as "d.s." to reduce the redundancy and avoid issues of similarity detection in text analysis. If $\mathcal{L}$ does not have a essential extension, then $\mathcal{L}$ is closed in $\mathcal{H}$, if for each nonzero submodule $\mathcal{L}$ of

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*Corresponding author

$\mathcal{H}$, denoted by $\mathcal{N} \leqslant_e \mathcal{H}, \mathcal{L} \cap \mathcal{N} \neq 0$, then $\mathcal{N}$ is essential in $\mathcal{H}$, see [5]. A module $\mathcal{H}$ is said to be an extending (or CS) module if each of its submodules is essential in a d.s. of $\mathcal{H}$. Equivalently if all of its closed submodules are d.s, then $\mathcal{H}$ is extending, as shown in [3]. In [10], [9], [8], and [11], some generalizations of extending modules are given. Since the idea of extending modules is crucial to module theory, the two relations listed below on a collection of submodules of $\mathcal{H}$ should be recalled: (i) $\mathcal{L}\alpha\mathcal{B}$ "if and only if" there exists $\mathcal{K} \leqslant \mathcal{H}$ such that $\mathcal{L} \leqslant_e \mathcal{K}$ and $\mathcal{B} \leqslant_e \mathcal{K}$; (ii) $\mathcal{L}\beta\mathcal{B}$ "if and only if" $\mathcal{L} \cap \mathcal{B} \leqslant_e \mathcal{L}$ and $\mathcal{L} \cap \mathcal{B} \leqslant_e \mathcal{B}$. Observe that the equivalence relation β is equal to a relation specified in [4]. Remember that $\mathcal{H}$ is extending (or CS) module if, $\mathcal{L}\alpha\mathcal{B}$, for all submodules $\mathcal{L}$ of $\mathcal{H}, \mathcal{B}$ is a summand of $\mathcal{H}$, consider [1]. Additionally, a Goldie extending (G-extending) module is referred to as a module $\mathcal{H}$, if there is a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{L}\beta\mathcal{B}$ for every $\mathcal{L} \leqslant \mathcal{H}$, or, similarly, if there is a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{L}\beta\mathcal{B}$, for every closed submodule $\mathcal{L}$ of $\mathcal{H}$. Every extending module is obviously G-extending, [1]. CCLS modules are an additional practical extension of extending modules. Following [7], if each c-closed submodule of a module $\mathcal{H}$ is a d.s. of $\mathcal{H}$, then the module $\mathcal{H}$ is termed CCLS. A submodule $\mathcal{L}$ of $\mathcal{H}$ is c-closed if, given some submodule $\mathcal{B}$ of $\mathcal{H}, \mathcal{L} = \mathcal{B}$ whenever $\frac{\mathcal{B}}{\mathcal{L}}$ is singular. It is evident that all c-closed submodules are closed in $\mathcal{H}$. Additionally, in a nonsingular module, c-closed submodules and complement submodules of $\mathcal{H}$ coincide [7]. In order to construct a module that is analogous to G^z-extending and G^r-extending modules, which, respectively, were presented in [17] and [16]. We examine a condition of a module that includes the relation β on the collection of all c-closed submodules of a module, in this study. We say that a module $\mathcal{H}$ is Goldie CCLS if there is a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{L}\beta\mathcal{B}$ for each c-closed submodule $\mathcal{L}$ of $\mathcal{H}$, then $\mathcal{H}$ is Goldie CCLS (for short G-CCLS). We call a ring $\mathcal{R}$ right G-CCLS if $\mathcal{R}_{\mathcal{R}}$ is a G-CCLS, i.e. if each c-closed ideal $\mathcal{J}$ of $\mathcal{R}$ there is an idempotent element e of $\mathcal{R}$ satisfies $\mathcal{J}\beta e\mathcal{R}$. It is evident that the G-extending and CCLS classes are appropriately contained in the class of G-CCLS modules. The relationship between the G-CCLS property, extending, CCLS notion, and G-extending circumstances is examined in Section 2, Additionally, we derive the structural behavior and fundamental features of the class of G-CCLS module. The decomposition theory of the G-CCLS modules is covered in Section 3, we address the situation where a direct sum of G-CCLS modules is likewise a G-CCLS, demonstrating through an example that the direct sum of G-CCLS modules does not necessarily have to be G-CCLS. Additionally, we look into the circumstances that allow the G-CCLS property to be inherited by d.s. Specifically, we demonstrate that if $\mathcal{L}$ is a projection invariant c-closed submodule of $\mathcal{H}$, then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{L}$, for some $\mathcal{H}_1 \leqslant \mathcal{H}$ and both $\mathcal{H}_1, \mathcal{L}$ are G-CCLS. Following [12], if each submodule of $\mathcal{H}$ has a unique closure in $\mathcal{H}$, then $\mathcal{H}$ is referred to as a UC-module. Recall that if $e\mathcal{L} \subseteq \mathcal{L}$ for every $e^2 = e \in \mathrm{End}(\mathcal{H}_{\mathcal{R}})$, $\mathcal{L}$ is projection invariant in $\mathcal{H}$, see [12]. *A* ring $\mathcal{R}$ is said to be Abelian if each idempotent is central [3].

2. Preliminaries

This section contains some new results as well as a review of some c-closed submodule attributes and the relation of G-CCLS and some generalizations of CS modules is given in this part.

