Algebraic Structures and Their Applications Vol. X No. X (20XX) pp XX-XX.

Research Paper

EULER FUNCTION GRAPH: A NUMBER THEORETICAL APPROACH

MANAS PRATIM KALITA AND MRIDUL DUTTA*

Abstract. This paper explores several fundamental properties and theorems about bipartiteness, completeness, and connectedness of the Euler function graph $G(\phi(n))$. A key focus of the study is the novel connection between the completeness of the Euler function graph $G(\phi(n))$ and the set $\Phi(n)$. By investigating this connection, we establish new results that clarify the interplay between graph completeness and fundamental properties of prime numbers. Finally, we provide certain examples as well as non-examples.

1. Introduction

The Euler totient function, often denoted as $\phi(n)$, is a fundamental concept in number theory. The function $\phi(n)$ counts the number of positive integers up to n that are relatively prime to n. S. Shanmugavelan [7] has introduced the Euler function graph $G(\phi(n))$ for a natural number n, is a simple undirected graph such that the vertex set V$(G(\phi(n)))$=$\{a \mid$ g.c.d$(a,n) = 1$ and $a < n$ $\}$ and the edge set E$(G(\phi(n)))$=$\{am \mid$ g.c.d$(a,m) = 1$ and $a < m$

DOI: 10.22034/as.2026.23648.1827

MSC(2010): 05C99, 11A41, 11A25.

Keywords: Bipartite graph, Complete graph, Euler function graph $G(\phi(n))$, Reduced residue system.

Received:11 September 2025, Accepted: 20 February 2026.

*Corresponding author

or $m < a$ }. S. Shanmugavelan has investigated connectedness, bipartiteness, connectivity, and the Euler function subgraph [7]. J. B. Babujee [2] has developed an algorithmic method, using number theory techniques for prime labeling of the maximal planar graph Pl_n. The relationship between arithmetic functions and graph theory has attracted attention in recent years. Ghanbari and Alikhani [4] created a graph based on the Euler totient function, with adjacency found using totient iteration. In 2022, S. Shanmugavelan and C. Natarajan [8] have introduced a new category of graphs called the divisor function even (odd) sum graph and have proved $G_{eD(n)}$ is complete iff it has only odd divisors and $G_{eD(n)}$ is not a bipartite graph and other related results. T. Chalapathy and R. Kumar [3] also have introduced a graph related to Euler totient numbers, proving that these graphs are hamiltonian, eulerian, and disconnected. They also studied the domination number, clique number, chromatic number, and girth of these graphs. In the study of the connection between number theory and graph theory, Y. Kaneda, J. D. Laison, J. Schreiner-McGraw, and C. Starr relate the prime power distance graph and prime product graph to the twin prime conjecture, Fermat's last theorem, and the Green-Tao theorem [6].

This paper investigates various concepts related to the Euler function graph [7], establishes its connections with the Euler totient function, and further incorporates the reduced residue system into the analysis. In our investigation, we have found that the maximum value of n for which every prime p with $p^2 < n$ divides n is 30. Moreover, the set $S = \{6, 8, 12, 18, 24, 30\}$ is the unique maximal set of integers for which every prime p satisfying $p^2 < n$ divides n. The motivation behind this paper was to highlight some number-theoretical concepts through the medium of a graph.

2. Preliminaries

Definition 2.1. For any positive integer n, the set of all positive integers that are less than n and relatively prime to n is known as the Euler totient set $\Phi(n)$. The cardinality of the set $\Phi(n)$ is denoted by the Euler totient function $\phi(n)$. Some properties of the Euler totient function $\phi(n)$ are listed below.

1. $\phi(n)$ is even for all $n > 2$.

2. $\phi(p) = p - 1$, if p is a prime number.

3. If p is a prime and n is a positive integer, then $\phi(p^n) = p^n - p^{(n-1)}$.

Definition 2.2. A bigraph or bipartite graph G is a graph whose vertex set can be partitioned into two non-empty disjoint subsets V_1 and V_2 such that each edge of G connects a vertex of V_1 and a vertex of V_2.

Remark 2.3. A graph is bipartite if and only if it does not contain any odd-length cycle.

