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Research Paper

ON THE STRONG EDGE METRIC DIMENSION OF GRAPHS

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Abstract. A vertex v in a connected graph G strongly distinguishes two edges e_1 and e_2 if $\mathrm{d}(e_1, v) = \mathrm{d}(e_2, v) + \mathrm{d}(e_1, e_2)$ or $\mathrm{d}(e_2, v) = \mathrm{d}(e_1, v) + \mathrm{d}(e_1, e_2)$, where for an edge $e = xy$, $\mathrm{d}(e, v) = \min\{\mathrm{d}(x, v), \mathrm{d}(y, v)\}$ and also the distance between two edges $e_1 = xy$ and $e_2 = uv$ is $\mathrm{d}(e_1, e_2) = \min\{\mathrm{d}(e_1, u), \mathrm{d}(e_1, v)\}$. A nonempty set $S \subseteq V(G)$ is a strong edge metric generator of a graph G if for any two distinct edges $e_1, e_2 \in E(G)$, there exists a vertex $s \in S$ such that s strongly distinguishes e_1 and e_2. A strong edge metric generating set with the smallest number of elements is called a strong edge metric basis of G and the number of elements in a strong edge metric basis is called the strong edge metric dimension of G and it is denoted by $\mathrm{sedim}(G)$. In this paper, we propose this concept as an extension of the strong metric dimension and we determine the strong edge metric dimension of several classes of graphs. Moreover, the sedim for the G-generalized join graph is investigated. Furthermore, we study the strong edge metric dimension of the comaximal graph of the ring of integers modulo n. Finally, we pose an integer linear programming model for finding the strong edge metric dimension of graphs.

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1. INTRODUCTION

Let $G = (V, E)$ be a simple connected graph and $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We denote the number of vertices of G by $|G|$. For distinct vertices u and v of G, we write $u \sim v$ if u and v are adjacent in G and the edge e between u and v is denoted by $e = uv$. Also the *distance* between two distinct vertices u and v, denoted by $\mathrm{d}(u, v)$, is the length of the shortest path connecting u and v, if such a path exists; otherwise, we set $\mathrm{d}(u, v) := \infty$. The distance between an edge $e = xy$ and a vertex v is defined as follows:

$$\mathrm{d}(e, v) = \min\{\mathrm{d}(x, v), \mathrm{d}(y, v)\}.$$

Also the distance between two edges $e_1 = xy$ and $e_2 = uv$ is defined as follows:

$$\mathrm{d}(e_1, e_2) = \min\{\mathrm{d}(e_1, u), \mathrm{d}(e_1, v)\}.$$

A vertex x *resolves* two vertices u and v if $\mathrm{d}(x, u) \neq \mathrm{d}(x, v)$. A nonempty set $S \subseteq V(G)$ is a *resolving set* of a graph G if for any two distinct vertices of G there exists a vertex of S which resolves them. A resolving set with the smallest number of elements is called the *metric basis* of G and the number of elements in a metric basis is called the *metric dimension* of G and it is denoted by $\dim(G)$. Also a vertex w *strongly resolves* two vertices u and v if $\mathrm{d}(w, u) = \mathrm{d}(w, v) + \mathrm{d}(v, u)$ or $\mathrm{d}(w, v) = \mathrm{d}(w, u) + \mathrm{d}(u, v)$. A nonempty set $S \subseteq V(G)$ is a *strongly resolving set* of a graph G if for any two distinct vertices of G there exists a vertex of S which strongly resolves them. A strongly resolving set with the smallest number of elements is called the *strong metric basis* of G, and the number of elements in a strong metric basis is called the *strong metric dimension* of G. It is denoted by $\mathrm{sdim}(G)$. A vertex v *distinguishes* two edges e_1 and e_2 if $\mathrm{d}(e_1, v) \neq \mathrm{d}(e_2, v)$. A nonempty set $S \subseteq V(G)$ is an *edge metric generator* of a graph G if for any two distinct edges $e_1, e_2 \in E(G)$, there exists a vertex $s \in S$ such that s distinguishes e_1 and e_2. An edge metric generating set with the smallest number of elements is called an *edge metric basis* of G, and the number of elements in an edge metric basis is called the *edge metric dimension* of G. It is denoted by $\mathrm{edim}(G)$.

The concept of metric dimension was first introduced by Slater in 1975 [22]. Since then lots of work has been done on these topics because of their wide range of applications in modeling of real world problems [11, 9, 3]. For instance, Garey and Johnson in [8], and Epstein, Levin, and Woeginger in [7] studied NP-hardness of computing of metric dimension. Also, this invariant was investigated over the Cartesian product of graphs in [5], over the lexicographic product of graphs in [19], over the deleted lexicographic product of graphs in [6], and over the hierarchical product of graphs in [24]. A close parameter to the metric dimension is the strong metric dimension of a graph which was introduced in [21]. The problem of determining the strong metric dimension of a graph is NP-hard. Many researchers obtained the strong metric dimension for various classes of graphs, see [13, 14, 17]. For more detail on the study of strong metric dimension of a graph, one can refer to [12]. In 2018, Kelenc, Tratnik and Yero introduced the concept of edge metric dimension [10] in analogy with the classical metric dimension $\dim(G)$, which was introduced in [22]. See [27] for more details on $\mathrm{edim}(G)$. Recently in [1, 16], the edge metric dimension of some graph operations was investigated.

In the present work, we expand the concept of strong metric dimension as strong edge metric dimension which is a close parameter to the edge metric dimension. We say that a vertex v *strongly distinguishes* two edges e_1 and e_2 if $\mathrm{d}(e_1, v) = \mathrm{d}(e_2, v) + \mathrm{d}(e_1, e_2)$ or $\mathrm{d}(e_2, v) = \mathrm{d}(e_1, v) + \mathrm{d}(e_1, e_2)$. A nonempty set $S \subseteq V(G)$ is a *strong edge metric generator* of a graph G if for any two distinct edges $e_1, e_2 \in E(G)$, there exists a vertex $s \in S$ such that s strongly distinguishes e_1 and e_2. A strong edge metric generating set with the smallest number of elements is called a *strong edge metric basis* of G, and the number of elements in a strong edge metric basis is called the *strong edge metric dimension* of G and we denot it by $\mathrm{sedim}(G)$.

In Section 2 of this paper, we determine the strong edge metric dimension of several classes of graphs. Moreover, the sedim for the G-generalized join graph is investigated. In Section 3, we study the strong edge metric dimension of the comaximal graph of the ring of integers modulo n. In Section 4, we present an integer linear programming model for computing sedim of graphs.

Throughout this paper our notation is standard and taken mainly from [4].

2. Basic results

In this section, we present some basic results on the strong edge metric dimension of graphs and we determine the strong edge metric dimension of several classes of graphs.

The following proposition gives the sedim of the path P_n.

