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Research Paper

PRIME IDEALS AND SPECTRAL TOPOLOGICAL CONDITIONS ON HOOPS

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Abstract. This paper delves into the investigation of ideals and prime ideals within hoops, a distinct class of algebraic structures. It seeks to identify and analyze specific properties of prime ideals, which are essential for comprehending the behavior of hoops. The notion of a local hoop is introduced, accompanied by an examination of its various equivalent definitions, showcasing its importance within the broader framework of hoop theory. A notable result presented is the proof that if a maximal ideal, denoted as M, exists in a hoop, the resulting quotient structure is locally finite, shedding light on the constraints and characteristics of these ideals. Furthermore, the paper introduces a topology on the set of all prime ideals associated with a hoop and explores the conditions required for this topology to be Hausdorff, offering new insights into the topological organization of prime ideals.

1. Introduction

Non-classical logic has become a crucial formal tool in computer science, particularly for managing uncertain and fuzzy information. The algebraic counterparts of certain non-classical

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logics conform to the principle of residuation, placing them within the realm of residuated lattices [19]. Notable examples include Hájek's BL (Basic Logic [20]), Łukasiewicz's MV (Many-Valued Logic [17]), and MTL (Monoidal T-Norm-Based Logic [14]), which correspond to BL-algebras, MV-algebras, and MTL-algebras, respectively. These algebras are built upon a shared foundational structure: lattices with residuation. Therefore, exploring the properties of algebras that incorporate residuation is important.

Hoops, introduced by B. Bosbach in [17] and later examined by J. R. Büchi and T. M. Owens (although their work remains unpublished), are naturally ordered, commutative, residuated integral monoids. Over the years, hoop theory has been significantly enriched with important structural theorems (see [8, 17]), many of which have greatly influenced fuzzy logic. Notably, the structure theorem for finite basic hoops ([8], Corollary 2.10) offers a concise and elegant proof of the completeness theorem for propositional basic logic ([8], Theorem 3.8), which was initially introduced by Hájek in [20].

The algebraic structures associated with Hájek propositional fuzzy basic logic, known as BL-algebras, are specific instances of hoops. A key example of BL-algebras is the interval [0,1], equipped with a structure induced by a t-norm. Among the most notable classes of BL-algebras are MV-algebras, product algebras, and Gödel algebras. Recent research has increasingly focused on non-commutative generalizations of these structures.

Filter theory is crucial in the examination of these algebras. From a logical standpoint, different types of filters correspond to various sets of provable formulas. In the realm of hoops, filter theory has been thoroughly investigated, resulting in several notable findings. Specifically, types of filters such as (positive) implicative filters and fantastic filters [9] have been introduced and characterized in works like [5, 2, 3, 7, 10, 12, 13, 21, 22, 23]. In MV-algebras, filters and ideals represent dual concepts. However, it is essential to recognize that residuated lattices and hoops are not comparable; not all hoops qualify as residuated lattices. A hoop, while functioning as a meet semi-lattice with respect to the operator $d \wedge w = d \odot (d \rightsquigarrow w)$, does not possess a complete lattice structure.

In [1], the authors defined and characterized the concepts of implicative, maximal, and prime ideals in hoops. They also explored the relationships among these concepts and demonstrated that every maximal implicative ideal in a $\vee$-hoop satisfying the (DNP) property is also a prime ideal.

Additionally, the authors introduced a congruence relation on hoops induced by ideals and analyzed the resulting quotient structures. This framework enables the demonstration that an ideal is maximal if and only if the quotient hoop is a simple MV-algebra. Furthermore, they examined the relationship between ideals and filters by employing the set of complements.

In this paper, we investigate the concepts of ideals and prime ideals in hoops, characterizing various properties of prime ideals. We introduce the notion of a local hoop and explore different equivalent definitions of this concept. Additionally, we prove that if M is a maximal ideal of a hoop, the corresponding quotient structure is locally finite. Finally, we define a topology on the set of all prime ideals of a hoop and examine the conditions under which this topology is Hausdorff.

2. Preliminaries

This section outlines the default contents that will be used in subsequent discussions.

Definition 2.1. By a *hoop* we mean an algebraic structure $(\mathcal{H}, \circledcirc, \rightsquigarrow, 1)$ where, for all $c, v, q \in \mathcal{H}$:

$(HP1)$ $(\mathcal{H}, \circledcirc, 1)$ is a commutative monoid;

$(HP2)$ $c \rightsquigarrow c = 1$;

$(HP3)$ $(c \circledcirc v) \rightsquigarrow q = c \rightsquigarrow (v \rightsquigarrow q)$;

$(HP4)$ $c \circledcirc (c \rightsquigarrow v) = v \circledcirc (v \rightsquigarrow c)$.

On hoop $\mathcal{H}$, the relation "$c \leq v$ if and only if $c \rightsquigarrow v = 1$" is a partially order relation and $(\mathcal{H}, \leq)$ is a poset. A *bounded hoop* is a hoop with the least element $0 \in \mathcal{H}$.

If $(\mathcal{H}, \circledcirc, \rightsquigarrow, 0, 1)$ is a bounded hoop, then we can define a unary operation "$*$" on $\mathcal{H}$ by $c^* = c \rightsquigarrow 0$, for all $c \in \mathcal{H}$. If it holds that $c^{**} = c$, for all $c \in \mathcal{H}$, then the bounded hoop $\mathcal{H}$ is said to have the double negation property (abbreviated as (DNP)).

A hoop $\mathcal{H}$ is called commutative, if for any $c, v \in \mathcal{H}$, $(c \rightsquigarrow v) \rightsquigarrow v = (v \rightsquigarrow c) \rightsquigarrow c$. If $(\mathcal{H}, \circledcirc, \rightsquigarrow, 0, 1)$ is a bounded commutative hoop, then it is straightforward to see that $\mathcal{H}$ has (DNP) property.

