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Research Paper

ON (t,n)-ABSORBING δ-SEMIPRIMARY HYPERIDEALS

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Abstract. The δ-primary hyperideal is a concept unifing the n-ary prime and n-ary primary hyperideals under one frame where δ is a function which assigns to each hyperideal Q of a hyperring G a hyperideal $\delta(Q)$ of the same hyperring with specific properties. In this paper, for a commutative Krasner (m,n)-hyperring (G,h,k) with scalar identity 1, we aim to introduce and study the notion of (t,n)-absorbing δ-semiprimary hyperideals which is a more general structure than δ-primary hyperideals. We say that a proper hyperideal Q of G is an (t,n)-absorbing δ-semiprimary hyperideal if whenever $k(a_1^{tn-t+1}) \in Q$ for $a_1^{tn-t+1} \in G$, then there exist $(t-1)n-t+2$ of the a_is whose k-product is in $\delta(Q)$. Furthermore, we extend the concept to weakly (t,n)-absorbing δ-semiprimary hyperideals. Several properties and characterizations of these classes of hyperideals are determined. In particular, after defining srongly weakly (t,n)-absorbing δ-semiprimary hyperideals, we present the condition in which a weakly (t,n)-absorbing δ-semiprimary hyperideal is srongly. Moreover, we show that $k(Q^{(tn-t+1)}) = 0$ where the weakly (t,n)-absorbing δ-semiprimary hyperideal Q is not (t,n)-absorbing δ-semiprimary. We give a type of Nakayama's Lemma on a commutative Krasner (m,n)-hyperring. Also, we investigate the stability of the concepts under intersection, homomorphism and cartesian product of hyperrings.

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1. Introduction

For the first time, the idea of 2-absorbing ideals as an extension of prime ideals was presented by Badawi in [5]. This concept and it's generalizations have been widely studied by many researchers e.g. in [1], [11], [19] and [23]. The notion of 2-absorbing δ-semiprimary ideals in a commutative ring which is an expansion of the 2-absorbing ideals was introduced by Celikel. From [6], a proper ideal I of a commutative ring R refers to a (weakly) 2-absorbing δ-semiprimary ideal if $x, y, z \in R$ and $(0 \neq xyz \in I)$ $xyz \in I$ imply $xy \in \delta(I)$ or $yz \in \delta(I)$ or $xz \in \delta(I)$ where δ is a function that assigns to each ideal I an ideal $\delta(I)$ of the same ring.

Study on n-ary algebras goes back to Kasner's lecture [13] at a scientific meeting in 1904. In 1928, the first paper was written concerning the theory of n-ary groups by Dorente. The n-ary structures have been studied in [7], [10], [14], [15], [16], [17], [18] and [21]. There is a hyperring in which the addition is a hyperoperation, while the multiplication is an ordinary binary operation. The hyperring introduced by Krasner is called Krasner hyperring. The Krasner (m, n)-hyperring as a generalization of the Krasner hyperrings and a subclass of (m, n)-hyperrings was introduced by Mirvakili and Davvaz in [18]. (G, h, k), or simply G, is a Krasner (m, n)-hyperring if: (1) (G, h) is a canonical m-ary hypergroup; (2) (G, k) is a n-ary semigroup; (3) The operation k is distributive with respect to the hyperoperation h , i.e., for every $g_1^{i-1}, g_{i+1}^n, x_1^m \in G$, and $1 \leq i \leq n$, $k(g_1^{i-1}, h(x_1^m), g_{i+1}^n) = h(k(g_1^{i-1}, x_1, g_{i+1}^n), \ldots, k(g_1^{i-1}, x_m, g_{i+1}^n))$; (4) 0 is a zero element of the n-ary operation k, i.e., for every $g_2^n \in R$, $k(0, g_2^n) = k(g_2, 0, g_3^n) = \cdots = k(g_2^n, 0) = 0$. A non-empty subset H of G is a subhyperring of G if (H, h, k) is a Krasner (m, n)-hyperring. The non-empty subset I of G is a hyperideal if (I, h) is an m-ary subhypergroup of (G, h) and $k(g_1^{i-1}, I, g_{i+1}^n) \subseteq I$, for all $g_1^n \in G$ and $1 \leq i \leq n$. By g_i^j we mean the sequence $g_i, g_{i+1}, \ldots, g_j$ and it is the empty symbol if $j < i$. Then we have $h(a_1, \ldots, a_i, b_{i+1}, \ldots, b_j, c_{j+1}, \ldots, c_n) = h(a_1^i, b_{i+1}^j, c_{j+1}^n)$, and we write $h(a_1^i, b^{(j-i)}, c_{j+1}^n)$, where $b_{i+1} = \cdots = b_j = b$. We define $h(G_1^n) = \bigcup\{h(g_1^n) \mid g_i \in G_i, 1 \leq i \leq n\}$, where G_1^n are non-empty subsets of G. Recall from [2] that a proper hyperideal P of G is called n-ary prime if $k(G_1^n) \subseteq P$, for hyperideals G_1^n of R implies that $G_i \subseteq P$, for some $1 \leq i \leq n$. If $k(g_1^n) \in P$, for all $g_1^n \in G$ and some hyperideal P of G implies that $g_i \in P$, for some $1 \leq i \leq n$, then P is an n-ary prime hyperideal of G, by Lemma 4.5 in [2]. Let I be a hyperideal in G with scalar identity 1. By $rad(I)$, we mean the intersection is taken over all n-ary prime hyperideals P which contain I. $rad(I) = G$, if the set of all n-ary prime hyperideals containing I is empty . It was shown (Theorem 4.23 in [2]) that if $g \in rad(I)$, then there exists $s \in \mathbb{N}$ such that $k(g^{(s)}, 1^{(n-s)}) \in I$, for $s \leq n$, or $k_{(l)}(g^{(s)}) \in I$, for $s = l(n-1)+1$. A proper hyperideal I of G with the scalar identity 1 is an n-ary primary hyperideal if $k(g_1^n) \in I$ and $g_i \notin I$, imply $k(g_1^{i-1}, 1, g_{i+1}^n) \in rad(I)$, for some $1 \leq i \leq n$. Theorem 4.28 in [2] shows that the radical of an n-ary primary hyperideal of G is an n-ary prime hyperideal of G. Let $\mathcal{HI}(G)$ denote the set of