Definition 2.4. A graph G is called connected if every pair of vertices is joined by a path.

Definition 2.5. The complement $\bar{G}$ of a graph G has V(G) as its point set, but two points are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

Definition 2.6. A graph without a loop and a parallel edge is called a complete graph if there exists an edge between every two distinct vertices.

Definition 2.7. ([1]) A set of integers is said to be pairwise relatively prime if every pair of distinct integers in the set has the greatest common divisor of 1.

Definition 2.8. A tree is a connected acyclic graph.

Definition 2.9. The set of integers a_1, a_2, ..., a_k is called a reduced residue system modulo n (written as RRS $\pmod{n}$) if

1. $\gcd(a_i, n) = 1$, $\forall i = 1, 2, \ldots, k$.
2. $a_i \not\equiv a_j \pmod{n}$, $\forall i \neq j$ and
3. If an integer m is relatively prime to n, then $m \equiv a_i \pmod{n}$ for a unique i, $1 \leq i \leq k$

Theorem 2.10. *For any positive integer n, the Euler function graph $G(\phi(n))$ is connected* [7].

Theorem 2.11. *$G(\phi(n))$ is complete iff every pair of vertices of $G(\phi(n))$ are relatively prime* [7].

3. Main results

Theorem 3.1. *$G(\phi(p))$ is not a bipartite graph for a prime number $p \geq 5$.*

Proof. Let p is any prime number such that $p \geq 5$. Since, p is a prime number this implies vertex set, $V(G(\phi(p))) = \{1, 2, 3, ..., (p-1)\}$. If possible, let $G(\phi(p))$ form a bipartite graph. Clearly, in the graph $G(\phi(p))$, vertex 1 is adjacent to all the vertices 2, 3,..., $(p-1)$. Since, $(1, n) = 1$ for all n belongs to { 2, 3,..., $(p-1)$}. Hence, vertex 1 is the only element of one of the partitions, (say)P_1. Therefore, it is obvious that the remaining vertices 2, 3,..., $(p-1)$ are in the other partition, (say) P_2. But some vertices of P_2 are adjacent, for example, 2 and 3 are adjacent. Since, for a bipartite graph, the elements of the same partition are not adjacent. This implies that graph G($\phi(p)$) is not bipartite for a prime number $p \geq 5$. □

Theorem 3.2. *$G(\phi(n))$ is not a bipartite graph if $n = 2p$, where p is a prime number and $p \geq 5$.*

Proof. Given $n = 2p$, where $p \geq 5$. Therefore, $V(G(\phi(n)))$ contains $\phi(2p) = (p-1)$ members and $V(G(\phi(2p))) = \{1, 3, 5, 7, ..., (2p-1)\} - \{p\}$. For a bipartite graph, the vertex set can be partitioned into two disjoint sets, and there does not exist an edge between any two vertices within the same set. Since 1 is adjacent to all the vertices, 1 is the only member of one of the partitions, and the set $\{3, 5, 7, 9, ..., (2p-1)\}$ is in the other partition. We will prove this theorem in the following two cases.

Case 1: If $p = 5$, then $V(G(\phi(2p))) = \{1, 3, 7, 9\}$. Since $(3, 7) = 1$, this implies that there exists an edge between 3 and 7. Therefore, $\{3, 7, 9\}$ is not within one partition. Hence, for $p = 5$, $G(\phi(2p))$ is not a bipartite graph.

Case 2: If $p \geq 7$, then $V(G(\phi(2p)) = \{1, 3, 5, 7, 9, ..., (2p-1)\} - \{p\}$. Since $(3, 5) = 1$, this implies that there exists an edge between 3 and 5. Therefore, $\{3, 5, 7, 9, ...(2p-1)\} - \{p\}$ is not within one partition. Therefore, $G(\phi(2p))$ is not a bipartite graph.