Lemma 2.1. [7] *Consider $\mathcal{H}$ to be a module, and $\mathcal{L} \leqslant \mathcal{N} \leqslant \mathcal{H}$, subsequently.*

(i) *$\mathcal{N}$ is c-closed in $\mathcal{H}$ "if and only if" $\frac{\mathcal{N}}{\mathcal{L}}$ is c-closed in $\frac{\mathcal{H}}{\mathcal{L}}$.*

(ii) *If $\mathcal{L}$ is c-closed in $\mathcal{H}$, then $\mathcal{L}$ is c-closed in $\mathcal{N}$.*

Lemma 2.2. [7] *Let $\mathcal{L}$ be c-closed in $\mathcal{H}$ and $\varphi : \mathcal{H} \to \mathcal{H}'$ be a homomorphism if $\operatorname{Ker}\varphi \leqslant \mathcal{L}$, then $\varphi(\mathcal{L})$ is c-closed in $\mathcal{H}'$.*

Lemma 2.3. *Assume that $\mathcal{H} = \bigoplus_{i=1}^{n} \mathcal{H}_i$, if $\mathcal{L}_i \leqslant \mathcal{H}_i \forall i = 1, \ldots, n$, then $\mathcal{L}_i$ is c-closed in $\mathcal{H}_i$ "if and only if" $\bigoplus_{i=1}^{n} \mathcal{L}_i$ is c-closed in $\mathcal{H}$.*

Proof. An analogous induction argument to that in ([7] Proposition 5) leads to this conclusion. □

Lemma 2.4. *Take $\mathcal{H}$ to be a module, let $\mathcal{X} \leqslant \mathcal{L} \leqslant \mathcal{H}$, if $\mathcal{X}$ is c-closed in $\mathcal{L}$ and $\mathcal{L}$ is c-closed in $\mathcal{H}$, then $\mathcal{X}$ is c-closed in $\mathcal{H}$.*

Proof. Let $\mathcal{K}$ be a submodule of $\mathcal{H}$ such that $\frac{\mathcal{K}}{\mathcal{X}}$ is singular, it is easy to see that $\frac{\mathcal{K}+\mathcal{L}}{\mathcal{L}}$ is singular. However, $\mathcal{L}$ is c-closed in $\mathcal{H}$, therefore $\mathcal{L} = \mathcal{L} + \mathcal{K}$ it follows that $\mathcal{K} \leqslant \mathcal{L}$. Since $\mathcal{X}$ is c-closed in $\mathcal{L}$, then $\mathcal{X} = \mathcal{K}$. Thus, $\mathcal{X}$ is c-closed in $\mathcal{H}$. □

Now, remember that the set of submodules of $\mathcal{H}$ has the following relations which are stated in [1].

(i) $\mathcal{L}\alpha\mathcal{N}$ "if and only if" there is $\mathcal{X} \leqslant \mathcal{H}$ such that $\mathcal{L} \leqslant_e \mathcal{X}$ and $\mathcal{N} \leqslant_e \mathcal{X}$.

(ii) $\mathcal{L}\beta\mathcal{N}$ "if and only if" $\mathcal{L} \cap \mathcal{N} \leqslant_e \mathcal{L}$ and $\mathcal{L} \cap \mathcal{N} \leqslant_e \mathcal{N}$.

Lemma 2.5. *Let $\mathcal{H}$ be a module, let $\mathcal{L}$ be c-closed in $\mathcal{H}$ then $\mathcal{L} \cap \mathcal{X}$ is c-closed in $\mathcal{X}$, for every submodule $\mathcal{X}$ of $\mathcal{H}$.*

Proof. Let $\mathcal{K}$ be a submodule of $\mathcal{X}$ such that $\frac{\mathcal{K}}{\mathcal{L}\cap\mathcal{X}}$ is singular. Consider the epimorphism $\psi : \frac{\mathcal{K}}{\mathcal{L}\cap\mathcal{X}} \to \frac{\mathcal{K}}{\mathcal{L}\cap\mathcal{K}}$, hence $\frac{\mathcal{K}}{\mathcal{L}\cap\mathcal{K}}$ is singular. So, $\frac{\mathcal{L}+\mathcal{K}}{\mathcal{L}}$ is singular, and the c-closed property of $\mathcal{L}$ yields that $\mathcal{L} + \mathcal{K} = \mathcal{L}$ and hence $\mathcal{K} \leqslant \mathcal{L}$. Thus, $\mathcal{L} \cap \mathcal{X} = \mathcal{K}$. □

The following Proposition gives an equivalent condition to CCLS in terms of α.

Proposition 2.6. *A module $\mathcal{H}$ is CCLS "if and only if" there is a summand $\mathcal{B}$ of $\mathcal{H}$ satisfies $\mathcal{L}\alpha\mathcal{B}$ for each c-closed submodule $\mathcal{L}$ of $\mathcal{H}$.*

Proof. The proof follows directly from the definitions. □

Inspired by the idea and the definitions of Goldie, Smith, Akalan, and Birkenmeier, we arrive at the definition that follows:

Definition 2.7. We say that a module $\mathcal{H}$ is Goldie CCLS (briefly G-CCLS) if, for any c-closed submodule $\mathcal{L}$ of $\mathcal{H}$, there is a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{L}\beta\mathcal{B}$.

It is evident that both G-extending and CCLS modules are included in the G-CCLS modules class. We then identify the G-CCLS condition in relation to a number of well-known extensions of the extending property in general.