all hyperideals of G. Recall from [3] that a function δ from $\mathcal{HI}(G)$ to $\mathcal{HI}(G)$ is a hyperideal expansion of G if $I \subseteq \delta(I)$, and if $I \subseteq J$, for any hyperideals I, J of G, then $\delta(I) \subseteq \delta(J)$. For example, the mapping δ_0, δ_1 and δ_R, defined by $\delta_0(I) = I$, $\delta_1(I) = rad(I)$ and $\delta_R(I) = R$, for all hyperideals I of G, are hyperideal expansions of G. Furthermore, the function δ_q, defined by $\delta_q(I/J) = \delta(I)/J$, for any hyperideal I of G containing hyperideal J and an expansion function δ of G, is a hyperideal expansion of G/J. A proper hyperideal P of G is an (t,n)-absorbing hyperideal as in [12] if $k(g_1^{tn-t+1}) \in P$, for some $g_1^{tn-t+1} \in G$, implies that there exist $(t-1)n-t+2$ of the g_is whose k-product is in P. From the paper, a proper P of G is called an (t,n)-absorbing primary hyperideal if $k(g_1^{tn-t+1}) \in P$, for some $g_1^{tn-t+1} \in G$ implies that $k(g_1^{(t-1)n-t+2}) \in P$ or a k-prodect of $(t-1)n-t+2$ of the g_is except $k(g_1^{tn-t+2})$ is in $rad(P)$. The notion of n-ary δ-primary hyperideals was introduced in a Krasner (m,n)-hyperring in [3]. This concept unifies the n-ary prime and n-ary primary hyperideals under one frame. A proper hyperideal Q of G is called δ-primary hyperideal if $k(a_1^n) \in Q$ for $a_1^n \in G$, implies that $a_i \in Q$ or $k(a_1^{i-1}, 1, a_{i+1}^n) \in \delta(Q)$, for some $1 \leq i \leq n$. Also, a proper hyperideal Q of G is (t,n)-absorbing δ-primary if for $a_1^{tn-t+1} \in G$, $k(a_1^{tn-t+1}) \in Q$, implies that $k(a_1^{(t-1)n-t+2}) \in Q$, or a k-product of $(t-1)n-t+2$ of a_i s except $k(a_1^{(t-1)n-t+2})$ is in $\delta(Q)$. Morover, the idea of weakly (t,n)-absorbing (primary) hyperideals in Krasner (m,n)-hyperrings was proposed by Davvaz et al. in [9].

Now in this paper, we aim to define and study the notion of (t,n)-absorbing δ-semiprimary hyperideals which is more general than the concept of δ-primary hyperideals in a Krasner (m,n)-hyperring. Additionally, we present an extension of the notion called weakly (t,n)-absorbing δ-semiprimary hyperideals. Among many results in this paper, it is shown (Theorem 2.22) that if Q is a weakly (t,n)-absorbing δ-semiprimary hyperideal of G that is not (t,n)-absorbing δ-semiprimary, then $k(Q^{(tn-t+1)}) = 0$. A version of Nakayama's Lemma is considered for strongly weakly (t,n)-absorbing δ-semiprimary hyperideals in Theorem 2.27. Moreover, the stability of these concepts are examined under intersection, homomorphism and cartesian product of hyperrings.

2. (t,n)-absorbing δ-semiprimary hyperideals

Throughout this section, G is a commutative Krasner (m,n)-hyperring with scalar identity 1. Initially, we give the definition of (t,n)-absorbing δ-semiprimary hyperideals of G.

Definition 2.1. Let δ be a hyperideal expansion of G and t be a positive integer. A proper hyperideal Q of G is called an (t,n)-absorbing δ-semiprimary hyperideal if whenever $k(a_1^{tn-t+1}) \in Q$, for $a_1^{tn-t+1} \in G$, then there exist $(t-1)n-t+2$ of the a_is whose k-product is in $\delta(Q)$.

Example 2.2. Assume that $(\mathbb{Z}, +, \cdot)$ is the ring of integers and $K = \{-1, 1\}$. Consider the Krasner (m, n)-hyperring $(G = \{\bar{g} \mid g \in \mathbb{Z}\}, h, k)$, where $\bar{g} = gK$, and the 3-ary hyperoperation h and 5-ary multiplication k are defined as follows:

$$\begin{cases} h(\bar{g}_1^3) = \{\bar{x} \mid \bar{x} \subseteq \bar{g}_1 + \bar{g}_2 + \bar{g}_3\}, \\ k(\bar{g}_1^5) = \overline{g_1 g_2 g_3 g_4 g_5} \ . \end{cases}$$

Then $\langle pK \rangle$ is an $(3, 5)$-absorbing δ_0-semiprimary hyperideal of G for all prime numbers.

Example 2.3. Suppose that $G = \{0, 1, 2\}$, and define 3-ary hyperoperation h and 3-ary operation k on G as follows:

$h(0,0,0) = 0$, $h(0,0,2) = 2$, $h(1,1,2) = G$,
$h(0,0,1) = 1$, $h(2,2,2) = 2$, $h(0,1,1) = 1$,
$h(1,1,1) = 1$, $h(0,1,2) = G$, $h(0,2,2) = 2$,
$h(1,2,2) = G$,

and

$k(1,1,1) = 1$,
$k(1,2,2) = k(1,1,2) = k(2,2,2) = 2$,
$k(r_1, r_2, 0) = 0$,

for $r_1, r_2 \in G$, such that h and k are commutative. Then, (G, h, k) is a Krasner $(3, 3)$-hyperring by Example 4.10 in [2]. In the hyperring G, $Q = \{0, 2\}$ is an $(2, 3)$-absorbing δ_1-semiprimary hyperideal of G.

Remark 2.4. For all n-ary prime hyperideal of G, we have
n-ary prime $\Longrightarrow$ n-ary δ-primary $\Longrightarrow$ (t, n)-absorbing δ-primary $\Longrightarrow$ (t, n)-absorbing δ-semiprimary

The next example shows that an (t, n)-absorbing δ-semiprimary hyperideal may not be prime.

Example 2.5. In the Krasner $(2, 2)$-hyperring $(G = [0, 1], \boxplus, \circ)$, that "$\circ$" is the usual multiplication on real numbers and 2-ary hyperoperation "$\boxplus$" is defined by

$$a \boxplus b = \begin{cases} \{max\{a, b\}\}, & \text{if} a \neq b, \\ [0, a], & \text{if} a = b, \end{cases}$$

the hyperideal $Q = [0, 0.5]$ is a $(2, 2)$-absorbing δ_1-semiprimary hyperideal of G but it is not 2-ary prime.

Example 2.6. Assume that Q is a hyperideal of (G, h, k). Then the set $G/Q = \{h(g_1^{i-1}, Q, g_{i+1}^m) \mid g_1^{i-1}, g_{i+1}^m \in G\}$, with m-ary hyperoperation h and n-ary operation k defind as $h(h(g_{11}^{1(i-1)}, Q, g_{1(i+1)}^{1m}), \ldots, h(g_{m1}^{m(i-1)}, Q, g_{m(i+1)}^{mm})) = h(h(g_{11}^{m1}), \ldots, h(g_{1(i-1)}^{m(i-1)}), Q, h(g_{1(i+1)}^{m(i+1)}), \ldots, h(g_{1m}^{mm}))$, for all $g_{11}^{1m}, \ldots, g_{m1}^{mm} \in G$, and $k(h(g_{11}^{1(i-1)}, Q, g_{1(i+1)}^{1m}), \ldots, h(g_{n1}^{n(i-1)}, Q, g_{n(i+1)}^{nm})) = h(k(g_{11}^{n1}), \ldots, k(g_{1(i-1)}^{n(i-1)}), Q, k(g_{1(i+1)}^{n(i+1)}), \ldots, k(g_{1m}^{nm}))$, for all $g_{11}^{1m}, \ldots, g_{n1}^{nm} \in G$, construct a Krasner (m, n)-hyperring [2]. Now, let us assume that $G = \mathbb{Z}_3[X, Y, Z]$ and $Q = \langle X^3Y^3Z^3 \rangle$. Then G/Q is a Krasner (m, n)-hyperring with ordinary addition and ordinary multiplication. In this hyperring, $rad(0_{G/Q})$ is not a $(1, 3)$-absorbing δ_0-semiprimary hyperideal of G/Q, where $0_{G/Q} = Q/Q$, because $2XYZ + Q = (2X + Q)(Y + Q)(Z + Q) \in rad(0_{G/Q})$, but no power of elements $2X, Y$ and Z are in Q which means none of the elements $(2X + Q), (Y + Q)$ and $(Z + Q)$ are in $rad(0_{G/Q})$.