Hence, $G(\phi(n))$ is not a bipartite graph if $n = 2p$, where p is a prime number, and $p \geq 5$. □

Theorem 3.3. *If $G(\phi(n))$ is bipartite, then the sum of degrees of vertices of the graph $G(\phi(n))$ is $2(\phi(n) - 1)$.*

Proof. We know that, $G(\phi(n))$ is a graph with $\phi(n)$ vertices. If $G(\phi(n))$ is bipartite, then the vertices of $G(\phi(n))$ can be partitioned into two sets, say V_1 and V_2. Since the vertex 1 is adjacent to all the other vertices of $G(\phi(n))$, thus 1 is the only element in one of the sets, say V_1, and the other $(\phi(n) - 1)$ vertices are in the set, say V_2. Since there does not exist any edge between the vertices of set V_2. So, the bipartite graph has only $(\phi(n) - 1)$ edges, because 1 is adjacent to all the other $(\phi(n) - 1)$ vertices. Since we are aware that a graph's total degree of vertices equals twice its total number of edges. Hence, the total degree of vertices equals $2(\phi(n) - 1)$. □

Theorem 3.4. *$G(\phi(n))$ is not bipartite graph for an odd number $n \geq 5$, if there exist a prime $p \geq 3$ such that $p < n$ and p does not divide n.*

Proof. If n is an odd number, and $n \geq 5$ then V$(G(\phi(n)))$ contains 2, since $(2, n) = 1$. Again, if there exists a prime, $p \geq 3$ such that $p < n$ and p does not divide n, then V$(G(\phi(n)))$ also contains p. Since $(1, p) = (2, p) = (1, 2) = 1$, the vertices 1, 2 and p form a cycle of length 3. Since the graph $G(\phi(n))$ contains a cycle of odd length, it is not bipartite. □

Example 3.5. $G(\phi(9))$ is not bipartite. Since there exists a prime number 5, such that $5 < 9$ and 5 does not divide 9. The graph of $G(\phi(9))$ is shown in **Figure 1**.

Lemma 3.6. *$\phi(n) \geq 4$ for all natural numbers $n > 6$.*

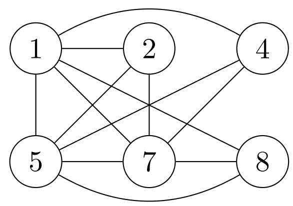

FIGURE 1. $G(\phi(9))$ graph.

Proof. We will prove this theorem in the following three cases for $n > 6$.

Case 1: Let $n = p^k$, where p is the only prime factor, and k belongs to natural number.

If $p = 2$, and $n = 2^k$, where $k \geq 3$, for $n > 6$. Therefore, $\phi(n) = \phi(2^k) = 2^{k-1} \geq 2^{(3-1)} = 4$. Since $k \geq 3$, this implies $\phi(n) \geq 4$.

If $p = 3$, and $n = 3^k$, where $k \geq 2$, for $n > 6$. Therefore, $\phi(n) = \phi(3^k) = 3^{k-1}(3-1) \geq 3.2 = 6$. Since $k \geq 2$, this implies $\phi(n) \geq 6$.

If $p = 5$, and $n = 5^k$, where $k \geq 2$, since $n > 6$. Therefore, $\phi(n) = \phi(5^k) = 5^{k-1}(5-1) \geq 5.4 = 20$. Since $k \geq 2$, this implies $\phi(n) \geq 20$.

If $p \geq 7$ and $n = p^k$, where $k \geq 1$ for $n > 6$. Therefore, $\phi(n) = \phi(p^k) = p^{k-1}(p-1) \geq 7^{(1-1)}(7-1) = 6$. Since $k \geq 1$ and this implies $\phi(n) \geq 6$. Therefore, if $n = p^k$ for $n > 6$. Then, $\phi(n) \geq 4$.

Case 2: Let $n = p_1^{k_1} \cdot p_2^{k_2}$, where p_1, p_2 are the only distinct primes, and k_1 and k_2 are natural numbers.

Sub-case 1: If $p_1 = 2$ and $p_2 = 3$, then at least one of k_1 and k_2 are greater than 1, let $k_1 \geq 2$ and $k_2 \geq 1$. Therefore, $\phi(n) = \phi(p_1^{k_1} \cdot p_2^{k_2})$=$p_1^{k_1} \cdot p_2^{k_2}(1-1/p_1)(1-1/p_2)$=$(p_1^{k_1} - p_1^{k_1-1}) \cdot (p_2^{k_2} - p_2^{k_2-1}) \geq (2^2 - 2^{2-1})(3-1) = 4$. This implies $\phi(n) \geq 4$. Again, let $k_1 \geq 1$ and $k_2 \geq 2$. Then, $\phi(n) = \phi(p_1^{k_1} \cdot p_2^{k_2}) \geq (2^1 - 2^{1-1})(3^2 - 3^{2-1}) = 6$. Therefore, $\phi(n) \geq 6$.