Proposition 2.8. *Let us examine the subsequent requirements for module $\mathcal{H}$.*

- (i) *$\mathcal{H}$ is extending.*
- (ii) *$\mathcal{H}$ is G-extending.*
- (iii) *$\mathcal{H}$ is CCLS.*
- (iv) *$\mathcal{H}$ is G-CCLS.*
- (v) *$\mathcal{H}$ is singular.*

Following that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (iii).

Proof. $(i) \Rightarrow (ii)$ From [1].
$(ii) \Rightarrow (iii)$ It follows from [7].
$(iii) \Rightarrow (iv)$ It is straightforward.
$(v) \Rightarrow (iii)$ It follows from the fact that the trivial submodules are the only c-closed submodules of a singular module $\mathcal{H}$ which are d.s. of $\mathcal{H}$.
$(ii) \nRightarrow$ (i) Choose any prime integer p, and $\mathcal{H}$ be the $\mathbb{Z}$-module $\mathcal{H} = \mathbb{Z}_p \oplus \mathbb{Q}$ is G-extending by ([1], Corollary 3.3(i)) $\mathcal{H}_{\mathbb{Z}}$ is not extending ([14], Example 4.2).
$(iii) \nRightarrow (ii)$ Consider $\mathcal{H}$ with the unique composition series $\mathcal{H} \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset 0$, $\mathcal{H}$ is singular (not G-extending) $\mathcal{R}$-module, $\mathcal{H} \oplus \frac{\mathcal{H}_1}{\mathcal{H}_2}$ is not F.I-extending, from [18]. Hence it is not G-extending. We can offer the following as a particular illustration. Let $\mathcal{J}$ be the ideal of $\mathcal{R}$ produced by a, b, a^2, b^2, and let $\mathcal{R} = \mathbb{Z}_2[a, b]$ be the commutative polynomial ring with the indeterminates a, b over the field $\mathbb{Z}_2$. In such case, $\frac{\mathcal{R}}{\mathcal{J}}$ is a singular $\mathcal{R}$-module that is not G-extending. As a result, $\frac{\mathcal{R}}{\mathcal{J}}$ is CCLS and not G-extending. Consequently, $(iii) \nRightarrow (ii)$.
$(iv) \nRightarrow (iii\ )$ Let $p > 0$ be the characteristic of the field $\mathbb{F}$. Assume that the cyclic group of order p is $\mathcal{P} =< a : a^p = 1 >$. The group algebra $\mathbb{F}[\mathcal{P}]$ is represented by $\mathcal{R}$. $\mathcal{R}$ is a self-injective Artinian ring since it is a quasi-Frobenius algebra. Specifically, $\mathcal{R}$ is extending because R_R is

uniform. Remember that $\mathcal{R} > \mathcal{J} > \mathcal{J}^2 > \cdots > \mathcal{J}^p = 0$ are the only ideals of $\mathcal{R}$, and thus the unique maximum ideal in $\mathcal{R}$ is $\mathcal{J} = \mathcal{R}(a-1)$. Let $\mathcal{H} = \mathcal{R} \oplus \frac{\mathcal{R}}{\mathcal{J}}$. From Corollary 3.3, $\mathcal{H}$ is G-CCLS, but $\mathcal{H}$ is not CCLS.
$(iii) \not\Rightarrow (v)$ The nonsingular $\mathbb{Z}$-module $\mathbb{Z}$ is CCLS. □

Corollary 2.9. *The statements that follow are identical in a nonsingular module $\mathcal{H}$.*

- (i) *$\mathcal{H}$ has extending property.*
- (ii) *$\mathcal{H}$ has G-extending property.*
- (iii) *$\mathcal{H}$ is CCLS.*
- (iv) *$\mathcal{H}$ is G-CCLS.*

Proof. From Proposition 2.8, it is sufficient to prove that $(iv) \Rightarrow (i)$, let $\mathcal{H}$ be a G-CCLS, let $\mathcal{L}$ be a closed submodule of $\mathcal{H}$, hence $\mathcal{L}$ is c-closed in $\mathcal{H}$, by [7], then there is a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{L} \cap \mathcal{B} \leqslant_e \mathcal{L}$ and $\mathcal{L} \cap \mathcal{B} \leqslant_e \mathcal{B}$. Now, $\frac{\mathcal{B}}{\mathcal{L} \cap \mathcal{B}} \cong \frac{\mathcal{L}+\mathcal{B}}{\mathcal{L}}$ is singular. But $\mathcal{L}$ is c-closed in $\mathcal{H}$, therefore $\mathcal{L} = \mathcal{L} + \mathcal{B}$ implies $\mathcal{B} \leqslant \mathcal{L}$, so $\mathcal{B} = \mathcal{L}$. Thus, $\mathcal{H}$ is an extending module. □

The example that follows demonstrates that there are modules not satisfy the G-CCLS condition.

Example 2.10. (i) Let $\mathcal{H} = \mathbb{Z}[m] \oplus \mathbb{Z}[m]$ as $\mathbb{Z}[m]$-module, as $\mathcal{H}$ is nonsingular not extending, Corollary 2.9 yields that $\mathcal{H}$ is neither CCLS nor G-CCLS.
(ii) The $\mathbb{Z}$-module $\oplus\mathbb{Z}$ is neither CCLS nor G-CCLS.