In the following, we investigate the relationship between an (t, n)-absorbing δ-semiprimary hyperideal of G and its radical.

Theorem 2.7. *Let Q be an (t, n)-absorbing δ-semiprimary hyperideal of G with $rad(\delta(Q)) \subseteq \delta(rad(Q))$. Then $rad(Q)$ is an (t, n)-absorbing δ-semiprimary hyperideal of G.*

Proof. Let $k(a_1^{tn-t+1}) \in rad(Q)$, for $a_1^{tn-t+1} \in G$, such that all products of $(t-1)n - t + 2$ of the a_i's, other than $k(a_1^{(t-1)n-t+2})$, are not in $\delta(rad(Q))$. By the assumption, we conclude that none of the k-products of the a_i's are in $rad(\delta(Q))$. Since $k(a_1^{tn-t+1}) \in rad(Q)$, there exists $s \in \mathbb{N}$, with $k(k(a_1^{tn-t+1})^{(s)}, 1^{(n-s)}) \in Q$, for $s \leq n$, or $k_{(l)}(k(a_1^{tn-t+1})^{(s)}) \in Q$, for $s > n$, and $s = l(n-1) + 1$. In the first possibility, we get $k(k(a_1)^{(s)}, k(a_2)^{(s)}, \ldots, k(a_{tn-t+1})^{(s)}, 1^{(n-s)}) \in Q$. Since Q is an (t, n)-absorbing δ-semiprimary hyperideal of G, we conclude that $k(k(a_1)^{(s)}, k(a_2)^{(s)}, \ldots, k(a_{(t-1)n-t+2})^{(s)}, 1^{(n-s)}) = k(k(a_1^{(t-1)n-t+2})^{(s)}, 1^{(n-s)}) \in \delta(Q)$, because none of the k-products of the a_i's are in $rad(\delta(Q))$. Then, we get the result that $k(a_1^{(t-1)n-t+2}) \in \delta(rad(Q))$, as $k(a_1^{(t-1)n-t+2}) \in rad(\delta(Q))$ and $rad(\delta(Q)) \subseteq \delta(rad(Q))$. If $k_{(l)}(k(a_1^{tn-t+1})^{(s)}) \in Q$, for $s > n$, and $s = l(n-1) + 1$, then we are done similarly. Thus $rad(Q)$ is an (t, n)-absorbing δ-semiprimary hyperideal of G. □

The following result is a direct consequence of the previous theorem.

Corollary 2.8. *If Q is an (t, n)-absorbing δ_1-semiprimary hyperideal of G, then $rad(Q)$ is an (t, n)-absorbing hyperideal of G.*

From [2], the hyperideal generated by an element g in G is defined by $< g >= k(G, g, 1^{(n-2)}) = \{k(r, g, 1^{(n-2)}) \mid r \in G\}$. The following theorem will give us a characterization of (t, n)-absorbing δ-semiprimary hyperideals.

Theorem 2.9. *Every proper hyperideal is an (t,n)-absorbing δ-semiprimary hyperideal of G if and only if every proper principal hyperideal is an (t,n)-absorbing δ-semiprimary hyperideal of G.*

Proof. $\Longrightarrow$ It is obvious.

$\Longleftarrow$ Assume that Q is a proper hyperideal of G and $k(a_1^{tn-t+1}) \in Q$, for $a_1^{tn-t+1} \in G$. Therefore $k(a_1^{tn-t+1}) \in < k(a_1^{tn-t+1}) >$. Since every proper principal hyperideal is an (t,n)-absorbing δ-semiprimary hyperideal of G, there exist $(t-1)n-t+2$ of the a_i's whose k-product is in $\delta(< k(a_1^{tn-t+1}) >) \subseteq \delta(Q)$. Hence Q is an (t,n)-absorbing δ-semiprimary hyperideal of G. □

Recall from [3] that a hyperideal expansion δ of G is called intersection preserving if it satisfies $\delta(P \cap Q) = \delta(P) \cap \delta(Q)$, for all hyperideals P and Q of G. For example, hyperideal expansion δ_1 of G is intersection preserving.

Theorem 2.10. *Let the hyperideal expansion δ of G be intersection preserving. If Q_1^s are (t,n)-absorbing δ-semiprimary hyperideals of G Such that $\delta(Q_i) = P$, for each $1 \leq i \leq s$, then $Q = \bigcap_{i=1}^{s} Q_i$ is an (t,n)-absorbing δ-semiprimary hyperideal of G with $\delta(Q) = P$.*

Proof. Assume that $k(a_1^{tn-t+1}) \in Q$ for $a_1^{tn-t+1} \in G$, such that $k(a_1^{(t-1)n-t+2}) \notin \delta(Q)$. Since $\delta(Q) = \delta(\cap_{i=1}^{s} Q_i) = \cap_{i=1}^{s} \delta(Q_i) = P$, then there exists $1 \leq u \leq s$, such that $k(a_1^{(t-1)n-t+2}) \notin \delta(Q_u)$. Since Q_u is an (t,n)-absorbing δ-semiprimary hyperideal of G and $k(a_1^{tn-t+1}) \in Q_u$, then there is a k-product of $(t-1)n-t+2$ of the a_i's is in $\delta(Q_u) = P = \delta(Q)$. Thus $Q = \bigcap_{i=1}^{s} Q_i$ is an (t,n)-absorbing δ-semiprimary hyperideal of G with $\delta(Q) = P$. □

Let (G_1, h_1, k_1) and (G_2, h_2, k_2) be two Krasner (m,n)-hyperrings such that 1_{G_1} and 1_{G_2} be scalar identities of G_1 and G_2, respectively. Then $(G_1 \times G_2, h = h_1 \times h_2, k = k_1 \times k_2)$ is a Krasner (m,n)-hyperring where

$$h_1 \times h_2((a_1,b_1),\ldots,(a_m,b_m)) = \{(a,b) \mid a \in h_1(a_1^m), b \in h_2(b_1^m)\},$$

$$k_1 \times k_2((a_1,b_1),\ldots,(a_n,b_n)) = (k_1(a_1^n), k_2(b_1^n)),$$

for all $a_i \in G_1$ and $b_i \in G_2$ [8].