Sub-case 2: Let $n = p_1^{k_1} \cdot p_2^{k_2}$ for $n > 6$ let $p_1 \geq 2$ and $p_2 \geq 5$ and $k_1, k_2 \geq 1$. Therefore, $\phi(n)$=$\phi(p_1^{k_1} \cdot p_2^{k_2})$ =$(p_1^{k_1} - p_1^{k_1-1}) \cdot (p_2^{k_2} - p_2^{k_2-1}) \geq (2^1 - 1)(5^1 - 1) = 4$. Then, $\phi(n) \geq 4$. This implies $\phi(n) \geq 4$ for $n = p_1^{k_1} \cdot p_2^{k_2}$.

Case 3: Let $n = p_1^{l_1} \cdot p_2^{l_2} ... \cdot p_s^{l_s}$, where $s \geq 3$ and at least three l_i, $1 \leq i \leq s$ nonzero and all p_i are distinct primes, (Say) $p_1 < p_2 <...< p_s$ and l_i belong to whole numbers.

Therefore, $\phi(n)$=$\phi(p_1^{l_1} \cdot p_2^{l_2} ... p_s^{l_s}) = (p_1^{l_1} - p_1^{l_1-1}) \cdot (p_2^{l_2} - p_2^{l_2-1}) ... \cdot (p_s^{l_s} - P_s^{l_s-1})$ =$p_1^{l_1-1}(p_1 - 1) \cdot p_2^{l_2-1}(p_2-1) ... \cdot p_s^{l_s-1}(p_s-1) \geq 1.2.4 = 8$. This implies $\phi(n) \geq 8$.

In all three cases, we deduce $\phi(n) \geq 4$. Hence, $\phi(n) \geq 4$ for all natural numbers $n > 6$. □

Remark 3.7. The lower bound in **Lemma 3.6** is strict. Indeed, for all integers $n > 6$, $\phi(n) \geq 4$, and this bound is achieved for $n = 8$ and $n = 12$, because $\phi(8) = \phi(12) = 4$. As a result, inequality cannot be improved in general.

Remark 3.8. $G(\phi(n))$ has at least 4 vertices for all $n > 6$, since the number of vertices of $G(\phi(n))$ is $\phi(n)$.

Theorem 3.9. *Let,* $X_n = \{\, n_i \mid \gcd(n_i, n) = 1,\ 1 < n_i < n,\ n_i, n \in \mathbb{N}\,\}$*, then there exists atleast two elements* $n_i, n_j \in X_n$ *such that* $\gcd(n_i, n_j) = 1$ *, for* $i \neq j$*,* $\forall n > 6$.

Proof. It is evident that $\Phi(n) = \{\, n_i \mid \gcd(n_i, n) = 1,\ 1 < n_i < n,\ n_i, n \in \mathbb{N}\,\}$. Therefore, $\Phi(n) = 1 \cup X_n$ and already we established that $|\Phi(n)| = \phi(n) \geq 4, \forall n > 6$. This implies $|X_n| \geq 3$, $\forall n > 6$.

Since 1 is excluded from X_n, there exists at least one integer $a \in X_n$ with $2 \leq a \leq n-2$. The proof is carried out in the following steps.

Step 1: We create an additional element in X_n without sacrificing generality. Take the integer $b = n - a$, $b \in (1, n)$ since $2 \leq n - a = b$. Therefore, $\gcd(a, n) = 1 \implies \gcd(n-a, n) = 1 \implies \gcd(b, n) = 1$. So, $b \in X_n$.

Step 2: In this step, we demonstrate that a and b are distinct. If $a = b$, then $a = n/2$, implying that $\gcd(a, n) = \gcd(n/2, n) = n/2 \neq 1$, a contradiction. Thus, a and b are distinct integers.