Suppose $\mathcal{H}$ has another binary operation $\vee$ such that for any $c, v \in \mathcal{H}$, we have $c \vee v = ((c \rightsquigarrow v) \rightsquigarrow v) \wedge ((v \rightsquigarrow c) \rightsquigarrow c)$. If $\vee$ is a join operation on $\mathcal{H}$, then the hoop $\mathcal{H}$ is called a $\vee$-*hoop*, which is a distributive lattice.

The following proposition provides some properties of hoops.

Proposition 2.2. [14, 15] *Let $(\mathcal{H}, \circledcirc, \rightsquigarrow, 1)$ be a hoop. Then the following conditions hold, for all $c, v, q \in \mathcal{H}$:*

(i) $(\mathcal{H}, \leq)$ *is a meet-semilattice, with* $c \wedge v = c \circledcirc (c \rightsquigarrow v)$;

(ii) $c \circledcirc v \leq q$ *if and only if* $c \leq v \rightsquigarrow q$;

(iii) $c \circledcirc v \leq c, v$ *and* $c \leq v \rightsquigarrow c$;

(iv) $c \rightsquigarrow c = 1$ *and* $1 \rightsquigarrow c = c$;

(v) $c \leq v \rightsquigarrow (c \circledcirc v)$;

(vi) $c \rightsquigarrow v \leq (v \rightsquigarrow q) \rightsquigarrow (c \rightsquigarrow q)$;

(vii) $c \leq v$ *implies* $c \circledcirc q \leq v \circledcirc q$, $q \rightsquigarrow c \leq q \rightsquigarrow v$ *and* $v \rightsquigarrow q \leq c \rightsquigarrow q$;

(viii) *if* $\mathcal{H}$ *is bounded, then* $c \leq c^{**}$, $c \circledcirc c^* = 0$ *and* $c^{***} = c^*$;

(ix) *if* $\mathcal{H}$ *is a* $\vee$*-hoop, then for any* $n \in \mathbb{N}$, $(c \vee v)^n \rightsquigarrow q = \bigwedge\{(d_1 \circledcirc d_2 \circledcirc \cdots \circledcirc d_n) \rightsquigarrow q \mid d_i \in \{c, v\}\}$;

(x) *if* $\mathcal{H}$ *is a* $\vee$*-hoop, then* $c \circledcirc (v \vee q) = (c \circledcirc v) \vee (c \circledcirc q)$.

Note. From now on, suppose $(\mathcal{H}, \circledcirc, \rightsquigarrow, 0, 1)$ or $\mathcal{H}$, for short, is bounded.

Definition 2.3. [1] Let $\emptyset \neq I \subseteq \mathcal{H}$. Then I is called an *ideal* of $\mathcal{H}$ if we have:

$(I1)$ $0 \in I$,

$(I2)$ for any $c, v \in I$, $c^* \rightsquigarrow v \in I$,

$(I3)$ for any $c, v \in \mathcal{H}$, if $c \leq v$ and $v \in I$, then $c \in I$.

Clearly, $\mathcal{H}$ and $\{0\}$ are the trivial ideals of $\mathcal{H}$. The set of all ideals of $\mathcal{H}$ is denoted by $\mathcal{ID}(\mathcal{H})$. I is called a *proper ideal* if I is an ideal of $\mathcal{H}$ and $I \neq \mathcal{H}$. Obviously, an ideal I is proper if and only if it is not containing 1.

Let $\{I_\lambda \mid \lambda \in \Delta\}$ be a family of ideals in $\mathcal{H}$. Then $\bigcap_{\lambda \in \Delta} I_\lambda \in \mathcal{ID}(\mathcal{H})$ but $\bigcup_{\lambda \in \Delta} I_\lambda \notin \mathcal{ID}(\mathcal{H})$.

Remark 2.4. [1] Let $I \in \mathcal{ID}(\mathcal{H})$. Then for any $c \in \mathcal{H}$, $c \in I$ if and only if $c^{**} \in I$. By Proposition 2.2(viii) and (I3), if $c^{**} \in I$, then $c \in I$. Let $c \in I$, since $0 \in I$, by (I2), $c^{**} = c \oplus 0 \in I$, where $c \oplus v = c^* \rightsquigarrow v$.

Proposition 2.5. [1] *Consider* $\emptyset \neq I \subseteq \mathcal{H}$. *Then, for any* $c, v \in \mathcal{H}$, *the following statements are equivalent:*

(i) $I \in \mathcal{ID}(\mathcal{H})$,

(ii) $0 \in I$, *for any* $c, v \in I$, $c \oplus v \in I$ *and if* $c^* \circledcirc v \in I$ *and* $c \in I$, *then* $v \in I$.

(iii) $0 \in I$, *for any* $c, v \in I$, $c \oplus v \in I$ *and if* $(c^* \rightsquigarrow v^*)^* \in I$ *and* $c \in I$, *then* $v \in I$.

Proposition 2.6. [1] *Let* $\mathcal{H}$ *be a bounded* $\vee$*-hoop and* $I \in \mathcal{ID}(\mathcal{H})$. *Then for any* $c, v \in \mathcal{H}$, *we have*

(i) $c, v \in I$ *if and only if* $c \vee v \in I$;

(ii) *if* $c, v \in I$, *then* $c \wedge v \in I$.

Definition 2.7. [1] Let $\emptyset \neq X \subseteq \mathcal{H}$. We recall that the smallest ideal containing X in $\mathcal{H}$ is called the *generated ideal by* X in $\mathcal{H}$ and it is denoted by $(X]$. It is also the intersection of all ideals of $\mathcal{H}$ contain X.

Theorem 2.8. [1] *Let* $\emptyset \neq X \subseteq \mathcal{H}$. *Then*

$$(X] = \{d \in \mathcal{H} \mid \exists n \in \mathbb{N} \ s.t. \ for \ c_1, c_2, \cdots, c_n \in X, \ d \leq c_1 \oplus (c_2 \oplus \cdots \oplus (c_{n-1} \oplus c_n) \cdots)\}.$$

Notation 1. Consider $d \oplus (d \oplus \cdots \oplus (d \oplus d) \cdots) = nd = ((n-1)d)^* \rightsquigarrow d$. If $\mathcal{H}$ has (DNP), then $c \oplus v = v \oplus c$ and $nd = ((d^*)^n)^*$.