Theorem 2.11. *Let δ_1 and δ_2 be two hyperideal expansions of Krasner (m,n)-hyperrings G_1 and G_2, respectively, such that $\delta(Q_1 \times Q_2) = \delta_1(Q_1) \times \delta_2(Q_2)$, for hyperideals Q_1 and Q_2 of G_1 and G_2, respectively. If $Q = Q_1 \times Q_2$ is an $(t+1,n)$-absorbing δ-semiprimary hyperideal of $G = G_1 \times G_2$, then either Q_1 is an $(t+1,n)$-absorbing δ_1-semiprimary hyperideal of G_1 and $\delta_2(Q_2) = G_2$, or Q_2 is an $(t+1,n)$-absorbing δ_2-semiprimary hyperideal of G_2 and $\delta_1(Q_1) = G_1$, or Q_i is an (t,n)-absorbing δ_i-semiprimary hyperideal of G_i for each $i \in \{1,2\}$.*

Proof. Let $Q = Q_1 \times Q_2$ be an $(t+1,n)$-absorbing δ-semiprimary hyperideal of $G = G_1 \times G_2$. Assume that $\delta_1(Q_1) \neq G_1$ and $\delta_2(Q_2) = G_2$. Let us suppose that $k_1(a_1^{(t+1)n-t}) \in Q_1$, for some $a_1^{(t+1)n-t} \in G_1$, such that all products of $tn-t+1$ of the a_i's except $k_1(a_1^{tn-t+1})$ are not in $\delta(Q_1)$. Note that $k((a_1,0),\ldots,(a_{(t+1)n-t},0)) \in Q$, and all products of $tn-t+1$ of the $(a_i,0)$'s are not in $\delta(Q)$. Since Q is an $(t+1,n)$-absorbing δ-semiprimary hyperideal of G, we get $k((a_1,0),\ldots,(a_{tn-t+1},0)) \in \delta(Q) = \delta_1(Q_1) \times \delta_2(Q_2)$, which means $k_1(a_1^{tn-t+1}) \in \delta(Q_1)$. Thus Q_1 is an $(t+1,n)$-absorbing δ_1-semiprimary hyperideal of G_1. Similiar for the second assertion. For the third assertion, assume $\delta_1(Q_1) \neq G_1$ and $\delta_2(Q_2) \neq G_2$. Moreover, let us suppose that Q_1 is not an (t,n)-absorbing δ-semiprimary hyperideal of G_1 and $k_1(a_1^{tn-t+1}) \in Q_1$. We define the following elements of G: $x_1 = (a_1, 1_{G_2}), x_2 = (a_2, 1_{G_2}), \ldots, x_{tn-t+1} = (a_{tn-t+1}, 1_{G_2}), x_{(t-1)n-t+2} = (1_{G_1}, 0)$. Therefore we have $k(x_1^{(t-1)n-t+2}) = (k_1(a_1^{tn-t+1}), 0) \in Q$, $k(x_1^{tn-t+1}) = (k_1(a_1^{tn-t+1}), 1_{G_2}) \notin \delta(Q)$ and $k(x_1,\ldots,\widehat{x}_i,\ldots,x_{(t-1)n-t+2}) = (k_1(a_1,\ldots,\widehat{a}_i,\ldots,a_{(t-1)n-t+2}),0) \notin \delta(Q)$, for some $1 \leq i \leq tn-t+1$, a contradiction. Thus Q_1 is an (t,n)-absorbing δ_1-semiprimary hyperideal of G_1. Similarly, we conclude that Q_2 is an (t,n)-absorbing δ_2-semiprimary hyperideal of G_2. □

Theorem 2.12. *Let $\delta_1,\ldots,\delta_{tn-t+1}$ be hyperideal expansions of Krasner (m,n)-hyperrings $G_1,\ldots,G_{tn-t+1}$, such that $\delta(Q_1 \times \cdots \times Q_{tn-t+1}) = \delta_1(Q_1) \times \cdots \times \delta_{tn-t+1}(Q_{tn-t+1})$, for hyperideals $Q_1,\ldots,Q_{tn-t+1}$ of $G_1,\ldots,G_{tn-t+1}$, respectively. If $Q = Q_1 \times \cdots \times Q_{tn-t+1}$ is an $(t+1,n)$-absorbing δ-semiprimary hyperideal of $G = G_1 \times \cdots \times G_{tn-t+1}$, then either Q_u is an $(t+1,n)$-absorbing δ-semiprimary hyperideal of G_u for some $1 \leq u \leq tn-t+1$, and $\delta_i(Q_i) = G_i$, for each $1 \leq i \leq tn-t+1$ and $i \neq u$ or Q_u and Q_v are (t,n)-absorbing $\delta_{u,v}$-semiprimary hyperideals of G_u and G_v, respectively, for some $u,v \in \{1,\ldots,tn-t+1\}$ and $\delta_i(Q_i) = G_i$, for all $1 \leq i \leq tn-t+1$, but $i \neq u,v$.*

Proof. It can be seen that the idea is true in a similar manner to the proof of Theorem 2.11. □

Now, we want to extend the notion of (t,n)-absorbing δ-semiprimary hyperideals to weakly (t,n)-absorbing δ-semiprimary hyperideal. Although different from each other in many aspects, they share quite a number of similar properties as well.

Definition 2.13. Let δ be a hyperideal expansion of G and t be a positive integer. A proper hyperideal Q of G refers to a weakly (t,n)-absorbing δ-semiprimary hyperideal if $a_1^{tn-t+1} \in G$, and $0 \neq k(a_1^{tn-t+1}) \in Q$, then there exist $(t-1)n-t+2$ of the a_i's whose k-product is in $\delta(Q)$.

Example 2.14. Suppose that $\mathbb{Z}_{12}$ is the set of all congruence classes of integers modulo 12 and $H = \{1,5,7,11\}$ is multiplicative subgroup of units $\mathbb{Z}_{12}$. Construct G as $\mathbb{Z}_{12}/H$. Then we have $G = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{6}\}$, in which $\bar{0} = \{0\}$, $\bar{1} = \{1,5,7,11\}$, $\bar{2} = \bar{10} = \{2,10\}$, $\bar{3} = \bar{9} = \{3,9\}$,

$\bar{4} = \bar{8} = \{4, 8\}$, $\bar{6} = \{6\}$. Consider Krasner hyperring $(G, \oplus, \star)$, that for all $\bar{a}, \bar{b} \in G$, $\bar{a} \star \bar{b} = \overline{ab}$ and 2-ary hyperoperation $\oplus$ is defined as follows:

$\oplus$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{6}$
$\bar{0}$	$\{\bar{0}\}$	$\{\bar{1}\}$	$\{\bar{2}\}$	$\{\bar{3}\}$	$\{\bar{4}\}$	$\{\bar{6}\}$
$\bar{1}$	$\{\bar{1}\}$	$\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{2}, \bar{4}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{1}\}$
$\bar{2}$	$\{\bar{2}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{0}, \bar{4}\}$	$\{\bar{1}\}$	$\{\bar{2}, \bar{6}\}$	$\{\bar{4}\}$
$\bar{3}$	$\{\bar{3}\}$	$\{\bar{2}, \bar{4}\}$	$\{\bar{1}\}$	$\{\bar{0}, \bar{6}\}$	$\{\bar{1}\}$	$\{\bar{3}\}$
$\bar{4}$	$\{\bar{4}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{2}, \bar{6}\}$	$\{\bar{1}\}$	$\{\bar{0}, \bar{4}\}$	$\{\bar{2}\}$
$\bar{6}$	$\{\bar{6}\}$	$\{\bar{1}\}$	$\{\bar{4}\}$	$\{\bar{3}\}$	$\{\bar{2}\}$	$\{\bar{0}\}$

It is easy to see that the hyperideal $Q = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ of G is a weakly $(2, 2)$-absorbing δ_1-semiprimary.