Proposition 2.11. *Given an indecomposable module $\mathcal{H}$, it is CCLS "if and only if" it is G-CCLS.*

Proof. Let $\mathcal{H}$ be a G-CCLS, let $\mathcal{L}$ be a c-closed submodule of $\mathcal{H}$, there is a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{L} \cap \mathcal{B} \leqslant_e \mathcal{L}$ and $\mathcal{L} \cap \mathcal{B} \leqslant_e \mathcal{B}$. However, $\mathcal{H}$ is indecomposable, either $\mathcal{B} = \mathcal{H}$ or $\mathcal{B} = 0$, it is not hard to see that either $\mathcal{L} = \mathcal{H}$ or $\mathcal{L} = 0$. So, $\mathcal{H}$ is CCLS, and the converse is from Proposition 2.8. □

Proposition 2.12. *Let $\mathcal{H}$ be a UC-module, then $\mathcal{H}$ is G-CCLS "if and only if" $\mathcal{H}$ is CCLS.*

Proof. Let $\mathcal{H}$ be a G-CCLS and let $\mathcal{L}$ be a c-closed submodule in $\mathcal{H}$, there is a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{L}\beta\mathcal{B}$. Take note that $(\mathcal{L}\cap\mathcal{B})\alpha\mathcal{L}$ and $(\mathcal{L}\cap\mathcal{B})\alpha\mathcal{B}$. Since $\mathcal{H}$ is a unique closure module, then α is transitive, so $\mathcal{L}\alpha\mathcal{B}$. Thus, $\mathcal{H}$ is CCLS, by Proposition 2.6. The converse is clear. □

Proposition 2.13. *Let End ($\mathcal{H}_{\mathcal{R}}$) be abelian and each c-closed submodule $\mathcal{L}$ of $\mathcal{H}$ can be written as $\mathcal{L}=\sum_{i\in I}\varphi_i(\mathcal{H})$, where each $\phi_i\in$ End ($\mathcal{H}_{\mathcal{R}}$). Then $\mathcal{H}$ is G-CCLS "if and only if" $\mathcal{H}$ is CCLS.*

Proof. Let $\mathcal{H}$ be a G-CCLS and let $\mathcal{L}$ be a c-closed submodule of $\mathcal{H}$. By the assumption, $\mathcal{L}=\sum_{i\in I}\varphi_i(\mathcal{H}),\varphi_i\in \operatorname{End}(\mathcal{H}_{\mathcal{R}})$, there exists $c^2=c\in \operatorname{End}(\mathcal{H}_{\mathcal{R}})$ such that $c\mathcal{H}\beta\mathcal{L}$, then each x in $\mathcal{L}$ can be written as $x=cx+(1-c)x=\sum_{i\in I}\phi_i\left(m_i\right),m_i\in\mathcal{H}$, hence $cx=c\left(\sum\phi_i\left(m_i\right)\right)=\sum_{i\in I}\varphi_i\left(c_{m_i}\right)\in\mathcal{L}\cap c\mathcal{H}=0$. Hence $\mathcal{L}\leqslant_e(1-c)\mathcal{H}$. According to the c-closed property of $\mathcal{L},\mathcal{L}=(1-c)\mathcal{H}$. Thus, $\mathcal{H}$ is CCLS. □

Next, we give an equivalent condition to G-CCLS.

Proposition 2.14. *For a module $\mathcal{H}$, the conditions listed below are equivalent.*

(*i*) *$\mathcal{H}$ is G-CCLS.*

(*ii*) *There exists a submodule $\mathcal{K}\leqslant\mathcal{H}$ and a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{K}\leqslant_e\mathcal{L}$ and $\mathcal{K}\leqslant_e\mathcal{B}$ for any c-closed submodule $\mathcal{L}$ of $\mathcal{H}$.*

(*iii*) *A complement $\mathcal{K}$ of $\mathcal{L}$ and a complement $\mathcal{W}$ of $\mathcal{K}$ exist for any c-closed submodule $\mathcal{L}$ of $\mathcal{H}$, so that $\mathcal{L}\beta\mathcal{W}$ and every homomorphism $\varphi:\mathcal{W}\oplus\mathcal{K}\to\mathcal{H}$ may be extended to a homomorphism $\psi:\mathcal{H}\to\mathcal{H}$.*

Proof. $(i)\Rightarrow(ii)$ Let $\mathcal{H}$ be a G-CCLS, let $\mathcal{L}$ be c-closed in $\mathcal{H}$, there is a summand $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{L}\cap\mathcal{B}\leqslant_e\mathcal{L}$ and $\mathcal{L}\cap\mathcal{B}\leqslant_e\mathcal{B}$. If we take $\mathcal{K}=\mathcal{L}\cap\mathcal{B}$, we are done.
$(ii)\Rightarrow(iii)$ Let $\mathcal{L}$ be c-closed submodule of $\mathcal{H}$. By (ii), there is a summand $\mathcal{B}$ of $\mathcal{H}$ and a submodule $\mathcal{K}$ of $\mathcal{H}$ such that $\mathcal{K}\leqslant_e\mathcal{L}$ and $\mathcal{K}\leqslant_e\mathcal{B}$. It is easy to see that $\mathcal{B}\cap\mathcal{L}\leqslant_e\mathcal{L}$ and $\mathcal{B}\cap\mathcal{L}\leqslant_e\mathcal{B}$.
$(iii)\Rightarrow(i)$ Let $\mathcal{L}$ be a c-closed submodule of $\mathcal{H}$. From (iii) and ([15], Lemma 3.97), $\mathcal{W}$ is a summand of $\mathcal{H}$. Thus, $\mathcal{H}$ is G-CCLS. □

A submodule of G-CCLS may not be G-CCLS again, as demonstrated by the example that follows.