Step 3: In this stage, we demonstrate that $\gcd(a, b) = 1$. This becomes apparent through a contradiction. If $\gcd(a, b) = d > 1$, then $d \mid a$ and $d \mid b$, which implies that $d \mid n - a$. Thus, $\gcd(a, n) \geq d > 1$ is a contradiction. Our assumption is wrong, which means $\gcd(a, b) = 1$.

From the preceding steps, it follows that there exist at least two distinct integers $a, b \in X_n$ satisfying $\gcd(a, b) = 1$, $\forall n > 6$. □

Remark 3.10. If $a_1, a_2, \ldots, a_{\phi(n)}$ is an RRS modulo n, provided that $1 \leq a_i < n$ and $n > 6$, then there exist at least two members of the RRS different from 1, such that their greatest common divisor is 1.

Corollary 3.11. *Suppose* $A_n = \{b_1, b_2, b_3, \ldots, b_{\phi(n)}\}$ *forms an RRS modulo n. Then, suppose that we remove the element that is congruent to* $1 \pmod{n}$ *from* X_n*. In that case, there will still exist at least two different numbers in* A_n *whose linear combination with integer coefficients is congruent to 1 modulo n, provided that* $n > 6$.

Proof. It is evident that, $1 \cup X_n$ is an RRS modulo n, where, $X_n = \{\, n_i \mid \gcd(n_i, n) = 1,\ 1 < n_i < n,\ n_i, n \in \mathbb{N}\,\}$. For convenience, let us assume $X_n = \{a_2, a_3, \ldots, a_{\phi(n)}\}$. Since A_n is also an RRS modulo n, $\exists$ a unique b_t in A_n, such that $b_t \equiv 1 \pmod{n}$. We can assume two elements a_i, a_j in X_n such that $\gcd(a_i, a_j) = 1$, for $i \neq j$ [as a consequence of **Theorem 3.9**]. We have to show that there exist at least two elements in A_n except b_t such that their greatest common divisor is 1.

As we know, $1 \cup X_n$ and A_n both are RRS modulo n, thus $a_i \equiv b_l \pmod{n}$ and $a_j \equiv b_m \pmod{n}$ and any of the b_l and b_m is not congruent to 1 modulo n. Therefore, $a_i = b_l + nk_1$ and $a_j = b_l + nk_2$ for some integers k_1 and k_2. As a consequence of $\gcd(a_i, a_j) = 1$, we can write $Aa_i + Ba_j = 1$ for some integers A, B. This implies $A(b_l + nk_1) + B(b_m + nk_2) = 1, \implies Ab_l + Bb_m + n(Ak_1 + Bk_2) = 1$. Hence, $Ab_l + Bb_m \equiv 1 \pmod{n}$. □

Corollary 3.12. *$G(\phi(n))$ is not a bipartite graph for $n > 6$.*

Proof. Let, the vertex set of $G(\phi(n))$ is $1 \cup X_n$, where $X_n = \{\, n_i \mid \gcd(n_i, n) = 1,\ 1 < n_i < n,\ n_i, n \in \mathbb{N} \,\}\ \forall\ n > 6$. If $G(\phi(n))$ is a bipartite $\forall\ n > 6$, then its vertices can be partitioned into two disjoint sets. Since 1 is adjacent to all other remaining vertices, it is the only element of one of the partitions. Thus, to exhibit bipartiteness, all other elements except 1 must be in the other partition. But there exist at least two distinct vertices of $G(\phi(n))$ different from 1 that are adjacent, because by **Theorem 3.9** , we find that there exist at least two different vertices of $G(\phi(n))$ exist, except 1, such that their gcd is 1 $\forall\ n > 6$. Therefore, at least one element is not in any of the two partitions. Therefore, $G(\phi(n))$ is not a bipartite graph $\forall$ $n > 6$. □

Corollary 3.13. *$G(\phi(n))$ is complete iff the Euler totient set $\Phi(n)$ is relatively prime in pairs.*

Proof. From the **Theorem 2.11**, we know that $G(\phi(n))$ is complete if and only if every pair of vertices of $G(\phi(n))$ is relatively prime. Again, we know that the set of vertices of $G(\phi(n))$ is $\Phi(n)$; this means that $G(\phi(n))$ is complete iff the Euler totient set $\Phi(n)$ is relatively prime in pairs. □