Proposition 2.1. *Let $n \geqslant 2$. We have*

$$\mathrm{sedim}(P_n) = \begin{cases} 1; & \textit{if } \ n = 2, \\ n-2; & \text{otherwise.} \end{cases}$$

Proof. If $n = 2$, then clearly $\mathrm{sedim}(P_n) = 1$. Assume that $n \geqslant 3$ and let S be a strong edge metric generating set. For each two adjacent edges e_1 and e_2, since $\mathrm{d}(e_1, e_2) = 0$, only their common vertex strongly distinguishes e_1 and e_2, and so it should belong to S. Hence all vertices of degree two belong to S. Thus $\mathrm{sedim}(P_n) \geqslant n - 2$. Now suppose that $X = V(P_n) \setminus \{v_1, v_n\}$, where v_1, v_n are the vertices of P_n of degree one. Then X is a strong edge metric generator of P_n and so $\mathrm{sedim}(P_n) \leqslant n - 2$. Therefore $\mathrm{sedim}(P_n) = n - 2$. $\square$

In the complete graph K_n, clearly $\mathrm{sedim}(K_2) = 1$. Also for $n = 3, 4$, since for each two adjacent edges, either their common vertex or a vertex whose distance to the edges are equal, can strongly distinguishe them, we have $\mathrm{sedim}(K_n) = 3$. In the following proposition we determine the sedim of the complete graph K_n, when $n \geqslant 5$.

Proposition 2.2. *Let $n \geqslant 5$. We have*

$$\mathrm{sedim}(K_n) = \begin{cases} 3; & \textit{if } n = 5, \\ n - 3; & \text{otherwise.} \end{cases}$$

Proof. Let $n \geqslant 5$ and S be a strong edge metric generating set. If $|S| \leqslant n - 4$, then there are vertices $a, b, c, d \notin S$. Consider the edges ab and cd. Now for each $x \in S$, we have $\mathrm{d}(x, ab) = \mathrm{d}(x, cd) = \mathrm{d}(ab, cd) = 1$. So none of the vertices of S can strongly distinguish the edges ab and cd, which is impossible. Hence $\mathrm{sedim}(K_n) \geqslant n - 3$. Let $n = 5$ and $V(K_5) = \{a, b, c, d, e\}$. If $S = \{d, e\}$ is a strong edge metric generating set, then none of the vertrices d or e can strongly distinguish the edges ae and ad. So $|S| \geqslant 3$. Also $S = \{a, d, e\}$ is a strong edge metric generating set for K_5, and so $\mathrm{sedim}(K_5) = 3$. Now let $n \geqslant 6$ and consider the set $S' = V(K_n) \setminus \{a, b, c\}$, where a, b, c are arbitrary vertices of K_n. Since $n \geqslant 6$, there exist vertices $s_1, s_2, s_3 \in S'$. Clearly each two edges of ab, ac, bc can be strongly distinguished by either of the elements of S' and the edges as_1 and as_2 can be strongly distinguished by s_3. Now one can see that S' is a strong edge metric generating set. So $\mathrm{sedim}(K_n) \leqslant n - 3$, and therefore $\mathrm{sedim}(K_n) = n - 3$, for $n \geqslant 6$. □

The following proposition gives the sedim of the cycle C_n.

Proposition 2.3. *Let $n \geqslant 3$. We have*

$$\mathrm{sedim}(C_n) = \begin{cases} n; & \textit{if } n \text{ is odd,} \\ \frac{n}{2}; & \textit{if } n \text{ is even.} \end{cases}$$

Proof. Let S be a strong edge metric generating set of C_n. We know that for each two adjacent edges, either their common vertex or a vertex whose distance to the edges are equal, can strongly distinguish them. So when n is odd, for each two adjacent edges of C_n, only their common vertex can strongly distinguish them and hence all vertices of C_n should be in S. Therefore $\mathrm{sedim}(C_n) = n$ when n is odd. Now suppose that n is even and $V(C_n) = \{g_1, g_2, \ldots, g_n\}$. In this situation, for strongly distinguishing adjacent edges, half of the vertices should be in S, and so $|S| \geqslant \frac{n}{2}$. On the other hand, one can easily see that the set $\{g_1, g_2, \ldots, g_{\frac{n}{2}}\}$ forms a strong edge metric generating set of order $\frac{n}{2}$. Thus $\mathrm{sedim}(C_n) = \frac{n}{2}$ when n is even. □

Recall that the *wheel graph* $W_{1,n}$ is the graph obtained from a cycle C_n and the graph K_1 by adding all the edges between the vertex of K_1 and every vertex of C_n.

The least integer greater than or equal to a number m is denoted by $\lceil m \rceil$. Also, the greatest integer less than or equal to a number m is denoted by $\lfloor m \rfloor$.

In the following proposition, we investigate the strong edge metric dimension $\mathrm{sedim}(W_{1,n})$.

Proposition 2.4. *Let* $W_{1,n}$ *be a wheel graph. Then*

$$\mathrm{sedim}(W_{1,n}) = \begin{cases} 3; & \text{if } n \leqslant 7, \\ 3 + \lceil \frac{n-9}{2} \rceil; & \text{if } n \geqslant 8. \end{cases}$$

Proof. It is clear that $W_{1,3} = K_4$. By Proposition 2.2, $\mathrm{sedim}(W_{1,3}) = 3$. So assume that $n \geqslant 4$. Let $\{g_1, g_2, \ldots, g_n\}$ be the vertices of degree 3 in $W_{1,n}$. For each two distinct edges e and e', and each vertex x in $W_{1,n}$, we have $\mathrm{d}(e, e') \in \{0, 1, 2\}$ and $\mathrm{d}(x, e) \in \{0, 1, 2\}$. If $\mathrm{d}(e, e') = 0$, then there exists a vertex that has a same distance from the edges e and e' and can strongly distinguish them. If $\mathrm{d}(e, e') = 1$, then either a vertex which is an end vertex of e or e' can strongly distinguish them, or a vertex which has distance one from one of the edges and distance two from the other one can strongly distinguish them. Also, if $\mathrm{d}(e, e') = 2$, then only an end vertex of either e or e' can strongly distinguish them. Now, for $n \leqslant 7$, one can see that the set $\{g_1, g_2, g_3\}$ is a strong edge metric basis for $W_{1,n}$. Assume that $n \geqslant 8$. Let S be a minimal strong edge metric generating set. Since $\mathrm{d}(g_1g_2, g_4g_5) = \mathrm{d}(g_3g_4, g_6g_7) = \mathrm{d}(g_5g_6, g_ng_1) = 2$ and also $\mathrm{d}(g_2g_3, g_5g_6) = \mathrm{d}(g_4g_5, g_7g_8) = \mathrm{d}(g_6g_7, g_1g_2) = 2$, without loss of generality, we assume that $g_2, g_4, g_6 \in S$. Since from each two edges of distance two on the path $g_7 \sim g_8 \sim \cdots \sim g_{n-2} \sim g_{n-1} \sim g_n \sim g_1$, at least one of their end vertices should belong to S, half of the vertices on the path $g_7 \sim g_8 \sim \cdots \sim g_{n-2}$, which is $\lceil \frac{(n-2)-7}{2} \rceil$ vertices should be in S. Hence $|S| \geqslant 3 + \lceil \frac{n-9}{2} \rceil$. Also one can easily see that if n is an even number, then $S = \{g_2, g_4, g_6, \ldots, g_{n-2}\}$ is a strong edge metric generating set, and for an odd number n, $S = \{g_2, g_4, g_6, \ldots, g_{n-3}\}$ is a strong edge metric generating set of order $3 + \lceil \frac{n-9}{2} \rceil$. So $\mathrm{sedim}(W_{1,n}) = 3 + \lceil \frac{n-9}{2} \rceil$, when $n \geqslant 8$. □

Similarly to the wheel graph, the *fan graph*, which is denoted by $F_{1,n}$, is the graph that is obtained from a path P_n and the graph K_1 by adding all the edges between the vertex of K_1 and every vertex of P_n.