Proposition 2.9. [1] *Let $I \in \mathcal{ID}(\mathcal{H})$ and $d \in \mathcal{H}$. Then*
(i) $(d] = \{c \in \mathcal{H} \mid \exists n \in \mathbb{N} \ s.t. \ c \leq nd\}$;
(ii) *if $\mathcal{H}$ has (DNP), then* $(I \cup \{d\}] = \{c \in \mathcal{H} \mid \exists n \in \mathbb{N} \ s.t. \ c \circledcirc (nd)^* \in I\}$;
(iii) *if $\mathcal{H}$ is a $\vee$-hoop with (DNP), then* $(I \cup \{c\}] \cap (I \cup \{v\}] = (I \cup \{c \wedge v\}]$.

Theorem 2.10. [11] *Let $I \in \mathcal{ID}(\mathcal{H})$. Define the relation $\sim_I$ on $\mathcal{H}$ by:*

$$c \sim_I v \ \Leftrightarrow \ (c \rightsquigarrow v)^* \in I, \ (v \rightsquigarrow c)^* \in I, \qquad \forall c, v \in \mathcal{H}.$$

Then $\sim_I$ is a congruence relation on $\mathcal{H}$. For any $c \in \mathcal{H}$, assume $\frac{c}{I} = \{v \in \mathcal{H} \mid c \sim_I v\}$. Consider $\dfrac{\mathcal{H}}{I} = \{\frac{c}{I} \mid c \in \mathcal{H}\}$ and we define some operations on $\dfrac{\mathcal{H}}{I}$ as follows:

$$\frac{c}{I} \circledcirc \frac{v}{I} = \frac{(c \circledcirc v)}{I}, \ \frac{c}{I} \rightsquigarrow \frac{v}{I} = \frac{c \rightsquigarrow v}{I}, \ \frac{0}{I} = I, \ \text{and} \ \frac{1}{I} = \{c^* \in I \mid c \in \mathcal{H}\}.$$

Also, define the order $\frac{c}{I} \leq \frac{v}{I}$ if and only if $(c \rightsquigarrow v)^* \in I$ on $\dfrac{\mathcal{H}}{I}$. Then $(\dfrac{\mathcal{H}}{I}, \circledcirc, \rightsquigarrow, \frac{0}{I}, \frac{1}{I})$ is a hoop and $\left(\dfrac{\mathcal{H}}{I}, \leq\right)$ is a poset.

It is straightforward to see that in $\dfrac{\mathcal{H}}{I}$, $\dfrac{0}{I} = \{c \in \mathcal{H} \mid c \in I\}$, $\dfrac{1}{I} = \{c \in \mathcal{H} \mid c^* \in I\}$.

Definition 2.11. [1] Let P be a proper ideal of $\mathcal{H}$. P is called a *prime ideal* of $\mathcal{H}$ if, $c \wedge v \in P$, then $c \in P$ or $v \in P$, for any $c, v \in \mathcal{H}$. The set of all prime ideals of $\mathcal{H}$ is denoted by $Spec(\mathcal{H})$.

Proposition 2.12. [1] *Let $\mathcal{H}$ be a $\vee$-hoop with (DNP) and P be a proper ideal of $\mathcal{H}$. Then*
(i) P is a prime ideal of $\mathcal{H}$ if and only if, for any $I, J \in \mathcal{ID}(\mathcal{H})$ such that $I \cap J \subseteq P$, we get $I \subseteq P$ or $J \subseteq P$.
(ii) for any proper ideal I of $\mathcal{H}$ there exists $P \in Spec(\mathcal{H})$ such that $I \subseteq P$.

Proposition 2.13. [1] *Let $\mathcal{H}$ be a $\vee$-hoop with (DNP), I be a proper ideal of $\mathcal{H}$ and $\emptyset \neq S \subseteq \mathcal{H}$ such that $I \cap S = \emptyset$. If S is $\wedge$-closed, then there exists $P \in Spec(\mathcal{H})$ such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Definition 2.14. [1] Let M be a proper ideal of $\mathcal{H}$. Then M is called a *maximal ideal* of $\mathcal{H}$ if, no proper ideal of $\mathcal{H}$ strictly containing M. It means that if there exists an ideal of $\mathcal{H}$ such as J that $M \subseteq J \subseteq \mathcal{H}$, then $M = J$ or $J = \mathcal{H}$. The set of all maximal ideals of $\mathcal{H}$ is denoted by $Max(\mathcal{H})$.

Proposition 2.15. [1] *Let $\mathcal{H}$ be a $\vee$-hoop with (DNP). Then every maximal ideal of $\mathcal{H}$ is a prime one.*

We recall that a *lattice-ordered group* (*ℓ-group*) is an algebra $(G; \vee, \wedge, +, -, 0)$ such that $(G; \vee, \wedge)$ is a lattice, $(G; +, -, 0)$ is a group, and $+$ is an order-preserving map. We denote by $G^+ = \{g \in G \mid g \geq 0\}$ and $G^- = \{g \in G \mid g \leq 0\}$.

Let H and G be ℓ-groups. We define the lexicographic product $H\overrightarrow{\times}G$ as the group addition on the direct product $H \times G$ endowed with the lexicographic order $(h_1, g_1) \leq (h_2, g_2)$ for $(h_1, g_1), (h_2, g_2) \in H \times G$ iff either $h_1 < h_2$ or $h_1 = h_2$ and $g_1 \leq g_2$. Then $H\overrightarrow{\times}G$ is a partially-ordered group that is an ℓ-group iff H is linearly ordered, see [18, (d) p. 26].

3. Prime ideals on hoops

In this section, we characterize certain properties of prime ideals. We introduce the notion of a local hoop and examine various equivalent definitions of this concept. Furthermore, we prove that if M is a maximal ideal of a hoop, the corresponding quotient structure is locally finite. Finally, we define a topology on the set of all prime ideals of a hoop and investigate the conditions under which this topology becomes Hausdorff.