Proposition 3.14. *Euler totient set $\Phi(n)$ is not pairwise relatively prime for all odd numbers $n \geq 5$.*

Proof. Let n be an odd number, and $n \geq 5$. Then, $\Phi(n)$ contains 2 and 4. Since, $(2, n) = (4, n) = 1$ and $(2, 4) \neq 1$. Therefore, $\Phi(n)$ is not relatively prime in pairs. □

Corollary 3.15. *$G(\phi(n))$ is not complete if n is an odd integer and $n \geq 5$, but the converse does not need to be true.*

Proof. Let n be any odd integer, and $n \geq 5$. Therefore, $V(G(\phi(n)))$ contains 2 and 4 for all odd integers $n \geq 5$. Since, $(2, n) = (4, n) = 1$ and $(2, 4) = 2 \neq 1$. Therefore, there is no edge between 2 and 4. Hence, $G(\phi(n))$ is not complete if n is an odd integer and $n \geq 5$. □

The Converse of this corollary is not true, since $G(\phi(14))$ is not complete, but 14 is not an odd number. The set of vertices of $G(\phi(14)) = \{1, 3, 5, 9, 11, 13\}$, and the graph of $G(\phi(14))$ is shown in **Figure 2**.

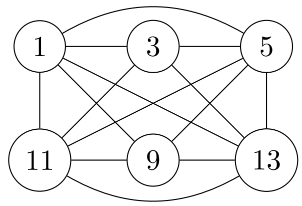

FIGURE 2. $G(\phi(14))$ graph.

Proposition 3.16. *Euler totient set $\Phi(n)$ is not pairwise relatively prime if there exists a prime p such that p does not divide n and $p^2 < n$.*

Proof. Given that, there exists a prime p such that p does not divide n and $p^2 < n$. Then, the set $\Phi(n)$ contains p and p^2. Since, $(p, n) = (p^2, n) = 1$. But $(p, p^2) = p \neq 1$. Therefore, $\Phi(n)$ is not pairwise relatively prime if there exists a prime p such that p does not divide n, and $p^2 < n$. □

Corollary 3.17. *$G(\phi(n))$ is not complete if there exists a prime p such that p does not divide n and $p^2 < n$.*

Proof. Given that, there exists a prime p such that p does not divide n and $p^2 < n$. Therefore, $V(G(\phi(n)))$ contains p and p^2. Since $(p, n) = (p^2, n) = 1$ and p, $p^2 < n$. But $(p, p^2) = p \neq 1$. This implies that there is no edge between the vertices p and p^2. Hence, $G(\phi(n))$ is not complete if there exists a prime p such that p does not divide n and $p^2 < n$. □

Example 3.18. $G(\phi(50))$ is not complete. Since there exists a prime number 7 such that 7 does not divide 50 and $7^2 = 49 < 50$. As a result, there is no edge between vertices 7 and 49.

Proposition 3.19. *Euler totient set $\Phi(n)$ is not pairwise relatively prime if $n = 2p$, where p is any prime and $p \geq 5$.*

Proof. Given $n = 2p$, where p is any prime, and $p \geq 5$. Therefore, the set $\Phi(2p)$ contains $\phi(2p) = (p - 1)$ members and the set $\Phi(2p) = \{1, 3, 5, 7, ..., (2p - 1)\} - \{p\}$. Therefore, the set $\Phi(2p)$ contains 3 and 9 for $p \geq 5$. Since, $(3, 9) = 3 \neq 1$. Hence, the set $\Phi(2p)$ is not pairwise relatively prime for $p \geq 5$. □

Corollary 3.20. *$G(\phi(n))$ is not complete if $n = 2p$ for $p \geq 5$, where p be any prime.*

Proof. Given $n = 2p$, where $p \geq 5$. Therefore, $V(G(\phi(n)))$ contains $\phi(n) = \phi(2p) = p-1$ members. Therefore, $V(G(\phi(2p))) = \{1, 3, 5, 7, ..., (2p-1)\} - \{p\}$. This implies that $V(G(\phi(2p)))$ contains 3 and 9, for $p \geq 5$. Since, $(3, 9) = 3 \neq 1$. Therefore, there is no edge between 3 and 9. Hence, $G(\phi(n))$ is not complete for $n = 2p$, where p is any prime number, and $p \geq 5$. □

Example 3.21. $G(\phi(10))$ is not complete. Since $10 = 2p$, where $p = 5$ is a prime number. It is also clear from the following Figure 3 that the graph $G(\phi(10))$ is not complete.