In the following proposition, we investigate the sedim of $F_{1,n}$.

Proposition 2.5. *For the fan graph $F_{1,n}$ we have*

$$\operatorname{sedim}(F_{1,n}) = \begin{cases} 1; & \textit{if } n=1,\\ 3; & \textit{if } n=2,\\ 2; & \textit{if } n=3,4,5,\\ 2\lfloor \frac{n}{5} \rfloor + 1; & \textit{if } n \equiv 1(\text{mod}5),\\ 2\lceil \frac{n}{5} \rceil; & \text{otherwise.} \end{cases}$$

Proof. By Propositions 2.3 and 2.2, we have $\operatorname{sedim}(F_{1,1}) = 1$ and $\operatorname{sedim}(F_{1,2}) = 3$. Let $\{g_1, g_2, \ldots, g_n\}$ be the vertices of the path P_n in the structure of $F_{1,n}$ and x be the vertex of degree n. For $n = 3,4,5$, clearly the set $S = \{g_2, g_3\}$ is a strong edge metric basis for $F_{1,n}$. Let $n \geqslant 6$ and S be a minimal strong edge metric generating set. For each two distinct edges e and e' with zero distance, only a vertex that has a same distance from the edges e and e' can strongly distinguish them. Since $\mathrm{d}(g_1g_2, xg_1) = \mathrm{d}(g_2g_3, xg_2) = \mathrm{d}(g_2g_3, xg_3) = \mathrm{d}(g_3g_4, xg_3) = 0$, we have $g_2, g_3 \in S$. Note that g_3 strongly distinguishes the edges g_4g_5 and xg_5. Now we consider the following cases:

Case 1. $n = 5k$, for some positive integer k. In this case the set

$$S = \{g_2, g_3, g_7, g_8, \ldots, g_{5t+2}, g_{5t+3}, \ldots, g_{5k-3}, g_{5k-2}\},$$

where $0 \leqslant t \leqslant (k-1)$, is a strong edge metric generating set. Also it is a strong edge metric basis for $F_{1,n}$ with $|S| = 2k = 2\lceil \frac{n}{5} \rceil$.

Case 2. $n = 5k+4$, for some positive integer k. In this case one can easily see that the set

$$S = \{g_2, g_3, g_7, g_8, \ldots, g_{5t+2}, g_{5t+3}, \ldots, g_{5k+2}, g_{5k+3}\},$$

where $0 \leqslant t \leqslant k$, is a strong edge metric basis for $F_{1,n}$ with $|S| = 2(k+1) = 2\lceil \frac{n}{5} \rceil$.

Case 3. $n = 5k+1$, for some positive integer k. In this case one can easily see that the set

$$S = \{g_2, g_3, \ldots, g_{5t+2}, g_{5t+3}, \ldots, g_{5k-3}, g_{5k-2}, g_{5k-1}\},$$

where $0 \leqslant t \leqslant (k-1)$, is a strong edge metric basis for $F_{1,n}$ with $|S| = 1 + 2k = 2\lfloor \frac{n}{5} \rfloor + 1$.

Case 4. $n \in \{5k+2, 5k+3\}$, for some positive integer k. In this case one can easily see that the set

$$S = \{g_2, g_3, \ldots, g_{5t+2}, g_{5t+3}, \ldots, g_{5k-3}, g_{5k-2}, g_{5k}, g_{5k+1}\},$$

where $0 \leqslant t \leqslant (k-1)$, is a strong edge metric basis for $F_{1,n}$ with $|S| = 2 + 2k = 2\lceil \frac{n}{5} \rceil$.

Now, by considering the above cases, the results hold. □

Proposition 2.6. *Let G be a connected graph. Then* $\operatorname{sedim}(G) = 1$ *if and only if G is a star graph.*

Proof. If G is a star graph with center x, then $S = \{x\}$ is a strong edge metric generating set and so $\mathrm{sedim}(G) = 1$. Now assume that G is a connected graph with n vertices and $\mathrm{sedim}(G) = 1$. So there exists a vertex x such that $S = \{x\}$ is a strong edge metric generating set. If $n = 2$, then we are done. Hence let $n \geqslant 3$. Assume on the contrary that G is not a star graph, and so there are at least two vertices of degree two. Let $a \neq x$ be a vertex of degree two. If $a \sim x$, then since $\deg(a) \geqslant 2$, there exists a vertex $b \neq x$ such that $a \sim b$. Since $\mathrm{d}(x, ax) = 0 = \mathrm{d}(ax, ab)$, the vertex x can not strongly distinguish the edges ax and ab, which is impossible. Now assume that a and x are not adjacent. Let $a = a_0 \sim a_1 \sim \cdots \sim a_n \sim x$ be a shortest path connecting a to x. Then one can easily see that the vertex x can not strongly distinguish the edges $a_n x$ and $a_{n-1}a_n$, which is again impossible. Therefore the result holds. □

Proposition 2.7. *For the complete bipartite graph $K_{m,n}$ we have*

$$\mathrm{sedim}(K_{m,n}) = \begin{cases} 1; & \textit{if } m = 1 \text{ or } n = 1, \\ 2; & \textit{if } 2 = m \leqslant n, \\ 3; & \textit{if } 3 = m \leqslant n, \\ \min\{m-1, n-1\} & \text{otherwise.} \end{cases}$$