Example 2.15. Assume that $\mathcal{R}=\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & \mathbb{Z}\end{pmatrix}$ is a submodule of the injective hull $\mathcal{S}_{\mathcal{R}}$. As $\mathcal{R}$ is nonsingular and not extending, Corollary 2.9 implies that $\mathcal{R}$ is not G-CCLS, while $\mathcal{S}_{\mathcal{R}}$ is G-CCLS.

Next, we give a condition under which the submodule of G-CCLS is again G-CCLS.

Proposition 2.16. *If $\mathcal{L}$ is a c-closed submodule of a G-CCLS module $\mathcal{H}$, then $\mathcal{L}$ is G-CCLS if its intersection with any d.s. of $\mathcal{H}$ is a d.s. of $\mathcal{L}$.*

Proof. Assuming that $\mathcal{X}$ is a c-closed submodule of $\mathcal{L}$, it is c-closed in $\mathcal{H}$, by Lemma 2.4. Consequently, there is a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{X}\beta\mathcal{B}$. Based on our presumption, $\mathcal{L}\cap\mathcal{B}$ is a d.s. of $\mathcal{L}$. Hence, $\mathcal{X} = (\mathcal{L}\cap\mathcal{X})\beta(\mathcal{L}\cap\mathcal{B})$. Thus, $\mathcal{L}$ is G-CCLS. □

3. Decompositions

There are nonsingular modules $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where each of $\mathcal{H}_1$ and $\mathcal{H}_2$ are both CCLS, however $\mathcal{H}$ is not CCLS. Example 2.10 shows that these modules demonstrate that the G-CCLS class is not closed under direct sums.

Finding the conditions under which the direct sum of G-CCLS is again G-CCLS is one of the section's primary goals.

Let $\mathcal{N}$ and $\mathcal{H}$ be modules, $\mathcal{N}$ is said to be $\mathcal{H}$-ejective if for all submodules $\mathcal{L}$ of $\mathcal{H}$ and all homomorphisms $\varphi : \mathcal{L} \to \mathcal{N}$, there exists a homomorphism $\psi : \mathcal{H} \to \mathcal{N}$ and $\mathcal{K} \leqslant_e \mathcal{L}$ such that $\varphi(k) = \psi(k), \forall k \in \mathcal{K}$, see [1]. Note that if $\mathcal{N}$ is $\mathcal{H}$-injective, then $\mathcal{N}$ is $\mathcal{H}$-ejective.

Lemma 3.1. [1] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be modules such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then, $\mathcal{H}_1$ is $\mathcal{H}_2$-ejective "if and only if" for every submodule $\mathcal{L}$ of $\mathcal{H}$ with $\mathcal{L}\cap\mathcal{H}_1 = 0$, there exist $\mathcal{H}_3 \leqslant \mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_3$ and $\mathcal{L}\cap\mathcal{H}_3 \leqslant_e \mathcal{L}$.*

Theorem 3.2. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be a direct sum of G-CCLS modules. If $\mathcal{H}_1$ is $\mathcal{H}_2$-ejective (or $\mathcal{H}_2$ is $\mathcal{H}_1$-ejective), then $\mathcal{H}$ is G-CCLS.*

Proof. Let $\mathcal{L}$ be a c-closed submodule of $\mathcal{H}$, we have the following two cases: If $\mathcal{L}\cap \mathcal{H}_1 = 0$, as $\mathcal{H}_1$ is $\mathcal{H}_2$-ejective, there is $\mathcal{H}_3 \leqslant \mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_3$ and $\mathcal{L}\cap \mathcal{H}_3 \leqslant_e \mathcal{L}$. Since $\mathcal{H}_2 \cong \mathcal{H}_3$, and $\mathcal{H}_2$ is G-CCLS, then $\mathcal{H}_3$ is G-CCLS. From Lemma 2.5, $\mathcal{L}\cap\mathcal{H}_3$ is c-closed in $\mathcal{H}_3$, then there exists a submodule $\mathcal{K}$ of $\mathcal{H}_3$ with $\mathcal{K} \leqslant_e \mathcal{L}\cap\mathcal{H}_3$ and d.s. $\mathcal{B}$ of $\mathcal{H}_3$ such that $\mathcal{K} \leqslant_e \mathcal{B}$. So, $\mathcal{B}$ is a summand of $\mathcal{H}$, and $\mathcal{L}\beta\mathcal{B}$. Now, if $\mathcal{L}\cap\mathcal{H}_1 \neq 0$, then $\mathcal{L}\cap\mathcal{H}_1$ has a complement in $\mathcal{L}$ say $\mathcal{N}$, hence $(\mathcal{L}\cap\mathcal{H}_1)\oplus\mathcal{N} \leqslant_e \mathcal{L}$. Since $\mathcal{N}\cap\mathcal{H}_1 = 0$, again Lemma 3.1, implies that there exists $\mathcal{H}_4 \leqslant \mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_4$ and $\mathcal{N}\cap\mathcal{H}_4 \leqslant_e \mathcal{N}$. Since $\mathcal{H}_4 \cong \mathcal{H}_2$ and $\mathcal{H}_2$ is G-CCLS, then $\mathcal{H}_4$ is G-CCLS. From Lemma 2.5 $\mathcal{L}\cap\mathcal{H}_1$ and $\mathcal{L}\cap\mathcal{H}_4$ are c-closed submodules of $\mathcal{H}_1$ and $\mathcal{H}_4$ respectively, so there exists $\mathcal{N}_1 \leqslant_e \mathcal{L}\cap\mathcal{H}_1, \mathcal{N}_2 \leqslant_e \mathcal{L}\cap\mathcal{H}_4$, and d.s. $\mathcal{B}_1$ of $\mathcal{H}_1$ and $\mathcal{B}_2$ of $\mathcal{H}_4$ satisfies $\mathcal{N}_1 \leqslant_e \mathcal{B}_1$ and $\mathcal{N}_2 \leqslant_e \mathcal{B}_2$. It is easy to show that $\mathcal{N}_1 \oplus (\mathcal{N}_2 \oplus \mathcal{N}) \leqslant_e (\mathcal{L}\cap\mathcal{H}_1)\oplus\mathcal{N} \leqslant_e \mathcal{L}$, hence $\mathcal{N}_1 \oplus \mathcal{N}_2 \leqslant_e \mathcal{L}$ and $\mathcal{N}_1 \oplus \mathcal{N}_2 \leqslant_e \mathcal{B}_1 \oplus \mathcal{B}_2$, where $\mathcal{B}_1 \oplus \mathcal{B}_2$ is a d.s. of $\mathcal{H}$. Thus, $\mathcal{H}$ is G-CCLS. □