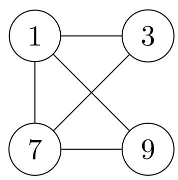

FIGURE 3. $G(\phi(10))$ graph.

Theorem 3.22. *$\bar{G}(\phi(n))$ is a disconnected graph for all natural number, $n \geq 3$.*

Proof. From **Theorem 2.10**, we know that $G(\phi(n))$ is a connected graph for any positive integer n, and also we know that $\phi(n) \geq 2$ for all $n \geq 3$; which implies that $G(\phi(n))$ contains at least 2 members. Since vertex 1 of V$(G(\phi(n)))$ is adjacent to all other vertices of V$(G(\phi(n)))$, so the vertex 1 is isolated vertex of $\bar{G}(\phi(n))$. Hence, $\bar{G}(\phi(n))$ is disconnected. □

Lemma 3.23. *Let p_i denote the i-th prime number. Then $\forall\ n \geq 4$, $\prod_{i=1}^{n} p_i > p_{n+1}^2$.*

Proof. We prove the inequality $\prod_{i=1}^{n} p_i > p_{n+1}^2$, by using the principle of mathematical induction.

Let $n = 4$, then $\prod_{i=1}^{4} p_i = p_1.p_2.p_3.p_4 = 2.3.5.7 = 210 > p_5^2 = 11^2 = 121$. Therefore the inequality under consideration is true for $n = 4$. Assume that the inequality under consideration is true for $n = k$. Then

$$\prod_{i=1}^{k} p_i > p_{k+1}^2. \tag{1}$$

Thus, we have to prove that the inequality under consideration is true for $n = k + 1$. After multiplying both sides of inequality (1) by p_{k+1} yields

$$\prod_{i=1}^{k+1} p_i > p_{k+1}^3. \tag{2}$$

According to **Bertrand's Postulate**, there exists atleast one prime in the interval $(p_{k+1}, 2p_{k+1})$, as a result

$$p_{k+2} < 2p_{k+1}. \tag{3}$$

Squaring both sides of inequality (3) yields

$$p_{k+2}^2 < 4p_{k+1}^2, \tag{4}$$

$$\implies p_{k+2}^2 < p_{k+1}p_{k+1}^2 = p_{k+1}^3. \tag{5}$$

Since $p_{k+1} > p_4 = 7 > 4$.

Using inequalities (2) and (5), we obtain $\prod_{i=1}^{k+1} p_i > p_{k+2}^2$. As a result, the principle of mathematical induction states that $\prod_{i=1}^{n} p_i > p_{n+1}^2$, $\forall n \geq 4$. □

Theorem 3.24. *For all natural numbers $n > 30$, there exists a prime number p such that p does not divide n and $p^2 < n$.*

Proof. We argue by contradiction. Suppose that there exists an integer $n > 30$ such that every prime p divides n whenever $p^2 < n$. Consequently, n is divisible by $\prod_{p^2<n} p$. Thus, there exists an integer $n > 30$ serving as a counterexample with the property that all primes p satisfying $p^2 < n$ divide n. In the cases given below, we attempt to find a counterexample.

Case 1: Let $30 < n \leq 49$, then the primes 2, 3 and 5 are the only primes that satisfy $p^2 < n$. But, $\prod_{p^2<n} p = 2.3.5 = 30$ and $\prod_{p^2<n} p$ does not divide n for $30 < n \leq 49$. In this instance, a counterexample is not feasible.