Example 2.15. In Example 2.6, $0_{G/Q}$ is a weakly $(1, 3)$-absorbing δ_0-semiprimary hyperideal of G/Q, and $rad(0_{G/G})$ is not a weakly $(1, 3)$-absorbing δ_0-semiprimary hyperideal of G/Q.

Theorem 2.16. *If Q is a (weakly) (t, n)-absorbing δ-semiprimary hyperideal of G, then Q is (weakly) (v, n)-absorbing δ-semiprimary for all $v > n$.*

Proof. By using an argument similar to that in the proof of Theorem 4.4 in [12], one can complete the proof. □

Theorem 2.17. *Let Q be a proper hyperideal of G. If $\delta(Q)$ is a (weakly) (t, n)-absorbing hyperideal of G, then Q is a (weakly) (t, n)-absorbing δ-semiprimary hyperideal of G.*

Proof. Let $(0 \neq k(a_1^{tn-t+1}) \in Q)$ $k(a_1^{tn-t+1}) \in Q$, such that all products of $(t-1)n - t + 2$ of the a_i's, other than $k(a_1^{(t-1)n-t+2})$, are not in $\delta(Q)$. Since $\delta(Q)$ is a (weakly) (t, n)-absorbing hyperideal of G and $Q \subseteq \delta(Q)$, we conclude that $k(a_1^{(t-1)n-t+2}) \in \delta(Q)$. This shows that Q is a (weakly) (t, n)-absorbing δ-semiprimary hyperideal of G. □

Theorem 2.18. *Let Q be a proper hyperideal of G, such that $\delta(\delta(Q)) = \delta(Q)$. Then $\delta(Q)$ is a (weakly) (t, n)-absorbing hyperideal of G if and only if $\delta(Q)$ is a (weakly) (t, n)-absorbing δ-semiprimary hyperideal of G.*

Proof. $\Longrightarrow$ Assume that $\delta(Q)$ is a (weakly) (t, n)-absorbing hyperideal of G. Since $\delta(\delta(Q)) = \delta(Q)$, we are done by Theorem 2.17.

$\Longleftarrow$ Let $\delta(Q)$ be a (weakly) (t,n)-absorbing δ-semiprimary hyperideal of G. Suppose that $(0 \neq k(a_1^{tn-n+1}) \in \delta(Q))\ k(a_1^{tn-n+1}) \in \delta(Q)$. Since $\delta(Q)$ is a (weakly) (t,n)-absorbing δ-semiprimary hyperideal of G, then there exist $(t-1)n-t+2$ of the a_i's whose k-product is in $\delta(\delta(Q))$. Since $\delta(\delta(Q)) = \delta(Q)$, then the k-product of the $(t-1)n-t+2$ of the a_i's is in $\delta(Q)$, which means $\delta(Q)$ is a (weakly) (t,n)-absorbing hyperideal of G. □

The following theorem is a trivial relation. Hence the proof is omitted.

Theorem 2.19. *Assume that δ and γ are two hyperideal expansions of G. If Q is a (weakly) (t,n)-absorbing δ-semiprimary hyperideal of G such that $\delta(Q) \subseteq \gamma(Q)$, then Q is a (weakly) (t,n)-absorbing γ-semiprimary hyperideal of G.*

Theorem 2.20. *Let Q be a (weakly) (t,n)-absorbing δ-semiprimary hyperideal of G and P be a proper hyperideal of G such that $P \subseteq Q$. If $\delta(Q) = \delta(P)$, then P is a (weakly) (t,n)-absorbing δ-semiprimary hyperideal of G.*

Proof. Assume that $(0 \neq k(a_1^{tn-t+1}) \in P)\ k(a_1^{tn-t+1}) \in P$, for $a_1^{tn-t+1} \in G$. By the assumption, we get $(0 \neq k(a_1^{tn-t+1}) \in Q)\ k(a_1^{tn-t+1}) \in Q$, which implies there exist $(t-1)n-t+2$ of the a_i's whose k-product is in $\delta(Q)$ because Q is a (weakly) (t,n)-absorbing δ-semiprimary hyperideal of G. From $\delta(Q) = \delta(P)$, it follows that the k-product of $(t-1)n-t+2$ of the a_i's is in $\delta(P)$ which means P is a (weakly) (t,n)-absorbing δ-semiprimary hyperideal of G. □

A proper hyperideal Q of G refers to a strongly (weakly) (t,n)-absorbing δ-semiprimary hyperideal if $(0 \neq k(Q_1^{tn-t+1}) \subseteq Q)\ k(Q_1^{tn-t+1}) \subseteq Q$, for some hyperideals Q_1^{tn-t+1} of G, then there exist $(t-1)n-t+2$ of Q_i's whose k-product is a subset of $\delta(Q)$. Moreover, $(a_1,\ldots,a_{tn-t+1})$ is called a δ-(t,n)-zero of a weakly (k,n)-absorbing δ-semiprimary hyperideal Q of G if $k(a_1^{tn-t+1}) = 0$, and none k-product of the terms $(t-1)n-t+2$ of a_i's is in $\delta(Q)$.

Theorem 2.21. *If Q is a strongly weakly (t,n)-absorbing δ-semiprimary hyperideal of G and $(a_1,\ldots,a_{tn-t+1})$ is a δ-(t,n)-zero of Q, then for $1 \leq i_1,\cdots,i_s \leq tn-t+1$, and $1 \leq s \leq (t-1)n-t+2$,*

$$k(a_1,\ldots,\widehat{a}_{i_1},\ldots,\widehat{a}_{i_2},\ldots,\widehat{a}_{i_s},\ldots,a_{tn-t+1},Q^{(s)}) = 0.$$

Proof. We use the induction on s. Assume that $s = 1$. Then we will prove that $k(a_1,\ldots,\widehat{a}_{i_1},\ldots,a_{tn-t+1},Q) = 0$. Without loss of generality, we show that $k(a_2^{tn-t+1},q) = 0$, that is, we eliminate a_1. Suppose on the contrary $k(a_2^{tn-t+1},Q) \neq 0$. Therefore we get the resul that $0 \neq k(a_2^{tn-t+1},q) \subseteq Q$, for some $q \in Q$. Since Q is a strongly weakly (t,n)-absorbing δ-semiprimary hyperideal of G and $0 \neq k(a_2^{tn-t+1},h(a_1,q,0^{(m-2)}) = h(k(a_2^{tn-t+1},a_1),k(a_2^{tn-t+1},q),0^{(m-2)}) \subseteq Q$, we conclude that