Note. From now on, we let $(\mathcal{H}, \circledcirc, \rightsquigarrow, 0, 1)$ or $\mathcal{H}$ for short, is a bounded hoop.

Note. For any $c, v \in \mathcal{H}$, we define $c \oplus v = c^* \rightsquigarrow v$ and $c \boxplus v = c^* \rightsquigarrow v^{**}$. Clearly, $c \boxplus v = c \oplus v^{**}$.

Proposition 3.1. *(i) If $\mathcal{H}$ is commutative, then the operation $\oplus$ is commutative and associative.*
(ii) The operation $\boxplus$ is commutative and if $\mathcal{H}$ is commutative, then $\boxplus$ is associative.
(iii) $c, v \leq c \oplus v$ and $c, v \leq c \boxplus v$.
(iv) $1 \oplus c = c \oplus 1 = 1$, $1 \boxplus c = c \boxplus 1 = 1$ and $c \boxplus c^* = 1$.
(v) $c \oplus v \leq c \boxplus v$.
If $\mathcal{H}$ is commutative, then
(vi) $c \oplus v = c \boxplus v$ and $c^* \circledcirc v^* = (c \boxplus v)^*$.

Proof. (i) Consider $c, v, q \in \mathcal{H}$. Since every bounded commutative hoop, has (DNP) property we have

$$c \oplus v = c^* \rightsquigarrow v = c^* \rightsquigarrow v^{**} = v^* \rightsquigarrow c^{**} = v^* \rightsquigarrow c = v \oplus c,$$

and

$$(c \oplus v) \oplus q = (c^* \rightsquigarrow v)^* \rightsquigarrow q = (c^* \rightsquigarrow v^{**})^* \rightsquigarrow q = ((c^* \circledcirc v^*) \rightsquigarrow 0)^* \rightsquigarrow q$$
$$= (c^* \circledcirc v^*)^{**} \rightsquigarrow q = (c^* \circledcirc v^*) \rightsquigarrow q = c^* \rightsquigarrow (v^* \rightsquigarrow q) = c \oplus (v \oplus q).$$

(ii) Let $c, v \in \mathcal{H}$. Then

$$c \boxplus v = c^* \rightsquigarrow v^{**} = v^* \rightsquigarrow c^{**} = v \boxplus c.$$

The proof of other case is similar to (i).
(iii) By Proposition 2.2(viii) and (vii) the proof is clear.
(iv) By definition of operations and Proposition 2.2(iv) the proof is clear.

(v) By Proposition 2.2(viii) and (vii), $v \leq v^{**}$, and so $c \oplus v = c^* \rightsquigarrow v \leq c^* \rightsquigarrow v^{**} = c \boxplus v$.
(vi) Since $\mathcal{H}$ is commutative, the proof is straightforward. □

Proposition 3.2. *$I \in \mathcal{ID}(\mathcal{H})$ if and only if (I3) holds and for any $c, v \in I$, we have $c \boxplus v \in I$.*

Proof. Suppose $I \in \mathcal{ID}(\mathcal{H})$. Then (I3) holds. Let $c, v \in I$. By Remark 2.4, we know that since $v \in I$, then $v^{**} \in I$. By (I2), $I \ni c \oplus v^{**} = c^* \rightsquigarrow v^{**} = c \boxplus v$. Hence, $c \boxplus v \in I$.
Conversely, obviously (I3) holds. By Proposition 3.1(v), $c \oplus v \leq c \boxplus v$, since $c \boxplus v \in I$, by (I3) we get $c \oplus v \in I$. Hence, $I \in \mathcal{ID}(\mathcal{H})$. □

The order of an element $c \in \mathcal{H}$, denoted by $ord(c)$, is the smallest natural number $n \in \mathbb{N}$ such that $c^n = 0$ and we write $ord(c) = n$. If no such n exists (that is, $c^n \neq 0$ for every $n \in \mathbb{N}$) we say that the order of c is infinite and we write $ord(c) = \infty$.

A hoop $\mathcal{H}$ is called *locally finite* if every non-unit element of $\mathcal{H}$ $(1 \neq c \in \mathcal{H})$ has finite order.

Proposition 3.3. *Let $\mathcal{H}$ be commutative. Then for any $c \in \mathcal{H}$, there exists $I \in \mathcal{ID}(\mathcal{H})$ proper such that $c \in I$ if and only if $ord(c^*) = \infty$.*

Proof. Assume $I \in \mathcal{ID}(\mathcal{H})$ is proper and $c \in \mathcal{H}$ such that $ord(c^*) \neq \infty$. Then there is $n \in \mathbb{N}$ such that $(c^*)^n = 0$, and so $1 = ((c^*)^n)^* = nc \in I$, which is a contradiction. Hence, $ord(c^*) = \infty$.
Conversely, suppose $ord(c^*) = \infty$. If $(c]$ is not proper, we get $1 \in (c]$. By Proposition 2.9, we get $nc = 1$ and so $(nc)^* = 0$, for $n \in \mathbb{N}$. Since $\mathcal{H}$ is commutative, we get $(c^*)^n = 0$, and so $ord(c^*) \neq \infty$ a contradiction. Hence, $(c]$ is proper. □

Proposition 3.4. *Let $P \in \mathcal{ID}(\mathcal{H})$. If $P \in Spec(\mathcal{H})$, then $c^{**} \wedge v^{**} \in P$ implies $c \in P$ or $v \in P$.*

Proof. Consider $P \in Spec(\mathcal{H})$ and $c^{**} \wedge v^{**} \in P$. Since by Proposition 2.2(viii), $c \wedge v \leq c^{**} \wedge v^{**}$, we get $c \wedge v \in P$. Since $P \in Spec(\mathcal{H})$, we obtain $c \in P$ or $v \in P$. □

Theorem 3.5. *Let $\mathcal{H}$ be a commutative $\vee$-hoop and $P \in \mathcal{ID}(\mathcal{H})$. Then $P \in Spec(\mathcal{H})$, if and only if $c^{**} \wedge v^{**} \in P$ implies $c \in P$ or $v \in P$.*

Proof. By using Proposition 2.12 we suppose $I, J \in \mathcal{ID}(\mathcal{H})$ such that $I \cap J \subseteq P$. If $I \nsubseteq P$ and $J \nsubseteq P$, then there are $c \in I \setminus P$ and $v \in J \setminus P$. Since by Remark 2.4, $c^{**} \in I$ and $v^{**} \in J$, from $c^{**} \wedge v^{**} \leq c^{**}, v^{**}$, we get $c^{**} \wedge v^{**} \in I \cap J \subseteq P$. By assumption $c \in P$ or $v \in P$, which is a contradiction. Therefore, $P \in Spec(\mathcal{H})$.