Proof. By Proposition 2.6, we have $\mathrm{sedim}(K_{m,n}) = 1$, for $m = 1$ or $n = 1$. Now let $V(K_{m,n}) = V_1 \cup V_2$, where $|V_1| = m$, $|V_2| = n$ and $m \leqslant n$. Suppose that $V_1 = \{a_1, \ldots, a_m\}$ and $V_2 = \{b_1, \ldots, b_n\}$. Let S be a strong edge metric generating set. Suppose that $2 = m \leqslant n$ and without loss of generality, assume that $a_1 \in S$. Then the edges a_1b_1 and a_2b_1 can be strongly distinguished only by one of the elements in V_2. Therefore $S = \{a_1, b_1\}$ is a strong edge metric basis for $K_{m,n}$, where $2 = m \leqslant n$. Now assume that $3 = m \leqslant n$. Since $\mathrm{d}(a_1b_1, a_2b_2) = 1$ and also $\mathrm{d}(a_j, a_1b_1) = \mathrm{d}(a_j, a_2b_2) = 1 = \mathrm{d}(b_j, a_1b_1) = \mathrm{d}(b_j, a_2b_2)$, for $j > 2$, the only vertex that can strongly distinguish the edges a_1b_1 and a_2b_2 should belong to the set $\{a_1, a_2, b_1, b_2\}$. So from each two nonadjacent edges e_1 and e_2, at least one of their end points should belong to S. If there are $a_i, a_j \in V_1 \setminus S$ and there are $b_r, b_s \in V_2 \setminus S$, then none of the vertices in S can strongly distinguish the edges a_ib_r and a_jb_s, which is impossible. Hence without loss of generality, we may assume that $a_1, a_2 \in S$. Now the edges a_1b_1 and a_2b_1 can be strongly distinguished by either a_3 or an element in V_2. So $S = \{a_1, a_2, a_3\}$ is a strong edge metric basis for $K_{m,n}$, where $3 = m \leqslant n$. Let $4 \leqslant m \leqslant n$. First note that the set $V_1 \setminus \{a_i\}$ and also the set $V_2 \setminus \{b_j\}$ are strong edge metric generating sets, for some $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Now assume that S is a minimal strong edge metric generating set. If $V_1 \setminus \{a_1\} \subseteq S$, then the minimality of S implies that $S = V_1 \setminus \{a_1\}$. So suppose that there exists $a_i \in V_1 \setminus \{a_1\}$

such that $a_i \notin S$, for $2 \leqslant i \leqslant m$. If there exists $a_j \notin S$, for $1 \leqslant j \neq i \leqslant m$, then at most one element from V_2, say b_1, can be not in S. In other words, $V_2 \setminus \{b_1\} \subseteq S$, and the minimality of S implies that $S = V_2 \setminus \{b_1\}$. Hence we assume that $V_1 \setminus \{a_i\} \subseteq S$, which the minimality of S again implies that $S = V_1 \setminus \{a_i\}$. Therefore any minimal strong edge metric generating set of $K_{m,n}$ is of the form $V_1 \setminus \{a_i\}$ or $V_2 \setminus \{b_j\}$, for some $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. So $\operatorname{sedim}(K_{m,n}) = \min\{m-1, n-1\}$. □

If we consider disconnected graphs, then sedim could be easily defined by considering the distance between two vertices and therefore two edges in two different components as infinity. In fact we have the following result.

Remark 2.8. Let G be a disconnected graph with components $G_1, \ldots, G_r$. If $I = \{i \mid G_i \text{ has at least two edges}\}$, then

$$\operatorname{sedim}(G) = \begin{cases} 1; & \text{if } I = \emptyset, \\ \sum_{i \in I} \operatorname{sedim}(G_i); & \text{if } I \neq \emptyset. \end{cases}$$

Recall that for two graphs H_1 and H_2 with disjoint vertex sets, the *join* $H_1 \vee H_2$ of the graphs H_1 and H_2 is the graph obtained from the union of H_1 and H_2 by adding new edges from each vertex of H_1 to every vertex of H_2. The concept of join graph is generalized (in [18], it is called as a generalized composition graph). Assume that G is a graph on k vertices with $V(G) = \{v_1, v_2, \ldots, v_k\}$, and let $H_1, H_2, \ldots, H_k$ be k pairwise disjoint graphs. The *G-generalized join graph* $G[H_1, H_2, \ldots, H_k]$ of $H_1, H_2, \ldots, H_k$ is the graph formed by replacing each vertex v_i of G by the graph H_i and then joining each vertex of H_i to each vertex of H_j whenever $v_i \sim v_j$ in the graph G. Now, if the graph G consists of two adjacent vertices, then the G-generalized join graph $G[H_1, H_2]$ coincides with the join $H_1 \vee H_2$ of the graphs H_1 and H_2. For simplicity, we denote the order of a graph H, by $|H|$.

In the following proposition, we study the strong edge metric dimension of the G-generalized join graph $G[H_1, H_2, \ldots, H_k]$, in the case that H_i's are empty graphs.

Proposition 2.9. *Assume that G is a connected graph on k vertices with $V(G) = \{v_1, v_2, \ldots, v_k\}$, and let $H_1, H_2, \ldots, H_k$ be k pairwise disjoint empty graphs and the number of H_i's of order more than one is t. Then*

$$\operatorname{sedim}(G[H_1, H_2, \ldots, H_k]) \leqslant \sum_{i=1}^{k} |H_i| - t.$$

Proof. Clearly the induced subgraph of $G[H_1, H_2, \ldots, H_k]$ on the vertex set $H_i \cup H_j$ is either an empty graph or it is isomorphic to $K_{|H_i|,|H_j|}$ depending on wether v_i is adjacent to v_j or not, for $1 \leqslant i \neq j \leqslant k$. Assume that v_i is adjacent to v_j and there are two nonadjacent edges

h_ih_j and $h'_ih'_j$ in $K_{|H_i|,|H_j|}$. Since $\mathrm{d}(h_ih_j, h'_ih'_j) = 1$, and $\mathrm{d}(x, h_ih_j) = \mathrm{d}(x, h'_ih'_j)$, for each vertex $x \notin \{h_i, h_j, h'_i, h'_j\}$, we have that the edges h_ih_j and $h'_ih'_j$ only can be strongly distinguished by one of the vertices of the set $\{h_i, h_j, h'_i, h'_j\}$. Let $\{H_{i_1}, H_{i_2}, \dots, H_{i_t}\}$ be of order more than one. Then the set $S = \bigcup_{j=1}^{t}(H_{i_j} \setminus h_{i_j}) \bigcup_{j=t+1}^{k}\{h_{i_j}\}$, for some $h_{i_j} \in V(H_{i_j})$, is a strong edge metric generating set, and so $\mathrm{sedim}(G[H_1, H_2, \dots, H_k]) \leqslant \sum_{i=1}^{t}|H_i| - t + k - t = \sum_{i=1}^{k}|H_i| - t$. □

Assume that G is isomorphic to the graph K_3 and H_1, H_2, H_3 be pairwise disjoint empty graphs such that $|H_1| = |H_2| = 1$ and $|H_3| = 2$. Then it is easy to see that $\mathrm{sedim}(K_3[H_1, H_2, H_3]) = \sum_{i=1}^{3}|H_i| - 1 = 3$. Therefore the inequality in Proposition 2.9 is sharp.