Case 2: Let $49 < n \leq 121$, then the primes 2, 3, 5 and 7 are the only primes that satisfy $p^2 < n$. However, $\prod_{p^2<n} p = p_1.p_2.p_3.p_4 = 2.3.5.7 = 210$ and $\prod_{p^2<n} p > 11^2 = p_5^2 = 121 \geq n$. Therefore, for $49 < n \leq 121$, $\prod_{p^2<n} p$ does not divide n. Therefore, the counterexample is similarly impossible in this instance.

Case 3: For all $n > 121$ we get a largest prime number p_k with $p_k^2 < n \leq p_{k+1}^2$ and $\prod_{p^2<n} p = p_1.p_2...p_k$. Using the **Lemma 3.23** we get that $\prod_{p^2<n} p = \prod_{i=1}^{k} p_i > p_{k+1}^2$ and $p_{k+1}^2 \geq n$ then, $\prod_{p^2<n} p > n$ and it follows that $\prod_{p^2<n} p$ does not divide n. For each natural number $n > 30$, there is no counterexample.

Our assumption is therefore incorrect, meaning that for all natural numbers $n > 30$, there exists a prime number p such that p does not divide n and $p^2 < n$. □

Corollary 3.25. *$\Phi(n)$ is not pairwise relatively prime for any natural number $n > 30$.*

Proof. By **Theorem 3.24**, for every natural numbers $n > 30$, there exists a prime p such that $p \nmid n$ and $p^2 < n$. Consequently, both p and p^2 belong to the set $\Phi(n)$. Since $\gcd(p, p^2) = p \neq 1$, it follows that $\Phi(n)$ is not pairwise relatively prime for any natural number $n > 30$. □

Theorem 3.26. *$G(\phi(n))$ is not complete for any natural number $n > 30$*

Proof. **Theorem 3.24** states that for all natural numbers $n > 30$, there exists a prime number p that does not divide n and holds the property $p^2 < n$, so p and $p^2 \in \Phi(n)$, $\gcd(p, p^2) = p \neq 1$. According to the definition of $G(\phi(n))$, p and p^2 are the vertices of $G(\phi(n))$, and there is no edge between p and p^2, as $\gcd(p, p^2) = p \neq 1$. Therefore, $G(\phi(n))$ is not complete. □

4. Discussions

In this article, we investigate the bipartiteness and completeness of the Euler function graph. After that, we studied the Euler function and the reduced residue system with its help. During the investigation, we found a connection between the graph of the Euler function and the reduced residue system. These results contribute to the theoretical understanding of graph properties and suggest directions for future research in the intersection of graph theory and number theory.

In this article, we look at some new prime number features. We hope that this graph, and others of its kind connected to number theory, can help uncover new characteristics or concepts in the future.

5. Acknowledgments

The authors are grateful to the anonymous referees for their insightful comments and detailed suggestions that significantly enhanced the quality of this manuscript.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer Science and Business Media, 2013.

[2] J. B. Babujee, *Euler's phi function and graph labeling*, Int. J. Contemp. Math. Sci, **5** No. 20 (2010) 977-984.

[3] T. Chalapathi and R. Kumar, *Graph structures of Euler totient numbers*, DIUJST, **11** No. 2 (2016) 19-29.

[4] N. Ghanbari and S. Alikhani, *A graph related to the Euler ϕ function*, Math. Gaz., **107** No. 569 (2023) 263-272.

[5] F. Harary, *Graph Theory (on Demand Printing of 02787)*, CRC Press, 2018.

[6] Y. Kaneda, J.D. Laison, J. Schreiner-McGraw and C. Starr, *Prime power and prime product distance graphs*, Discrete Appl. Math., **255** (2019) 334-338.

[7] S. Shanmugavelan, *The Euler function graph G ($\phi(n)$)*, Int. J. Pure Appl. Math, **116** (2017) 45-48.

[8] S. Shanmugavelan and C. Natarajan, On Divisor Function Even (Odd) Sum Graphs, In *In Nonlinear Dynamics and Applications: Proceedings of the ICNDA 2022*, Springer International Publishing, 2022.

Manas Pratim Kalita

Department of Mathematics, Gauhati University,

Guwahati, Assam, India.

kalitamanaspratim6@gmail.com

Mridul Dutta

Department of Mathematics, Dudhnoi college,

Goalpara, Assam, India.

mridulduttamc@gmail.com