Theorem 3.2 directly leads to the following consequence.

Corollary 3.3. *Let $\mathcal{H}_1$ be a G-CCLS and $\mathcal{H}_2$ is semisimple, then $\mathcal{H}$ is G-CCLS.*

Example 3.4.

(i) Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Abelian groups ($\mathbb{Z}$-modules), $\mathcal{H}_1$ is divisible and $\mathcal{H}_2 = \mathbb{Z}_{p^n}, n$ is any positive integer and p is a prime. According to Corollary 3.3, $\mathcal{H}$ is G-CCLS, which is not CS, from [13], when $\mathcal{H}_1 = \mathbb{Q}$.

(ii) Take the composition series $0 = \mathcal{H}_0 \leqslant \mathcal{H}_1 \leqslant \mathcal{H}_n = \mathcal{H}_1$ and let $\mathcal{H}_2 = \frac{\mathcal{H}_n}{\mathcal{H}_{n-1}} \oplus \ldots \oplus \frac{\mathcal{H}_1}{\mathcal{H}_0}$. Put $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, Corollary 3.3, yields that $\mathcal{H}$ is G-CCLS, however, in general, it is not CS, ([3], Corollary 7.4).

Proposition 3.5. *Let $\mathcal{L}$ be a c-closed submodule of a G-CCLS $\mathcal{H}$.*

(i) *If for each $c^2 = c \in \mathrm{End}(\mathcal{H})$, there exists $g^2 = g \in \mathrm{End}(\mathcal{L})$ such that $\mathcal{L} \cap c\mathcal{H} \leqslant_e g\mathcal{L}$, then $\mathcal{L}$ is G-CCLS.*

(ii) *If for each $c^2 = c \in \mathrm{End}(\mathcal{H})$, there exists $g^2 = g \in \mathrm{End}(\mathcal{H})$ such that $c\mathcal{H}\beta g\mathcal{H}$ and $g\mathcal{L} \leqslant_e \mathcal{L}$, then $\mathcal{L}$ is G-CCLS. Specifically, every projection invariant c-closed submodule of $\mathcal{H}$ is G-CCLS.*

Proof. (i) Let $\mathcal{X}$ be a c-closed submodule of $\mathcal{L}$, hence $\mathcal{X}$ is c-closed in $\mathcal{H}$, there exists $\mathcal{N} \leqslant \mathcal{H}$ such that $\mathcal{N} \leqslant_e \mathcal{X}$ and $c^2 = c \in \mathrm{End}\ (\mathcal{H}_R)$ such that $\mathcal{N} \leqslant_e c\mathcal{H}$, hence $\mathcal{N} \leqslant_e \mathcal{L} \cap c\mathcal{H} \leqslant_e g\mathcal{L}$, for some $g^2 = g \in End\,(\mathcal{L}_\mathcal{R})$. Thus, $\mathcal{L}$ is G-CCLS.

(ii) Let $\mathcal{X}$ be a c-closed submodule of $\mathcal{L}$, hence $\mathcal{X}$ is c-closed in $\mathcal{H}$. So, there exists $c^2 = c \in \mathrm{End}\ (\mathcal{H}_R)$ such that $\mathcal{X}\beta c\mathcal{H}$, then $\mathcal{X}\beta g\mathcal{H}$, by assumption. Since g$\mathcal{L} \leqslant \mathcal{L}, \mathcal{L}$ is G-CCLS.