a k-product of the $(t-1)n-t+2$ of a_i's including q is in $\delta(Q)$. We may assume $k(a_3^{tn-t+1}, h(a_1, q, 0^{(m-2)})) = h(k(a_3^{tn-t+1}, a_1), k(a_3^{tn-t+1}, q), 0^{(m-2)}) \subseteq \delta(Q)$, which implies $k(a_3^{tn-t+1}, a_1) \in h(-k(a_3^{tn-t+1}, q), 0^{(m-1)}) \subseteq \delta(Q)$, a contradiction. This means that $k(a_2^{tn-t+1}, q) = 0$, and so $k(a_1, \dots, \widehat{a}_{i_1}, \dots, a_{tn-t+1}, Q) = 0$. Now, let us suppose that $k(a_1, \dots, \widehat{a}_{i_1}, \dots, \widehat{a}_{i_2}, \dots, \widehat{a}_{i_s}, \dots, a_{tn-t+1}, Q^{(s)}) \neq 0$. We may assume that $k(a_{s+1}^{tn-t+1}, Q^{(s)}) \neq 0$, that is, we omit a_1^s. Hence $0 \neq k(a_{s+1}^{tn-t+1}, q_1^s) \in Q$, for some $q_1^s \in Q$. Thus $0 \neq k(a_{s+1}^{tn-t+1}, h(a_1, q_1, 0^{(m-2)}), \dots, h(a_s, q_s, 0^{(m-2)})) \subseteq Q$, by induction hypothesis. So,

$$k(a_{s+1}^{tn-t+1}, h(a_1, q_1, 0^{(m-2)}), \dots, h(a_1, \widehat{q_1, 0^{(m-2)}})_{i_1}, \dots, h(a_2, \widehat{q_2, 0^{(m-2)}})_{i_2}, \dots,$$
$$h(a_{n-1}, \widehat{q_{n-1}, 0^{(m-2)}})_{i_{n-1}}, \dots, h(a_s, q_s, 0^{(m-2)})) \subseteq \delta(Q),$$

or

$$k(a_{s+1}, \dots, \widehat{a}_{i_{s+1}}, \dots, \widehat{a}_{i_{s+2}}, \dots, \widehat{a}_{i_{s+n-1}}, \dots, a_{tn-t+1}, h(a_1, q_1, 0^{(m-2)}), \dots, h(a_s, q_s, 0^{(m-2)}))$$
$$\subseteq \delta(Q),$$

for some $i \in \{1, \dots, s\}$. Therefore, we get the result that $k(a_{s+1}, \dots, a_{tn-t+1}, \dots, a_n, \dots, a_s) \in \delta(Q)$ or $k(a_{s+n}, \dots, a_{tn-t+1}, \dots, a_1^s) \in \delta(Q)$, a contradiction. Thus, we have $k(a_1, \dots, \widehat{a}_{i_1}, \dots, \widehat{a}_{i_2}, \dots, \widehat{a}_{i_s}, \dots, a_{tn-t+1}, Q^{(s)}) = 0$. □

Theorem 2.22. *Let Q be a strongly weakly (t,n)-absorbing δ-semiprimary hyperideal of G but is not (t,n)-absorbing δ-semiprimary. Then $k(Q^{(tn-t+1)}) = 0$.*

Proof. Since Q is not (t,n)-absorbing δ-semiprimary, it has a δ-(t,n)-zero of Q, say (a_1^{tn-t+1}). Suppose on the contrary that $k(Q^{(tn-t+1)}) \neq 0$. Then we conclude that $0 \neq g(q_1^{tn-t+1}) \in Q$, for some $q_1^{tn-t+1} \in Q$. Then we get the result that $0 \neq k(h(a_1, q_1, 0^{(m-2)}), h(a_2, q_2, 0^{(m-2)}), \cdots, h(a_{tn-t+1}, q_{tn-t+1}, 0^{(m-2)})) \subseteq Q$, by Theorem 2.21. Since Q is a strongly weakly (t,n)-absorbing δ-semiprimary hyperideal of G, there exists $(t-1)n-t+2$ of $h(a_i, q_i, 0^{(m-2)})$'s whose k-product of is a subset of $\delta(Q)$. We may assume that $k(h(a_2, q_2, 0^{(m-2)}), h(a_2, q_3, 0^{(m-2)}), \cdots, h(a_{(t-1)n-t+3}, q_{tn-n-t+3}, 0^{(m-2)})) \subseteq rad(Q)$, without loss of generailty. Then, we conclude that $h(\underbrace{k(a_2^{(t-1)n-t+3}), k(a_2^u, q_1^v), k(q_1^{(t-1)n-t+2})}_{w}, 0^{(m-t)}) \subseteq rad(Q)$, such that $u+v = (t-1)n-t+2$. Therefore, $k(a_2^{(t-1)n-t+3}) \in h(\underbrace{-k(a_2^u, q_1^v, -k(q_1^{(t-1)n-t+2})}_{w-1}, 0^{(m-w+1)}) \subseteq rad(Q)$, which means $k(a_2^{(t-1)n-t+3}) \in rad(Q)$, which is impossible. Consequently, $k(Q^{(tn-t+1)}) = 0$. □

As an instant consequence of the previous theorem, we have the following explicit results.

Corollary 2.23. *Let Q be a weakly (t,n)-absorbing δ-semiprimary hyperideal of G but is not (t,n)-absorbing δ-semiprimary. Then $Q \subseteq rad(0)$.*

Corollary 2.24. *Assume that the commutative Krasner (m,n)-hyperring G has no non-zero nilpotent elements. If Q is a weakly (t,n)-absorbing δ-semiprimary hyperideal of G, then Q is an (t,n)-absorbing δ-semiprimary hyperideal of G.*

The following example verifies that a proper hyperideal Q staisfying $k(Q^{(tn-t+1)}) = 0$, may not be a weakly (t,n)-absorbing δ-semiprimary hyperideal.

Example 2.25. Consider Krasner $(4,3)$-hyperring $(\mathbb{Z}_{5^{5t}}, +, \cdot)$, where $+$ and $\cdot$ are usual addition and multiplication and $t > 4$. In the hyperring, $Q = \langle 5^t \rangle$ is not a weakly $(2,3)$-absorbing δ_0-semiprimary hyperideal of $\mathbb{Z}_{5^{5t}}$, since $5.5.5.5.5^{t-4} \in Q$, but $5.5.5, 5.5.5^{t-4} \notin \delta_0(Q)$. However, $Q^5 = 0$.

The next theorem provides us how to determine weakly (t,n)-absorbing δ-semiprimary hyperideal to be (t,n)-absorbing δ-semiprimary.

Theorem 2.26. *Let Q be a weakly (t,n)-absorbing δ-semiprimary hyperideal of G such that $\delta(Q) = \delta(0)$. Then Q is not (t,n)-absorbing δ-semiprimary if and only if there exists a δ-(t,n)-zero of 0.*

Proof. $\Longrightarrow$ Assume that Q is not an (t,n)-absorbing δ-semiprimary hyperideal of G. This implies that $k(a_1^{tn-t+1}) = 0$, and none k-product of the terms $(t-1)n-t+2$ of a_i's is in $\delta(Q)$, for some $a_1^{tn-t+1} \in G$. From $\delta(Q) = \delta(0)$, it follows that (a_1^{tn-t+1}) is a δ-(t,n)-zero of 0.