Lemma 2.10. *Assume that G is a connected graph on k vertices with $V(G) = \{v_1, v_2, \dots, v_k\}$, and let $H_1, H_2, \dots, H_k$ be k pairwise disjoint graphs. Suppose that the edges e and e' of G are strongly distinguished by a vertex v in G. Then we have the following statements.*

(i) *Let v be not an end vertex of the edges $e = h_ih_j$ and $e' = h_rh_s$ in $G[H_1, H_2, \dots, H_k]$, where $h_i \in H_i$, $h_j \in H_j$, $h_r \in H_r$ and $h_s \in H_s$. Then if $\mathrm{d}(e, e') \neq 0$, then v strongly distinguishes all pair of edges $h'_ih'_j$ and $h'_rh'_s$, and if $\mathrm{d}(e, e') = 0$, then v strongly distinguishes all pair of edges $h'_ih'_j$ and $h'_rh'_s$ with zero distance, where $h'_i \in H_i$, $h'_j \in H_j$, $h'_r \in H_r$ and $h'_s \in H_s$.*

(ii) *If v is an end vertex of one of the edges $e = h_ih_j$ or $e' = h_rh_s$ in $G[H_1, H_2, \dots, H_k]$, then v strongly distinguishes all pair of edges $h'_ih'_j$ and $h'_rh'_s$, where $h_i, h'_i \in H_i$, $h_j, h'_j \in H_j$, $h_r, h'_r \in H_r$, $h_s, h'_s \in H_s$, in the case that $\mathrm{d}(e, e') = 0$.*

Proof. (i) Suppose that v is not an end vertex of the edges $e = h_ih_j$ and $e' = h_rh_s$ in $G[H_1, H_2, \dots, H_k]$, where $h_i \in H_i$, $h_j \in H_j$, $h_r \in H_r$ and $h_s \in H_s$. We have $\mathrm{d}(v, e) = \mathrm{d}(v, h'_ih'_j)$ and $\mathrm{d}(v, e') = \mathrm{d}(v, h'_rh'_s)$, for each $h'_i \in H_i$, $h'_j \in H_j$, $h'_r \in H_r$ and $h'_s \in H_s$. Now if $\mathrm{d}(e, e') \neq 0$, then $\mathrm{d}(e, e') = \mathrm{d}(h'_ih'_j, h'_rh'_s)$, and so in this situation v strongly distinguishes every pair of edges $h'_ih'_j$ and $h'_rh'_s$, for each $h'_i \in H_i$, $h'_j \in H_j$, $h'_r \in H_r$ and $h'_s \in H_s$. If $\mathrm{d}(e, e') = 0$, then for each pair of edges $h'_ih'_j$ and $h'_rh'_s$ with zero distance, where $h'_i \in H_i$, $h'_j \in H_j$, $h'_r \in H_r$, $h'_s \in H_s$, since $\mathrm{d}(e, e') = \mathrm{d}(h'_ih'_j, h'_rh'_s)$, we have that v strongly distinguishes them.

(ii) Assume that $\mathrm{d}(e, e') = 0$. Then v should be the common end vertex of the edges e and e'. Now one can easily see that v strongly distinguishes all pair of edges $h'_ih'_j$ and $h'_rh'_s$, where $h_i, h'_i \in H_i$, $h_j, h'_j \in H_j$, $h_r, h'_r \in H_r$, $h_s, h'_s \in H_s$. □

Example 2.11. Consider the graph $G[H_1, H_2, H_3]$, where $G \cong K_3$ with vertex set v_1, v_2, v_3 and $H_1 \cong K_2$, $|H_2| = |H_3| = 1$. Let h_1, h'_1 be the vertices of H_1 and h_2, h_3 be the vertices of H_2 and H_3, respectively. Suppose that the induced subgraph on $\{h_1, h_2, h_3\}$ is isomorphic to

G. Consider the edges $e = h_1h_2$ and $e' = h_1h_3$. Then $\mathrm{d}(e, e') = 0$ and h_1 strongly distinguishes the edges e and e'. By Lemma 2.10 (i), h_1 also strongly distinguishes the edges h'_1h_2 and h'_1h_3.

In the follwing proposition, we determine the strong edge metric dimension of the G-generalized join graph $G[H_1, H_2, \ldots, H_n]$, in the case that H_i's are empty graphs and G is the path P_3.

Proposition 2.12. *Assume that G is the path P_3 with $V(P_3) = \{v_1, v_2, v_3\}$ and let H_1, H_2, H_3 be 3 pairwise disjoint empty graphs of order m, n, t, respectively. Then*

$$\mathrm{sedim}(P_3[H_1, H_2, H_3]) = \begin{cases} 1; & \textit{if } n = 1, \\ 2; & \textit{if } n > 1, m = t = 1, \\ n; & \textit{if } n \in \{2, 3\} \\ \mathrm{sedim}(K_{t,n}); & \textit{if } t, n > 1, m = 1, \\ \mathrm{sedim}(K_{m,n}); & \textit{if } m, n > 1, t = 1, \\ \min\{m + t, n - 1\}; & \textit{if } n \notin \{2, 3\}, m, t \in \{2, 3\}, \\ \min\{m + t - 1, n - 1\}; & \textit{if } m \in \{2, 3\}, n, t \notin \{2, 3\}, \\ \min\{m + t - 1, n - 1\}; & \textit{if } t \in \{2, 3\}, m, n \notin \{2, 3\}, \\ \min\{m + t - 2, n - 1\}; & \textit{if } m, n, t \geqslant 4. \end{cases}$$

Proof. By Propositions 2.6, 2.7 and Lemma 2.10, the result holds. □

In the following theorem, we determine the strong edge metric dimension of the G-generalized join graph $G[H_1, H_2, \ldots, H_n]$, in the case that H_i's are empty graphs and G is the complete graph K_n.

Theorem 2.13. *Assume that $G \cong K_n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n > 2$, and let $H_1, H_2, \ldots, H_n$ be n pairwise disjoint empty graphs such that $|H_n| = \max\{|H_i| \mid 1 \leqslant i \leqslant n\} > 1$ and set $l = \sum_{i=1}^{n-1} |H_i| - 1$. Then we have*

$$\mathrm{sedim}(G[H_1, H_2, \ldots, H_n]) = \begin{cases} \sum_{i=1}^{n-1} |H_i| - 1; & \textit{if } l \geqslant 3, \\ 3; & \textit{if } l < 3. \end{cases}$$

Proof. For each two distinct edges e and e', we have $\mathrm{d}(e, e') \in \{0, 1\}$. Note that any subset of $V(G[H_1, H_2, \ldots, H_n])$, say S with $|S| < 3$ can not strongly distinguish all pair of adjacent edges, and so can not be a strong metric generating set. Because if $S = \{a\}$ or $S = \{a, b\}$, then the edges ac and cb are not strongly distinguished by the elements of S, for some $c \neq a, b$. Hence $\mathrm{sedim}(G[H_1, H_2, \ldots, H_n]) \geqslant 3$. Now let $S \subseteq H_1 \cup H_2 \cup H_3$ with $|S| = 3$. Suppose that