□

Proposition 3.6. *Let $\mathcal{X}$ be a projection invariant c-closed submodule of a G-CCLS module $\mathcal{H}$, then $\frac{\mathcal{H}}{\mathcal{X}}$ is CCLS.*

Proof. Let $\frac{\mathcal{L}}{\mathcal{X}}$ be c-closed in $\frac{\mathcal{H}}{\mathcal{X}}$, hence $\mathcal{L}$ is c-closed in $\mathcal{H}$, by Lemma 2.1. Since $\mathcal{H}$ is G-CCLS, there exists $d^2 = d \in \mathrm{End}\ (\mathcal{H}_R)$ such that $\mathcal{L}\beta d\mathcal{H}$. Since $= \mathcal{X} \cap (d\mathcal{H} \oplus\ (1-d)\mathcal{H}) = (\mathcal{X} \cap d\mathcal{H}) \oplus (\mathcal{X} \cap (1-d)\mathcal{H})$. Observe that $\frac{\mathcal{L}}{\mathcal{X}}\beta\frac{d\mathcal{H}}{\mathcal{X}}$, where $\frac{d\mathcal{H}}{\mathcal{X}}$ is d.s. of $\frac{\mathcal{H}}{\mathcal{X}}$. Thus, $\frac{\mathcal{H}}{\mathcal{X}}$ is G-CCLS.

□

Corollary 3.7. *If $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$ is G-CCLS and $\mathcal{H}_i$ is projection invariant c-closed in $\mathcal{H}, \forall i \in I$. Then $\mathcal{H}_i$ is G-CCLS,* ∀i ∈ I.

Theorem 3.8. *Let $\mathcal{L}$ be a c-closed submodule of a G-CCLS module $\mathcal{H}$ that is projection invariant. There exists $\mathcal{H}_1 \leqslant \mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{L}$ and $\mathcal{H}_1, \mathcal{L}$ are G-CCLS.*

Proof. Since $\mathcal{H}$ is G-CCLS and $\mathcal{L}$ is c-closed in $\mathcal{H}$, there is $c^2 = c \in \mathrm{End}\,(\mathcal{H}_\mathcal{R})$ such that $\mathcal{L}\beta c\mathcal{H}$. Since $\mathcal{L} = c\mathcal{L} \bigoplus (1-c)\mathcal{L}, c\mathcal{L} = \mathcal{L} \cap c\mathcal{H}$ and $(1-c)\mathcal{L} = \mathcal{L} \cap (1-\ c)\mathcal{H}$, according to the projection invariant property of $\mathcal{L}$. Note that $c\mathcal{L} \leqslant_e c\mathcal{H}$ and $c\mathcal{L} \leqslant_e \mathcal{L}$, then $\mathcal{L} \cap (1-c)\mathcal{H} = 0$, hence $(1-c)\mathcal{L} = 0$ implies that $\mathcal{L} = c\mathcal{L} \leqslant_e c\mathcal{H}$, however, $\mathcal{L}$ is c-closed in $\mathcal{H}$, therefore $\mathcal{L} = c\mathcal{H}$. By Proposition 3.5$(ii)$, $\mathcal{L}$ is G-CCLS. Now, $\mathcal{H}_1 \cong \frac{\mathcal{H}}{\mathcal{L}}$ is G-CCLS, according to Proposition 3.6. □

Remember that a module $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ has exchangeable property if, for any d.s. $\mathcal{X}$ of $\mathcal{H}$, there exist $\mathcal{L} \leqslant \mathcal{H}_1$ and $\mathcal{B} \leqslant \mathcal{H}_2$ such that $\mathcal{H} = \mathcal{X} \oplus \mathcal{L} \oplus \mathcal{B}$. If $\mathcal{H}$ is Goldie extending and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is exchangeable, then $\mathcal{H}_1$ and $\mathcal{H}_2$ are also Goldie extending, as shown in ([6], Lemma 2.3). We then demonstrate that a similar outcome to the previously mentioned fact holds for the G-CCLS module.

Proposition 3.9. *If $\mathcal{H}$ is G-CCLS and the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is exchangeable, then $\mathcal{H}_1$ is also G-CCLS. In particular, if $\mathcal{H}$ is G-CCLS with the finite internal exchange property, then so is any d.s. of $\mathcal{H}$.*

Proof. Let $\mathcal{L}$ be c-closed in $\mathcal{H}_1$, hence $\mathcal{L} \oplus \mathcal{H}_2$ is c-closed in $\mathcal{H}$, by ([7], Proposition 3). As $\mathcal{H}$ has the exchangeable and G-CCLS property, there exists $Y \leqslant \mathcal{H}$ and a decomposition $\mathcal{H} = \mathcal{B} \oplus \mathcal{H}_1' \oplus \mathcal{H}_2', Y \leqslant_e \mathcal{B}$ and $Y \leqslant_e \mathcal{L} \oplus \mathcal{H}_2, \mathcal{H}_1' \leqslant \mathcal{H}_1$ and $\mathcal{H}_2' \leqslant \mathcal{H}_2$. It is easy to verify that $Y \cap \mathcal{H}_2' = 0$ implies that $\mathcal{H}_2' = 0$, hence $\mathcal{H}_1 = \mathcal{H}_1' \oplus (\mathcal{B} \cap \mathcal{H}_1)$. Now, $Y \leqslant_e \mathcal{B}$ and $Y \leqslant_e \mathcal{L} \oplus \mathcal{H}_2$ yields that $Y \leqslant_e \mathcal{B} \cap \mathcal{H}_1$ and $Y \cap \mathcal{H}_1 \leqslant_e \mathcal{B} \cap \mathcal{H}_1$. It follows that $\mathcal{H}_1$ is G-CCLS. □

As partner conditions for the extending condition and its generalizations, the following well-known features are effective; see to [15]. A module $\mathcal{H}$ is said to have the C_3 property if whenever $\mathcal{H}_1$ and $\mathcal{H}_2$ are d.s. of $\mathcal{H}$ with $\mathcal{H}_1 \cap \mathcal{H}_2 = 0$, then $\mathcal{H}_1 \oplus \mathcal{H}_2$ is a d.s. of $\mathcal{H}$. When $\mathcal{H}_1$ and $\mathcal{H}_2$ are d.s. of $\mathcal{H}$, then $\mathcal{H}_1 \cap \mathcal{H}_2$ is also a d.s. of $\mathcal{H}$, this is the summand intersection property (SIP) of a module $\mathcal{H}$.