$\Longleftarrow$ Straightforward. □

Assume that M is a non-empty set and (G,h,k) is a commutative Krasner (m,n)-hyperring . If (M,h') is an m-ary hypergroup and the map

$$k' : \underbrace{G \times \ldots \times G}_{n-1} \times M \longrightarrow P^*(M),$$

statisfied the following conditions:

(1) $k'(x_1^{n-1}, h'(a_1^m)) = h'(k'(x_1^{n-1}, a_1), \ldots, k'(a_1^{n-1}, a_m))$,
(2) $k'(x_1^{i-1}, h(y_1^m), x_{i+1}^{n-1}, a) = h'(k'(x_1^{i-1}, y_1, x_{i+1}^{n-1}, a), \ldots, k'(x_1^{i-1} y_m, x_{i+1}^{n-1}, a))$,
(3) $k'(x_1^{i-1}, k(x_i^{i+n-1}), x_{i+m}^{n+m-2}, a) = k'(x_1^{n-1}, k'(x_m^{n+m-2}, a))$,
(4) $0 = k'(x_1^{i-1}, 0, x_{i+1}^{n-1}, a)$,

then the triple (M,h',k') is called an (m,n)-hypermodule over (G,h,k) [4]. In following theorem, Nakayama's Lemma is considered for a weakly (t,n)-absorbing δ-semiprimary hyperideal of G.

Theorem 2.27. *Let (M, h', k') be an (m,n)-hypermodule over (G, h, k) and Q be a weakly (t,n)-absorbing δ-semiprimary hyperideal of G that is not (t,n)-absorbing δ-semiprimary. If $M = k'(Q, 1^{(n-2)}, M)$, then $M = \{0\}$.*

Proof. Assume that Q is a weakly (t,n)-absorbing δ-semiprimary hyperideal of G that is not (t,n)-absorbing δ-semiprimary. and $M = k'(Q, 1^{(n-2)}, M)$. By Theorem 2.22, we get the result that $k'(k(Q^{(n)}), 1^{(n-2)}, M) = \{0\}$. On the other hand,

$$\begin{aligned} k'(k(Q^{(n)}), 1^{(n-2)}, M) &= k'(k(Q^{(n-1)}, 1), 1^{(n-2)}, k'(Q, 1^{(n-2)}, M)) \\ &= k'(k(Q^{(n-1)}, 1), 1^{(n-2)}, M) \\ &= k'(k(Q^{(n-2)}, 1^{(2)}), 1^{(n-2)}, k'(Q, 1^{(n-2)}, M)) \\ &= k'(k(Q^{(n-2)}, 1^{(2)}), 1^{(n-2)}, M) \\ &= \cdots \\ &= k'(Q, 1^{(n-2)}, k'(Q, 1^{(n-2)}, M)) \\ &= k(Q, 1^{(n-2)}, M) \\ &= M. \end{aligned}$$

Then, we conclude that $M = \{0\}$. □

Let (G_1, h_1, k_1) and (G_2, h_2, k_2) be two commutative Krasner (m,n)-hyperrings. Recall from [18] that a mapping $f : G_1 \longrightarrow G_2$ is called a homomorphism if we have $f(h_1(a_1^m)) = h_2(f(a_1), \ldots, f(a_m))$ and $f(k_1(b_1^n)) = k_2(f(b_1), \ldots, f(b_n))$, for all $a_1^m \in G_1$ and $b_1^n \in G_1$. Let δ and δ' be hyperideal expansions of G_1 and G_2, respectively. Recall from [3] that $f : G_1 \longrightarrow G_2$ is called a $\delta\delta'$-homomorphism if $\delta(f^{-1}(Q_2)) = f^{-1}(\delta'(Q_2))$, for hyperideal Q_2 of G_2. Note that $\delta'(h(Q_1) = h(\delta(Q_1)$, for $\delta\delta'$-epimorphism f and for hyperideal Q_1 of G_1 with $Ker(f) \subseteq Q_1$.

Theorem 2.28. *Let (G_1, h_1, k_1) and (G_2, h_2, k_2) be two Krasner (m,n)-hyperrings and $f : G_1 \longrightarrow G_2$ be a $\delta\delta'$-homomorphism. Then the followings hold:*

(i) *If Q_2 is an (t,n)-absorbing δ'-semiprimary hyperideal of G_2, then $f^{-1}(Q_2)$ is an (t,n)-absorbing δ-semiprimary hyperideal of G_1.*

(ii) *If Q_2 is a weakly (t,n)-absorbing δ'-semiprimary hyperideal of G_2 and $Kerf$ is a weakly (t,n)-absorbing δ-semiprimary hyperideal of G_1, then $f^{-1}(Q_2)$ is a weakly (t,n)-absorbing δ-semiprimary hyperideal of G_1.*

(iii) *Let f be an epimorphism and Q_1 be a proper hyperideal of G_1 containing $Kerf$. If Q_1 is a (weakly) (t,n)-absorbing δ-semiprimary hyperideal of G_1, then $f(Q_1)$ is a (weakly) δ'-semiprimary hyperideal of G_2.*

Proof. (i) Let $k_1(a_1^{tn-t+1}) \in f^{-1}(Q_2)$, for $a_1^{kn-k+1} \in G_1$. Therefore we obtain $f(k_1(a_1^{tn-t+1})) = k_2(f(a_1), \ldots, f(a_{tn-t+1})) \in Q_2$. Since Q_2 is an (t,n)-absorbing δ'-semiprimary hyperideal of G_2, then there exist $(t-1)n-t+2$ of $f(a_i)$'s whose k_2-product is an element in $\delta'(Q_2)$. It follows that the image f of $(t-1)n-t+2$ of a_i's whose k_2-product is in $\delta'(Q_2)$ which means there exist $(t-1)n-t+2$ of a_i's whose k_1-product is in $f^{-1}(\delta'(Q_2)) = \delta(f^{-1}(Q_2))$. Thus $f^{-1}(Q_2)$ is an (t,n)-absorbing δ-semiprimary hyperideal of G_1.

(ii) Assume that $k_1(a_1^{tn-t+1}) \in f^{-1}(Q_2)$, for $a_1^{kn-k+1} \in G_1$. Therefore we get $f(k_1(a_1^{tn-t+1})) = k_2(f(a_1), \ldots, f(a_{tn-t+1})) \in Q_2$. If $0 \neq f(k_1(a_1^{tn-t+1}))$, then it can be proved by using an argument similar to that in the proof of the part (i). Let us assume that $f(k_1(a_1^{tn-t+1})) = 0$. Then we obtain $k_1(a_1^{tn-t+1}) \in Kerf$. Since $Kerf$ is a weakly (t,n)-absorbing δ-semiprimary hyperideal of G_1, then there exist $(t-1)n-t+2$ of a_i's whose k_1-product is an element in $\delta(Kerf)$. From $\delta(Kerf) = \delta(f^{-1}(0)) \subseteq \delta(f^{-1}(Q_2))$, it follows that $f^{-1}(Q_2)$ is a weakly (t,n)-absorbing δ-semiprimary hyperideal of G_1. (iii) Let $(0 \neq k_2(b_1^{tn-t+1}) \in f(Q_1))$ $k_2(b_1^{tn-t+1}) \in f(Q_1)$, for some $b_1^{tn-t+1} \in G_2$. Since f be an epimorphism, then there exists $a_i \in G_1$, for each $1 \leq i \leq tn-t+1$, such that $f(a_i) = b_i$. Hence, $k_2(b_1^{tn-t+1}) = k_2(f(a_1), \ldots, f(a_{tn-t+1})) = f(k_1(a_1^{tn-t+1})) \in f(Q_1)$. Since Q_1 containing $Kerf$, we conclude that $(0 \neq k_1(a_1^{tn-t+1}) \in Q_1)$ $k_1(a_1^{tn-t+1}) \in Q_1$. As Q_1 is a (weakly) (t,n)-absorbing δ-semiprimary hyperideal of G_1, then there exist $(t-1)n-t+2$ of a_i's whose k_1-product is in $\delta(Q_1)$. Now, since f is a homomorphism and $f(\delta(Q_1)) = \delta'(f(Q_1))$, the proof is completed. □