$S = \{a, b, c\}$ and e, e' are two adjacent edges in $G[H_1, H_2, \ldots, H_n]$. If a, b, c are the end points of the edges e and e', then clearly their common end point strongly distinguishes them. Now assume that there exists an element of S, say a, which is not an end poin of the edges e and e'. Then since $\mathrm{d}(a, e) = 1 = \mathrm{d}(a, e')$, we have that a strongly distinguishes the edges e and e'. So the minimum number of vertices for strongly distinguishing all pair of edges with zero distance, is three. Let e and e' be two nonadjacent edges in $G[H_1, H_2, \ldots, H_n]$. If x is a vertex which is not an end point of any of the edges e and e', then since $\mathrm{d}(e, e') = 1 = \mathrm{d}(x, e) = \mathrm{d}(x, e')$, the vertex x can not strongly distinguish the edges e and e'. Thus for each two edges with distance one from each other, only one of their end points can strongly distinguish them. Assume that S is a minimal strong edge metric generating set. If there are four H_i's such that $H_i \nsubseteq S$, then there will be two nonadjacent edges that non of their end points belong to S, which is impossible. Hence we have the following cases:

Case 1. There are three H_i's such that $H_i \nsubseteq S$, say H_1, H_2, H_3. Then for each $j > 3$ we have $H_j \subseteq S$, and if $|H_i \setminus S| > 1$, for some $i \in \{1, 2, 3\}$, then we have two nonadjacent edges such that none of their end points belong to S, which is impossible. Therefore $|S| = \sum_{i=1}^{n} |H_i| - 3$, and so $\mathrm{sedim}(G[H_1, H_2, \ldots, H_n]) \leqslant \sum_{i=1}^{n} |H_i| - 3$.

Case 2. There are two H_i's such that $H_i \nsubseteq S$, say H_1, H_2. Then if $|H_1 \setminus S| = 1$, then by the minimality of S, we should have $H_2 \cap S = \emptyset$ which implies that $|S| = \sum_{i=3}^{n} |H_i| + |H_1| - 1$. Now if $|H_1 \setminus S| > 1$, then we should have $|H_2 \setminus S| = 1$, and since S is minimal, we have $H_1 \cap S = \emptyset$, and so in this situation we have $|S| = \sum_{i=3}^{n} |H_i| + |H_2| - 1$. Therefore in this case by considering $|H_n| = \max\{|H_i| \mid 1 \leqslant i \leqslant n\}$, we have $\mathrm{sedim}(G[H_1, H_2, \ldots, H_n]) \leqslant \sum_{i=1}^{n-1} |H_i| - 1$.

Case 3. There is one H_i such that $H_i \nsubseteq S$, say H_1. Since S is minimal, $H_1 \cap S = \emptyset$. On the other hand, $S \setminus \{h_i\}$, for some $h_i \in H_i$, where $2 \leqslant i \leqslant n$, is a strong edge metric generating set, which is a contradiction with the fact that S is minimal. Hence this case never happen.

Now by considering the above cases, we have that $\mathrm{sedim}(G[H_1, H_2, \ldots, H_n]) = \max\{3, \sum_{i=1}^{n-1} |H_i| - 1\}$, where $|H_n| = \max\{|H_i| \mid 1 \leqslant i \leqslant n\} > 1$. □

Example 2.14. Consider the graph $K_3[H_1, H_2, H_3]$, where H_1, H_2, H_3 are pairwise disjoint empty graphs such tha $|H_1| = |H_1| = 1$ and $|H_3| = 2$. Then by Theorem 2.13, since $l = 1 < 3$, we have $\mathrm{sedim}(G[H_1, H_2, H_n]) = 3$. In this example, if we consider $|H_1| = |H_1| = 3$, then $l = 5$, and by Theorem 2.13, we have $\mathrm{sedim}(G[H_1, H_2, H_n]) = 5$.

In the following proposition, we determine the strong edge metric dimension of the G-generalized join graph $G[H_1, H_2, \ldots, H_n]$, in the case that H_i's are complete graphs and G is a path P_n.

Proposition 2.15. *Assume that G is a path on $n \geqslant 2$ vertices with $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let $H_1, H_2, \ldots, H_n$ be n pairwise disjoint complete graphs of order more than two. Then* $\mathrm{sedim}(G[H_1, H_2, \ldots, H_n]) = |\bigcup_{i=1}^{n} H_i| - 3\lceil \frac{n}{2} \rceil$.

Proof. The induced subgraph on $H_i \cup H_{i+1}$ is isomorphic to $K_{|H_i|+|H_{i+1}|}$, and by Proposition 2.2, its strong edge metric dimension is equal to $|H_i| + |H_{i+1}| - 3$. So in order to detrmine a strong edge metric basis for $G[H_1, H_2, \ldots, H_n]$, it is enough that from the vertex set $V(G[H_1, H_2, \ldots, H_n])$, ommit three vertices from each of the graphs H_{2k-1}, for $1 \leqslant k \leqslant \lceil \frac{n}{2} \rceil$. Hence we have $\mathrm{sedim}(G[H_1, H_2, \ldots, H_n]) = |\bigcup_{i=1}^{n} H_i| - 3\lceil \frac{n}{2} \rceil$. □

Example 2.16. Consider the graph $P_2[H_1, H_2]$, where H_1 and H_2 are pairwise disjoint complete graphs of order 3. Then by Theorem 2.15, we have $\mathrm{sedim}(G[H_1, H_2]) = 3$.

In the following proposition, we determine the strong edge metric dimension of the G-generalized join graph $G[H_1, H_2, \ldots, H_n]$, in the case that H_i's are complete graphs and G is isomorphic to the cycle C_n, where $n > 3$. Note that the case $n = 3$ is obtained by by Proposition 2.2.

Proposition 2.17. *Assume that G is a cycle C_n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n > 3$, and let $H_1, H_2, \ldots, H_n$ be n pairwise disjoint complete graphs of order more than two. Then* $\mathrm{sedim}(G[H_1, H_2, \ldots, H_n]) = |\bigcup_{i=1}^{n} H_i| - 3\lfloor \frac{n}{2} \rfloor$.

Proof. Since the induced subgraph on $H_i \cup H_{i+1}$ is isomorphic to $K_{|H_i|+|H_{i+1}|}$, by Proposition 2.2, its strong edge metric dimension is equal to $|H_i| + |H_{i+1}| - 3$. Now in order to detrmine a strong edge metric basis for $G[H_1, H_2, \ldots, H_n]$, it is enough that from the vertex set $V(G[H_1, H_2, \ldots, H_n])$, ommit three vertices from each of the graphs H_{2k-1}, for $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$. Hence we have $\mathrm{sedim}(G[H_1, H_2, \ldots, H_n]) = |\bigcup_{i=1}^{n} H_i| - 3\lfloor \frac{n}{2} \rfloor$. □

3. STRONG EDGE METRIC DIMENSION OF $\Gamma(\mathbb{Z}_n)$

Let R be a commutative ring with nonzero identity. We denote the set of all unit elements and zero divisors of R by $U(R)$ and $Z(R)$, respectively, and $Z^*(R) = Z(R) \setminus \{0\}$. Also, for $n > 1$, $\mathbb{Z}_n$ denotes the ring of integers modulo n. Sharma and Bhatwadekar [20] defined the comaximal graph of a commutative ring R. The *comaximal graph* of R is a simple graph whose vertices consists of all elements of R, and two distinct vertices a and b are adjacent if and only if $aR + bR = R$, where cR is the ideal generated by c, for $c \in R$. Let $\Gamma(R)$ be an induced subgraph of the comaximal graph with nonunit elements of R as vertices. The properties of the graph $\Gamma(R)$ were studied in [15], [23] and [25].