Converse by Proposition 3.4 holds. □

Proposition 3.6. *Let $P \in \mathcal{ID}(\mathcal{H})$. If for any $c, v \in \mathcal{H}$, $(c \rightsquigarrow v)^* \in P$ or $(c \rightsquigarrow v)^* \in P$, we get $P \in Spec(\mathcal{H})$.*

Proof. By assumption and Theorem 2.10, since for any $c, v \in \mathcal{H}$, $(c \rightsquigarrow v)^* \in P$ or $(c \rightsquigarrow v)^* \in P$, we get $\frac{c}{P} \leq \frac{v}{P}$ or $\frac{v}{P} \leq \frac{c}{P}$. Thus, $\frac{\mathcal{H}}{P}$ is a chain. Suppose $c \wedge v \in P$. Then $\frac{c \wedge v}{P} = \frac{0}{P}$, and so $\frac{c}{P} \wedge \frac{v}{P} = \frac{0}{P}$. Since $\frac{\mathcal{H}}{P}$ is a chain, we get $\frac{c}{P} = \frac{0}{P}$ or $\frac{v}{P} = \frac{0}{P}$. Hence, $c \in P$ or $v \in P$. Therefore, $P \in Spec(\mathcal{H})$. □

Remark 3.7. (i) Clearly, if $\mathcal{H}$ is a chain, then for any deal of $\mathcal{H}$. Since for any $c, v \in \mathcal{H}$, $c \leq v$ or $v \leq c$, thus $(c \rightsquigarrow v)^* = 0 \in P$ or $(v \rightsquigarrow c)^* = 0 \in P$. It means that if $\mathcal{H}$ is a chain, then every ideal of $\mathcal{H}$ is prime.
(ii) If $\mathcal{H}$ is a chain, then for any $c, v \in \mathcal{H}$, $c \circledcirc v^* \in P$ or $v \circledcirc c^* \in P$. Since $\mathcal{H}$ is a chain, we get $(c \rightsquigarrow v)^* = 0 \in P$ or $(v \rightsquigarrow c)^* = 0 \in P$. By Proposition 2.2(viii) and (vii), we have

$$c \circledcirc v^* \leq (c \circledcirc v^*)^{**} = (c \rightsquigarrow v^{**})^* \leq (c \rightsquigarrow v)^* \in P,$$

and so $c \circledcirc v^* \in P$. The proof of other case is similar.

Proposition 3.8. *Consider $\mathcal{H}$ to be a chain. For every $P \in Spec(\mathcal{H})$, the set $\mathcal{P} = \{I \in \mathcal{ID}(\mathcal{H}) \mid P \subseteq I$ and $I \neq \mathcal{H}\}$ is totally ordered by inclusion.*

Proof. Suppose $\mathcal{P}$ is not a chain. Then there exist $I_1, I_2 \in \mathcal{ID}(\mathcal{H})$ such that $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$. Thus, there are $c \in I_1 \setminus I_2$ and $v \in I_2 \setminus I_1$. By Remark 3.7(ii), we have $c \circledcirc v^* \in P$ or $v \circledcirc c^* \in P$. Thus by definition of $\mathcal{P}$, we have $c \circledcirc v^* \in I_1 \cap I_2$ or $v \circledcirc c^* \in I_1 \cap I_2$. Suppose $v \circledcirc c^* \in I_1$. Since $c \in I_1$, we get $c \oplus (v \circledcirc c^*) \in I_1$. By Proposition 2.2(ii), we get $v \leq c^* \rightsquigarrow (v \circledcirc c^*) = c \oplus (v \circledcirc c^*) \in I_1$. Then $v \in I_1$ which is a contradiction. The proof of other case is similar. Hence, $\mathcal{P}$ is a chain. □

By [19], $N(G) = (G^-, \odot, \rightarrow, 0)$ is a hoop, where $\odot$ and $\rightarrow$ are defined as follows:

$$g \odot h = g + h, \quad g \rightarrow h = (h - g) \wedge 0, \quad \forall g, h \in G^-. \tag{1}$$

Example 3.9. Let $\mathcal{G}$ be the Abelian ℓ-group of integers with natural ordering, $\mathbb{Z}$. Then $N(\mathcal{G})$ is a totally ordered hoop. Consider an element $\perp \notin \mathbb{Z}$ and set $X = \{\perp, 0, -1, -2, \ldots\}$. Define binary operation $\boxdot$ and $\Rightarrow$ on X as follows:

$$x \Rightarrow y := \begin{cases} x \rightarrow y & \text{if } x, y \in \{0, -1, -2, \ldots\}, \\ 1 & \text{if } x = \perp, \\ \perp & \text{if } x = \perp, y \in \{0, -1, -2, \ldots\}; \end{cases}$$

$$x \boxdot y := \begin{cases} x \odot y & \text{if } x, y \in \{0, -1, -2, \ldots\}, \\ \bot & \text{if } x = \bot \text{ or } y = \bot; \end{cases}$$

by routine calculation we can show that $(x, \boxdot, \Rightarrow, 0)$ is bounded hoop with the least element $\bot$ and the greatest element 0. In addition, $\{\bot\}$ is a maximal ideal of c.

Example 3.10. Consider the Abelain ℓ-group $\mathbb{R}$ of real numbers with natural partially ordered relation. Then $\mathcal{G} = \mathbb{R} \overrightarrow{\times} \mathbb{R}$ is a totally ordered group and $\mathcal{G}[(1,0)]$ bounded hoop. $\{(0,0)\}$ is a prime ideal of $\mathcal{G}[(1,0)]$. Thus this example satisfies the assumption Proposition 3.8.