Theorem 3.10. *Let $\mathcal{H}$ be a G-CCLS module, then all of its c-closed d.s. are G-CCLS when $\mathcal{H}$ has the SIP or the C_3 condition.*

Proof. Let $\mathcal{H} = \mathcal{L} \bigoplus \mathcal{L}', \mathcal{L}$ is closed in $\mathcal{H}$. According to Proposition 3.5(i), $\mathcal{L}$ is G-CCLS. Now, assume that $\mathcal{H}$ satisfies the C_3 condition. Let $\pi : \mathcal{H} \to \mathcal{L}$ be the canonical projection on $\mathcal{L}$ and let $\mathcal{K}$ be c-closed in $\mathcal{L}$, then $\mathcal{K}$ is c-closed in $\mathcal{H}$, by Lemma 2.4. Since $\mathcal{H}$ is G-CCLS, there is a d.s. $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{K} \cap \mathcal{B} \leqslant_e \mathcal{K}$ and $\mathcal{K} \cap \mathcal{B} \leqslant_e \mathcal{B}$. Since $\mathcal{L}'$ and $\mathcal{B}$ are both d.s. of $\mathcal{H}$ and $\mathcal{L}' \cap \mathcal{B} = 0$, then $\mathcal{L}' \oplus \mathcal{B}$ is a d.s. of $\mathcal{H}$. It can be seen that $\mathcal{L}' \oplus \mathcal{B} = \mathcal{L}' \oplus \pi(\mathcal{B})$ Hence $\pi(\mathcal{B})$ is a d.s. of $\mathcal{L}$. Let $0 \neq y \in \pi(\mathcal{B}), y = \pi(b), 0 \neq b \in \mathcal{B}$. There is $r \in \mathcal{R}$ such that $0 \neq rb \in \mathcal{K} \cap \mathcal{B}$, so $rb = k = x$, where $k \in \mathcal{K}$ and $x \in \mathcal{B}$. Now, $0 \neq rb = r\pi(b) = k = \pi(x) \in \mathcal{K} \cap \pi(\mathcal{B})$. Then $\mathcal{K} \cap \pi(\mathcal{B}) \leqslant_e \pi(\mathcal{B})$. Clearly $\pi(\mathcal{B}) = \mathcal{L} \cap (\mathcal{L}' \oplus \pi(\mathcal{B})) = \mathcal{L} \cap (\mathcal{L}' \oplus \mathcal{B})$. So $\mathcal{K} \cap \pi(\mathcal{B}) = \mathcal{K} \cap (\mathcal{L}' \oplus \mathcal{B}) \leqslant_e \mathcal{K}$. Thus, $\mathcal{L}$ is G-CCLS. □

Proposition 3.11. *If $\mathcal{H}$ is G-CCLS with uniform dimension 2, then every d.s. of $\mathcal{H}$ is G-CCLS.*

Proof. Let $\mathcal{B}$ be any d.s. of $\mathcal{H}$. If $\mathcal{B} = \mathcal{H}$, then $\mathcal{B}$ is G-CCLS, by assumption. Now, assume $\mathcal{B} \neq \mathcal{H}$, then $\mathcal{B}$ is uniform, so, $\mathcal{B}$ is G-CCLS. □

The relationship between G-CCLS and a few generalizations of extending property is summarized in the schematic in Fig. 1.

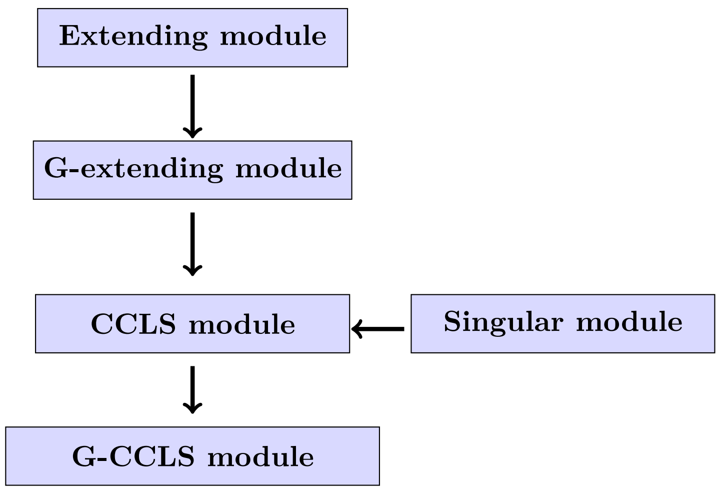

Figure 1. The connection between various generalizations of extending property and G-CCLS.

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Enas Mustafa Kamil

Department of Mathematics, College of Science for Women,

University of Baghdad, Baghdad, Iraq.

Inas.m@csw.uobaghdad.edu.iq