For two integers r and s, the notation (r, s) stands for the greatest common divisor of r and s. Also we denote the elements of the ring $\mathbb{Z}_n$, where $n > 1$, by $0, 1, 2, \ldots, n-1$. For every nonzero element a in $\mathbb{Z}_n$, if $(a, n) = 1$, then a is a unit element; otherwise, $(a, n) \neq 1$, and so a is a zerodivisor. Therefore, $|U(\mathbb{Z}_n)| = \varphi(n)$ and $|Z(\mathbb{Z}_n)| = n - \varphi(n)$, where φ is the Euler's totient function.

An integer d is said to be a *proper divisor* of n if $1 < d < n$ and $d \mid n$. Now let $d_1, d_2, \ldots, d_k$ be the distinct proper divisors of n. For $1 \leqslant i \leqslant k$, set

$$A_{d_i} := \{x \in \mathbb{Z}_n \mid (x, n) = d_i\}.$$

Clearly, the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_k}$ are pairwise disjoint and we have

$$Z^*(\mathbb{Z}_n) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_k},$$

and

$$V(\Gamma(\mathbb{Z}_n)) = \{0\} \cup A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_k}.$$

The following lemma is stated from [26].

Lemma 3.1. [26, Proposition 2.1] *Let* $1 \leqslant i \leqslant k$. *Then* $|A_{d_i}| = \varphi(\frac{n}{d_i})$.

In this section, the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the set A_{d_i} is denoted by $\Gamma(A_{d_i})$, where $1 \leqslant i \leqslant k$.

The following lemma states some adjacencies in $\Gamma(\mathbb{Z}_n)$.

Lemma 3.2. [2, Lemma 2.2] *The following statements hold:*

(i) *Two distinct vertices x and y are adjacent in $\Gamma(\mathbb{Z}_n)$ if and only if $(x, y) \in U(\mathbb{Z}_n)$.*

(ii) *For $1 \leqslant i \leqslant k$, $\Gamma(A_{d_i})$ is isomorphic to $\overline{K}_{\varphi(\frac{n}{d_i})}$.*

(iii) *For $1 \leqslant i \neq j \leqslant k$, a vertex of A_{d_i} is adjacent to a vertex of A_{d_j} if and only if $(d_i, d_j) = 1$.*

Now, we introduce a simple graph G_n, which plays an important role in the structure of $\Gamma(\mathbb{Z}_n)$. The graph G_n is the simple graph with vertex set $\{d_1, d_2, \ldots, d_k\}$, where d_i's, $1 \leqslant i \leqslant k$, are the proper divisors of n, and two distinct vertices d_i and d_j are adjacent if and only if $(d_i, d_j) = 1$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ be the factorization of n to its prime powers, where $t, \alpha_1, \ldots, \alpha_t$ are positive integers and $p_1, \ldots, p_t$ are distinct prime numbers. Every divisor of n is of the form $p_1^{\beta_1} p_2^{\beta_2} \ldots p_t^{\beta_t}$, for some integers $\beta_1, \ldots, \beta_t$, where $0 \leqslant \beta_i \leqslant \alpha_i$ for each $i \in \{1, 2, \ldots, t\}$. Hence the number of proper divisors of n is equal to $\prod_{i=1}^{t}(n_i + 1) - 2$. Therefore we have $k = |V(G_n)| = \prod_{i=1}^{t}(n_i + 1) - 2$.

Let $\Gamma^*(\mathbb{Z}_n) = \Gamma(\mathbb{Z}_n) \setminus \{0\}$. Consider the graph G_n and replace each vertex d_i of G_n by $\Gamma[A_{d_i}]$. In view of Lemma 3.1, we have

$$\Gamma^*(\mathbb{Z}_n) = G_n[\overline{K}_{\varphi(\frac{n}{d_1})}, \overline{K}_{\varphi(\frac{n}{d_2})}, \ldots, \overline{K}_{\varphi(\frac{n}{d_k})}].$$

Now, since the zero element is adjacent to non of the vertices of $\Gamma^*(\mathbb{Z}_n)$, we have

$$\Gamma(\mathbb{Z}_n) = (K_1 \cup \Gamma^*(\mathbb{Z}_n)).$$

The following theorem in which we study the Strong edge metric dimension of $\Gamma(\mathbb{Z}_n)$, follows from Proposition 2.9 and Remark 2.8.

Theorem 3.3. *Assume that $d_1, d_2, \ldots, d_k$, where $k \geqslant 1$, are the proper divisors of n. Then we have*

$$\operatorname{sedim}(\Gamma(\mathbb{Z}_n)) \leqslant \sum_{i=1}^{k} \varphi(\frac{n}{d_i}) - k,$$

where $n \neq 2d_i$ for all $1 \leqslant i \leqslant k$. Otherwise

$$\operatorname{sedim}(\Gamma(\mathbb{Z}_n)) \leqslant \sum_{i=1}^{k} \varphi(\frac{n}{d_i}) - k + 1.$$

Example 3.4. Consider the ring $\mathbb{Z}_{12}$. We have $d_1 = 2, d_2 = 3, d_3 = 4$, and $d_4 = 6$. Then G_{12} is the graph $2 \sim 3 \sim 4 \cup \{6\}$, which is isomorphic to $P_3 \cup K_1$. Hence we have

$$\Gamma(\mathbb{Z}_{12}) = K_1 \cup G_{12}[\overline{K}_2, \overline{K}_2, \overline{K}_2, K_1],$$

and in view of Remark 2.8, we have that the strong edge metric basis of $\Gamma(\mathbb{Z}_{12})$ is $\{2, 3\}$, and so $\operatorname{sedim}(\Gamma(\mathbb{Z}_{12})) = 2$.

In the rest of this section, we discuss the strong edge metric basis of $\Gamma(\mathbb{Z}_n)$, for (i) $n = p^t$, (ii) $n = pq$ and (iii) $n = p^2q$, where p and q are distinct prime numbers and t is a positive integer.

(i) Let $n = p^t$. Then $\Gamma(\mathbb{Z}_{p^t})$ is an empty graph with $p^t - \varphi(p^t) = p^{t-1}$ vertices, and so $\Gamma(\mathbb{Z}_{p^t}) = \overline{K_{p^{t-1}}}$. Now, by Remark 2.8 we have $\operatorname{sedim}(\Gamma(\mathbb{Z}_{p^t})) = 1$.