Remark 3.11. In a hoop $\mathcal{H}$, if $(P_i)_{i\in I} \subseteq Spec(\mathcal{H})$ is a totally ordered family of prime ideals of $\mathcal{H}$, then $P = \bigcap_{i\in I} P_i \in Spec(\mathcal{H})$ and $Q = \bigvee_{i\in I} P_i \in Spec(\mathcal{H})$. Indeed, let $c, v \in \mathcal{H}$ such that $c \wedge v \in P$, if $c \notin P$ and $v \notin P$, then there are $i_1, i_2 \in I$ such that $c \notin P_{i_1}$ and $v \notin P_{i_2}$. Since P_{i_1}, P_{i_2} are prime ideals and $c \wedge v \in P_{i_1}$, P_{i_2}, then $c \in P_{i_1}$ or $v \in P_{i_1}$ and $c \in P_{i_2}$ or $v \in P_{i_2}$. Since the family $(P_i)_{i\in I} \subseteq Spec(\mathcal{H})$ is a totally ordered, then $P_{i_1} \subseteq P_{i_2}$ or $P_{i_2} \subseteq P_{i_1}$. If $P_{i_1} \subseteq P_{i_2}$, then $v \in P_{i_2}$, a contradiction. The proof of other case is similar. It follows that $c \in P$ or $v \in P$, that is, $P \in Spec(\mathcal{H})$. Also, suppose $c \wedge v \in Q$. Then there is $i \in I$ such that $c \wedge v \in P_i$, and since $P_i \in Spec(\mathcal{H})$ we get $c \in P_i$ or $v \in P_i$. Hence, $c \in Q$ or $v \in Q$. Therefore, $Q \in Spec(\mathcal{H})$.

Corollary 3.12. *By all conditions in Remark 3.11, the set of all prime ideal of $\mathcal{H}$ makes a complete lattice.*

Proposition 3.13. *Suppose $\mathcal{H}$ is a commutative $\vee$-hoop. Then $ord(c^*) < \infty$ if and only if $c \notin P$, for any $P \in Spec(\mathcal{H})$.*

Proof. Assume $ord(c^*) < \infty$ and there is a prime ideal P such that $c \in P$. So, for any $n \in \mathbb{N}$, $nc \in P$. Also, since $ord(c^*) < \infty$, there exists $n \in \mathbb{N}$ such that $(c^*)^n = 0$ and so $((c^*)^n)^* = 1$. Since $\mathcal{H}$ is commutative, by Notation 1, $nc = ((c^*)^n)^* = 1 \in P$, which is a contradiction, since $P \in Spec(\mathcal{H})$.

Conversely, suppose for any $P \in Spec(\mathcal{H})$, $c \notin P$ and $ord(c^*) = \infty$. Clearly, $(c]$ is proper and by Proposition 2.12(ii), there is $P \in Spec(\mathcal{H})$ such that $(c] \subseteq P$, and so $c \in P$, a contradiction.

□

Proposition 3.14. *Suppose X is a subalgebra of a commutative $\vee$-hoop $\mathcal{H}$ and $P_1 \in Spec(X)$. Then there exists $P \in Spec(\mathcal{H})$ such that $P_1 = P \cap X$.*

Proof. Let $P_1 \in Spec(X)$ and $S = X \setminus P_1$. Clearly, $S \cap P_1 = \emptyset$. Also, for any $c, v \in S$, since $P_1 \in Spec(X)$, we get $c \wedge v \in S$. So, S is $\wedge$-closed. Then by Proposition 2.13, there is

$P \in Spec(\mathcal{H})$ such that $P_1 \subseteq P$ and $P \cap S = \emptyset$. Moreover, from $P \cap S = P \cap (X \setminus P_1)$, we get $P \subseteq P_1$, and so $P_1 = X \cap P$. □

Proposition 3.15. *Suppose $\mathcal{H}$ is commutative. Then $M \in Max(\mathcal{H})$ if and only if ($c \notin M \Leftrightarrow$ there exists $n \in \mathbb{N}$ such that $(nc)^* \in M$.*

Proof. Suppose $M \in Max(\mathcal{H})$ and $c \notin M$. Then $(M \cup \{c\}] = \mathcal{H}$, and so $1 \in (M \cup \{c\}]$. By Proposition 2.9(ii), there is $n \in \mathbb{N}$ such that $1 \circledcirc (nc)^* \in M$. Hence, $(nc)^* \in M$. Conversely, suppose $c \in M$ and there exists $n \in \mathbb{N}$ such that $(nc)^* \in M$. Since M is ideal, we get $nc \in M$ and so $1 = (nc) \oplus (nc)^* \in M$, a contradiction.

Suppose I is a proper ideal of $\mathcal{H}$ such that $M \subseteq I$. If $I \neq M$, then there is $c \in I \setminus M$. By assumption, there exists $n \in \mathbb{N}$ such that $(nc)^* \in M$, and so $(nc)^* \in I$. Since $c \in I$ and I is ideal, we get $nc \in I$ for any $n \in \mathbb{N}$, thus $1 = (nc) \oplus (nc)^* \in I$, which is a contradiction. Therefore, $I = M$, and so $M \in Max(\mathcal{H})$. □

Theorem 3.16. *Suppose M is a proper ideal of $\mathcal{H}$. Then $M \in Max(\mathcal{H})$ if and only if $\dfrac{\mathcal{H}}{M}$ is locally finite.*

Proof. Suppose $M \in Max(\mathcal{H})$ and $\frac{c}{M} \neq \frac{1}{M}$. Then $c^* \notin M$. By Proposition 3.15, there exists $n \in \mathbb{N}$ such that $(nc^*)^* \in M$. Thus, $((c^{**})^n)^{**} \in M$. Since M is ideal, by Remark 2.4, $c^n \in M$, and so $\left(\frac{c}{M}\right)^n = \frac{0}{M}$. Hence, $\dfrac{\mathcal{H}}{M}$ is locally finite.