Theorem 2.29. *Let P and Q be two proper hyperideals of G with $P \subseteq Q$. If Q is an (t,n)-absorbing δ-semiprimary hyperideal of G, then Q/P is an (t,n)-absorbing δ_q-semiprimary hyperideal of G/P.*

Proof. By considering the natural homomorphism $\pi : G \longrightarrow G/P$, defined by $\pi(a) = f(a, P, 0^{(m-2)})$, and using Theorem 2.28, we are done. □

Theorem 2.30. *Let Q be an (t,n)-absorbing δ-semiprimary hyperideal of G. If G' is a subhyperring of G such that $G' \nsubseteq Q$, then $Q \cap G'$ is an (t,n)-absorbing δ-semiprimary hyperideal of G'.*

Proof. It follows by Theorem 2.28. □

Theorem 2.31. *Let δ_1 and δ_2 be two hyperideal expansions of Krasner (m,n)-hyperrings G_1 and G_2, respectively, such that $\delta(Q_1 \times Q_2) = \delta_1(Q_1) \times \delta_2(Q_2)$, for hyperideals Q_1 and Q_2 of*

G_1 and G_2, respectively. If $Q = Q_1 \times G_2$ is a weakly (t,n)-absorbing δ-semiprimary hyperideal of $G_1 \times G_2$, then it is an (t,n)-absorbing δ-semiprimary hyperideal of $G_1 \times G_2$.

Proof. Assume that $Q_1 \times G_2$ is a weakly (t,n)-absorbing δ-semiprimary hyperideal of $G_1 \times G_2$. Since $k(Q^{(tn-t+1)}) \neq 0$, we conclude that $Q = Q_1 \times G_2$ is an (t,n)-absorbing δ-semiprimary hyperideal of $G_1 \times G_2$ by Theorem 2.22. □

We say that δ has $(\mathfrak{P})$ property if it satisfies the condition: $\delta(Q) = G$, if and only if $Q = G$, for all hyperideals Q of G.

Theorem 2.32. *Let $\delta_1, \ldots, \delta_{tn-t+1}$ be hyperideal expansions of Krasner (m,n)-hyperrings $G_1, \ldots, G_{tn-t+1}$, such that each δ_i has $(\mathfrak{P})$ property and $\delta(Q_1 \times \cdots \times Q_{tn-t+1}) = \delta_1(Q_1) \times \cdots \times \delta_{tn-t+1}(Q_{tn-t+1})$, for hyperideals $Q_1, \cdots, Q_{tn-t+1}$ of $G_1, \cdots, G_{tn-t+1}$, respectively. If $Q = Q_1 \times \cdots \times Q_{tn-t+1}$ is a weakly (t,n)-absorbing δ-semiprimary hyperideal of $G = G_1 \times \cdots \times G_{tn-t+1}$, then Q is an (t,n)-absorbing δ-semiprimary hyperideal of $G = G_1 \times \cdots \times G_{tn-t+1}$.*

Proof. Let Q is a weakly (t,n)-absorbing δ-semiprimary hyperideal of G. Let us consider the following elements of G: $x_i = (1_{G_1} \ldots, 1_{G_{i-1}}, a_i, 1_{G_{i+1}}, \ldots, 1_{G_{tn-t+1}})$, for all $1 \leq i \leq tn-t+1$. Then we have $0 \neq k(x_1^{tn-t+1}) \in Q$. Since $Q = Q_1 \times \cdots \times Q_{tn-t+1}$ is a weakly (t,n)-absorbing δ-semiprimary hyperideal of $G = G_1 \times \cdots \times G_{tn-t+1}$, then there exists $(t-1)n-t+2$ of the x_i's whose k-product is in $\delta(Q) = \delta_1(Q_1) \times \cdots \times \delta_{tn-t+1}(Q_{tn-t+1})$. This implies that there exists some $1 \leq j \leq tn-t+1$, such that $1_{G_j} \in \delta_j(Q_j)$, which means $\delta_j(Q_j) = G_j$. Since δ_j has $(\mathfrak{P})$ property, then $Q_j = G_j$. Hence we conclude that $k(Q^{(tn-t+1)}) \neq 0$, which implies Q is an (t,n)-absorbing δ-semiprimary hyperideal of G by Theorem 2.22. □

Theorem 2.33. *Let $\delta_1, \ldots, \delta_{tn-t+1}$ be hyperideal expansions of Krasner (m,n)-hyperrings $G_1, \ldots, G_{tn-t+1}$, such that each δ_i has $(\mathfrak{P})$ property and $\delta(Q_1 \times \cdots \times Q_{tn-t+1}) = \delta_1(Q_1) \times \cdots \times \delta_{tn-t+1}(Q_{tn-t+1})$, for hyperideals $Q_1, \ldots, Q_{tn-t+1}$ of $G_1, \ldots, G_{tn-t+1}$, respectively. If $Q = Q_1 \times \cdots \times Q_{tn-t+1}$ is a weakly $(t+1,n)$-absorbing δ-semiprimary hyperideal of $G = G_1 \times \cdots \times G_{tn-t+1}$, then either there exists $1 \leq u \leq tn-t+1$, such that Q_u is an $(t+1,n)$-absorbing δ-semiprimary hyperideal of G_u and $Q_i = G_i$, for each $1 \leq i \leq tn-t+1$ and $i \neq u$, or Q_u and Q_v are (t,n)-absorbing $\delta_{u,v}$-semiprimary hyperideals of G_u and G_v, respectively, for some $u, v \in \{1, \cdots, tn-t+1\}$ and $Q_i = G_i$, for all $1 \leq i \leq tn-t+1$, but $i \neq u, v$.*

Proof. Let $Q = Q_1 \times \cdots \times Q_{tn-t+1}$ be a weakly $(t+1,n)$-absorbing δ-semiprimary hyperideal of $G = G_1 \times \ldots \times G_{tn-t+1}$. Therefore we conclude that Q is an $(t+1,n)$-absorbing δ-semiprimary hyperideal of G by Theorem 2.32. Now, by using Theorem 2.12, we are done. □

3. conclusion

This research contributes to the idea of (t,n)-absorbing δ-semiprimary hyperideals in Krasner (m,n)-hyperrings, and gives a description of these hyperideals. We examined whether the radical of an (t,n)-absorbing δ-semiprimary hyperideal preserves the algebraic structure. Additionally, we generalized this notion to weakly (t,n)-absorbing δ-semiprimary hyperideals. We gave many special results illustrating these structures. Furthermore, a version of Nakayama's Lemma was studied on commutative Krasner (m,n)-hyperrings. We investigated the images and inverse images of (weakly) (t,n)-absorbing δ-semiprimary hyperideals under a homomorphism. Also, some characterizations of these concepts were given on cartesian product of Krasner (m,n)-hyperrings. In future work, one can study similar notions in the context of (m,n)- hypermodules.

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