(ii) Let $n = pq$, where p and q are distinct prime numbers with $p < q$. Since the only proper divisors of n are p and q, the graph G_{pq} is $p \sim q$. So we have

$$\Gamma(\mathbb{Z}_{pq}) = K_1 \cup G_{pq}[\overline{K}_{\varphi(q)}, \overline{K}_{\varphi(p)}].$$

Now, by Proposition 2.7, we have

$$\operatorname{sedim}(\Gamma(\mathbb{Z}_{pq})) = \begin{cases} p; & \text{if } p \in \{2, 3\}, \\ p - 1; & \text{otherwise.} \end{cases}$$

(iii) Let $n = p^2q$, where p and q are distinct prime numbers. Since p, q, and pq are the proper divisors of n, the graph G_{p^2q} is $p \sim q \sim p^2 \cup \{pq\}$. Hence we have

$$\Gamma(\mathbb{Z}_{p^2q}) = K_1 \cup G_{p^2q}[\overline{K}_{\varphi(pq)}, \overline{K}_{\varphi(p^2)}, \overline{K}_{\varphi(q)}, \overline{K}_{\varphi(p)}].$$

Now by Remark 2.8, we have $\mathrm{sedim}(\Gamma(\mathbb{Z}_{p^2q})) = \mathrm{sedim}(P_3[\overline{K}_{\varphi(pq)}, \overline{K}_{\varphi(p^2)}, \overline{K}_{\varphi(q)}])$ which can be obtained by Proposition 2.12.

4. Integer LP model

In this section, we present an ILPM for the strong edge metric basis. To do this, we need to introduced some notations.

Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. For $e_i, e_j \in E(G)$ and $v_k \in V(G)$, $d_{ij}^k = 1$ if $\mathrm{d}(e_1, v_k) = \mathrm{d}(e_2, v_k) + \mathrm{d}(e_1, e_2)$ or $\mathrm{d}(e_2, v_k) = \mathrm{d}(e_1, v_k) + \mathrm{d}(e_1, e_2)$, and $d_{ij}^k = 0$ otherwise. Now we are ready to present an ILPM for the strong edge metric basis. Let $F : \{0,1\}^n \to \mathbb{N}_0$ be defined by

$$F(x_1, \ldots, x_n) = x_1 + \cdots + x_n\,.$$

Then our aim is to determine $\min F$ subject to the constraints

$$\sum_{k=1}^{n} d_{ij}^k x_k > 0, \quad 1 \le i < j \le m.$$

Therefore, it is not difficult to see that if $x'_1, \ldots, x'_n$ is a set of values for which F attains its minimum, then $S = \{v_i \ : \ x'_i = 1\}$ is a strong edge metric basis for G.

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References

[1] M. Afkhami, *On the edge metric dimension and Wiener index of the blow up of graphs*, NOTE MAT., **40** No. 2 (2021) 99-110.

[2] M. Afkhami, Z. Barati and K. Khashyarmanesh, *On the signless Laplacian spectrum of the comaximal graphs*, Asian-Eur. J. Math., **16** (2023) 2350055.

[3] A. Behtoei and Y. Golkhandy Pour, *On two-dimensional Cayley graphs*, Alg. Struc. Appl., **4** No. 1 (2017) 45-52.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, Vol. 244, Springer, New York, 2008.

[5] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara and D. R. Wood, *On the metric dimension of cartesian products of graphs*, SIAM J. Discrete Math., **21** No. 2 (2007) 423-441.

[6] K. Ch. Das and M. Tavakoli, *Bounds for metric dimension and defensive k-alliance of graphs under deleted lexicographic product*, Trans. Comb., **9** No. 1 (2020) 31-39.

[7] Leah L. Epstein, A. Levin and G. J. Woeginger, *The (weighted) metric dimension of graphs: hard and easy cases*, Algorithmica, **72** (2015) 1130-1171.

[8] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness,* W.H. Freeman, New York, 1979.

[9] M. Johnson, *Structure-activity maps for visualizing the graph variables arising in drug design*, J. Biopharm. Stat., **3** (1993) 203-236.

[10] A. Kelenc, N. Tratnik and I. G. Yero, *Uniquely identifying the edges of a graph: The edge metric dimension*, Discrete Appl. Math., **256** (2018) 204-220.

[11] S. Khuller, B. Raghavachari and A. Rosenfeld, *Landmarks in graphs,* Discrete Appl. Math., **70** (1996) 217-229.

[12] J. Kratica, V. Kovačević-Vujčić, M. Čangalović and N. Mladenović, *Strong metric dimension: a survey,* Yugosl. J. Oper. Res., **24** No. 2 (2014) 187-198.

[13] D. Kuziak, I. G. Yero and J. A. Rodríguez-Velázquez, *On the strong metric dimension of corona product graphs and join graphs*, Discrete Appl. Math., **161** No. (7-8) (2013) 1022-1027.

[14] D. Kuziak, I. G. Yero and J. A. Rodríguez-Velázquez, *On the strong metric dimension of the strong products of graphs*, Open Math., **13** No. 1 (2015) 64-74.

[15] H. R. Maimani, M. Salimi, A. Sattari and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra, **319** (2008) 1801-1808.

[16] I. Peterin and I. G. Yero, *Edge metric dimension of some graph operations*, Bull. Malays. Math. Sci. Soc., **43** No. 3 (2020) 2465-2477.

[17] J. A. Rodríguez-Velázquez, I. G. Yero, D. Kuziak and O. R. Oellermann, *On the strong metric dimension of Cartesian and direct products of graphs*, Discrete Math., **335** (2014) 8-19.

[18] A. J. Schwenk, Computing the characteristic polynomial of a graph. In *Graphs and Combinatorics: Proceedings of the Capital Conference on Graph Theory and Combinatorics at the George Washington University June 18-22*, 1973, pp. 153-172. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.

[19] S. W. Saputro, R. Simanjuntak, S. Uttunggadewa, H. Assiyatun, E. T. Baskoro, A. N. M. Salman and M. Bača, *The metric dimension of the lexicographic product of graphs,* Discrete Math., **313** (2013) 1045-1051.

[20] P. K. Sharma and S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra, **176** (1995) 124-127.

[21] A. Sebo and E. M. Tannier, *On metric generators of graphs*, Math. Oper. Res., **29** No. 2 (2004) 383-393.

[22] P. J. Slater, *Leaves of trees*, Congr. Numer., **14** (1975) 549-559.

[23] M. M. Slavko and Z. Z. Petrovic, *On the structure of comaximal graphs of commutative rings with identity*, Bull. Aust. Math. Soc., **83** (2011) 11-21.

[24] M. Tavakoli, F. Rahbarnia and A. R. Ashrafi, *Distribution of some graph invariants over hierarchical product of graphs*, Appl. Math. Comput., **220** (2013) 405-413.

[25] H. J. Wang, *Graphs associated to co-maximal ideals of commutative rings*, J. Algebra, **320** (2008) 2917-2933.

[26] M. Young, *Adjacency matrices of zero-divisor graphs of integers modulo n*, Involve, **8** (2015) 753-761.

[27] N. Zubrilina, *On the edge dimension of a graph*, Discrete Math., **341** (2018) 2083-2088.

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