Conversely, suppose I is a proper ideal of $\mathcal{H}$ such that $M \subseteq I$. Then there is $c \in I \setminus M$. Thus, $\frac{c^*}{M} \neq \frac{1}{M}$. Since if $\frac{c^*}{M} = \frac{1}{M}$, then $c^{**} \in M$, and so $c \in M$, a contradiction. Also, $\dfrac{\mathcal{H}}{M}$ is locally finite, thus there is $n \in \mathbb{N}$ such that $\left(\frac{c^*}{M}\right)^n = \frac{0}{M}$, and so $(c^*)^n \in M$. Hence, $(c^*)^n \in I$, and so $1 \in I$, which is a contradiction. Therefore, $M = I$, and so $M \in Max(\mathcal{H})$. □

Theorem 3.17. *Every prime ideal of $\mathcal{H}$ is contained in a unique maximal ideal of $\mathcal{H}$*

Proof. Let $P \in Spec(\mathcal{H})$ and $\mathcal{P} = \{I \in \mathcal{ID}(\mathcal{H}) \mid P \subseteq I \text{ and } I \neq \mathcal{H}\}$. By Proposition 3.8, $\mathcal{P}$ is totally ordered by inclusion. Set $M = \bigcup_{i \in I} P_i$. Clearly, M is a proper ideal of $\mathcal{H}$ and it is the only maximal ideal containing P. □

Let $\mathcal{G}$ be an arbitrary Abelian ℓ-group with neutral element 0 and the binary operation $+$. For every $a \geq 0$, by [19], $\mathcal{G}[a] = ([0,a], \odot, \rightarrow, 0, u)$ is a bounded hoop, where

(2) $g \odot h = (g + h - a) \vee 0, \quad g \rightarrow h = (h - g + a) \wedge a, \quad \forall g, h \in [0,a] = \{g \in G \mid 0 \leq g \leq a\}.$

Example 3.18. Let $\mathcal{H}$ be an arbitrary Abelian ℓ-group with neutral element 0 and the binary operation $+$. Consider the lexicographic products of l-groups $\mathcal{G} = \mathbb{Z}\overrightarrow{\times}\mathcal{H}$. It is an Abelian l-group and we have $(0,0) < (1,0)$. Thus, $\mathcal{G}[(1,0)]$ is a Wajsberg hoop, where it underlying set is $(\{0\} \times H^+) \cup (\{1\} \times H^-)$. An easy calculation shows that $I = \{0\} \times H^+$ is unique maximal ideal of the hoop $\mathcal{G}[(1,0)]$. In addition, $\{(0,0)\}$ is a prime ideal of $\mathcal{G}[(1,0)]$.

Proposition 3.19. *Consider $\mathcal{H}$ is commutative. Set $\mathcal{J} = \{c \in \mathcal{H} \mid ord(c^*) = \infty\}$. Then the following statements are equivalent:*
(i) $\mathcal{J} \in \mathcal{ID}(\mathcal{H})$.
(ii) $(\mathcal{J}]$ is a proper ideal of $\mathcal{H}$.
(iii) $\mathcal{H}$ has just one maximal ideal.
(iv) $Max(\mathcal{H}) = \{\mathcal{J}\}$.

Proof. $(i) \Rightarrow (ii)$ Obviously, $(\mathcal{J}] = \mathcal{J}$. Since $1 \notin \mathcal{J}$, so $(\mathcal{J}]$ is proper.
$(ii) \Rightarrow (i)$ Clearly, $0 \in \mathcal{J}$ and so $\mathcal{J} \neq \emptyset$ and (I1) holds. Suppose $v \in \mathcal{J}$ and $c \leq v$. Since $ord(v^*) = \infty$ and by Proposition 2.2, we have $v^* \leq c^*$, thus $ord(c^*) = \infty$. Hence, $c \in \mathcal{J}$, and so (I3) holds. Assume $c, v \in \mathcal{J}$. Then $ord(c^*) = \infty$ and $ord(v^*) = \infty$. If $c \oplus v \notin \mathcal{J}$, then there is $n \in \mathbb{N}$ such that $((c \oplus v)^*)^n = 0$. By Proposition 2.2(viii) and (vii) we have

$$c^* \rightsquigarrow v \leq c^* \rightsquigarrow v^{**} \Rightarrow (c^* \rightsquigarrow v^{**})^* \leq (c^* \rightsquigarrow v)^* = (c \oplus v)^*.$$

Thus,

$$(c^* \circledcirc v^*)^n \leq ((c^* \circledcirc v^*)^{**})^n = ((c^* \rightsquigarrow v^{**})^*)^n \leq ((c \oplus v)^*)^n = 0.$$

So $(c^* \circledcirc v^*)^n = 0$. Since $\mathcal{H}$ is a commutative monoid, we get $(c^*)^n \circledcirc (v^*)^n = 0$, and so $((c^*)^n \circledcirc (v^*)^n)^* = 1$. By Proposition 2.2(viii) we have

$$1 = ((c^*)^n \circledcirc (v^*)^n)^* = (v^*)^n \rightsquigarrow ((c^*)^n)^* = (v^*)^n \rightsquigarrow ((c^*)^n)^{***} = ((c^*)^n)^{**} \rightsquigarrow ((v^*)^n)^* = (nc) \oplus (nv).$$

Hence, by Proposition 2.9, $1 \in (\mathcal{J}]$, which is a contradiction.
$(iv) \Rightarrow (iii)$ The proof is clear.
$(iii) \Rightarrow (iv)$ Let $\mathcal{H}$ has just one maximal ideal such as M. By definition of $\mathcal{J}$, for any $c \in \mathcal{J}$, $ord(c^*) = \infty$, we get $(c]$ is a proper ideal and it can extend to a maximal ideal such as M_c. Since $\mathcal{H}$ has just one maximal ideal such as M, we get $M_c = M$. Hence, $c \in M$, and so $\mathcal{J} \subseteq M$. On the other side, since M is proper, by Proposition 3.3, we get $M \subseteq \mathcal{J}$. Hence, $Max(\mathcal{H}) = \{\mathcal{J}\}$.
$(i) \Rightarrow (iv)$ For proving that $\mathcal{J}$ is maximal, we suppose $c \notin \mathcal{J}$. Then $ord(c^*) < \infty$, and so there is $n \in \mathbb{N}$ such that $(c^*)^n = 0$. By assumption, $(nc)^* = ((c^*)^n)^{**} = 0$. By Proposition 3.15, we get $\mathcal{J} \in Max(\mathcal{H})$. Suppose there exists a proper ideal I such that $I \nsubseteq \mathcal{J}$. Then there is $c \in I \setminus \mathcal{J}$. Thus there is $n \in \mathbb{N}$ such that $(c^*)^n = 0$, and so $nc = ((c^*)^n)^* = 1 \in I$, a contradiction. Hence, $Max(\mathcal{H}) = \{\mathcal{J}\}$. □

A hoop $\mathcal{H}$ is called *local* if it has just only one maximal ideal.

Theorem 3.20. *If $\mathcal{H}$ is a commutative local hoop, then for any $c \in \mathcal{H}$, $ord(c) < \infty$ or $ord(c^*) < \infty$.*

Proof. Suppose for any $c \in \mathcal{H}$ and $n \in \mathbb{N}$, we get $c^n > 0$ and $(c^*)^n > 0$. By Proposition 2.2(viii), $(c^{**})^n > 0$. Thus, by Proposition 3.19, $c, c^* \in \mathcal{J}$, and so $1 \in \mathcal{J}$, which is a contradiction. □

Now, by using the notion of prime ideal on hoop we can define spectral topology on hoops, as follows:

Suppose that $\mathcal{H}$ is a non-trivial hoop. For any $I \in \mathcal{ID}(\mathcal{H})$, define

$$\mathcal{V}(I) = \{P \in Spec(\mathcal{H}) \mid I \subseteq P\}.$$

Proposition 3.21. *Consider $\mathcal{H}$ is commutative. The family of $\{\mathcal{V}(I_i)\}_{i\in I}$ of subsets of $Spec(P)$ satisfies the axioms for closed sets in a topological space.*

Proof. Clearly, $\mathcal{V}(\mathcal{H}) = \emptyset$ and $\mathcal{V}(\{0\}) = Spec(\mathcal{H})$. We show that for any $I, J \in \mathcal{ID}(\mathcal{H})$, $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cap J)$. For this suppose $P \in \mathcal{V}(I) \cup \mathcal{V}(J)$. Then $I \subseteq P$ or $J \subseteq P$. Since $I \cap J \subseteq I, J$, we get $I \cap J \subseteq P$, and so $P \in \mathcal{V}(I \cap J)$. Conversely, suppose $P \in \mathcal{V}(I \cap J)$. Then $I \cap J \subseteq P$. Since $P \in Spec(\mathcal{H})$, by Proposition 2.12, we get $I \subseteq P$ or $J \subseteq P$. Hence, $P \in \mathcal{V}(I) \cup \mathcal{V}(J)$. Now, we show that $\bigcap_{i\in I} \mathcal{V}(I_i) = \mathcal{V}\left(\bigvee_{i\in I} I_i\right)$. For this, suppose $P \in \bigcap_{i\in I} \mathcal{V}(I_i)$. Then for any $i \in I$, $P \in \mathcal{V}(I_i)$, thus, for any $i \in I$, $I_i \subseteq P$. So, $\bigvee_{i\in I} I_i \subseteq P$ and so $P \in \mathcal{V}\left(\bigvee_{i\in I} I_i\right)$. In addition, if $P \in \mathcal{V}\left(\bigvee_{i\in I} I_i\right)$, we get $\bigvee_{i\in I} I_i \subseteq P$. Since $I_i \subseteq \bigvee_{i\in I} I_i \subseteq P$, we get $P \in \mathcal{V}(I_i)$, for any $i \in I$. Hence, $P \in \bigcap_{i\in I} \mathcal{V}(I_i)$. Therefore, all axioms for closed sets in a topological space hold. □

According to above proposition, the set of $\mathcal{V}(I)$ is closed subset so the complement of them is open. So, $\mathcal{V}(I)^c = \{P \in Spec(\mathcal{H}) \mid I \nsubseteq P\}$ is open subset of $\mathcal{H}$. Hence, $\Psi_{\mathcal{H}} = \{\mathcal{V}(I)^c \mid I \in \mathcal{ID}(\mathcal{H})\}$ is called a spectral topology on $\mathcal{H}$.

Theorem 3.22. *Suppose $\mathcal{H}$ is a hoop where for any $c \in \mathcal{H}$, $c \wedge c^* = 0$. Then $(Spec(\mathcal{H}), \Psi_{\mathcal{H}})$ is Hausdorff.*

Proof. Let P_1 and P_2 be two distinct prime ideals of $\mathcal{H}$ such that $P_1 \nsubseteq P_2$ or $P_2 \nsubseteq P_1$. Without loss of generality, we may assume that $P_1 \nsubseteq P_2$. So, there is $c \in P_1 \setminus P_2$. Consider $I = \mathcal{V}^c(c)$ and $J = \mathcal{V}^c(c^*)$. Clearly, I and J are neighborhoods of P_1 and P_2, respectively. Then $I \cap J = \mathcal{V}^c(c) \cap \mathcal{V}^c(c^*) = \mathcal{V}^c(c \wedge c^*) = \mathcal{V}^c(0) = \emptyset$. Therefore, $(Spec(\mathcal{H}), \Psi_{\mathcal{H}})$ is Hausdorff.

□

4. Conclusions and future works

In this paper, the concept of ideals and prime ideals in hoops are explored and characterized certain properties of prime ideals. The notion of a local hoop is introduced and examined various equivalent definitions of this concept. Furthermore, it is proved that if M is a maximal ideal of a hoop, the corresponding quotient structure is locally finite. Finally, a topology on the set of all prime ideals of a hoop is defined and investigated the conditions under which this topology becomes Hausdorff.

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