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Research Paper

STRUCTURE OF LINEAR CODES INVARIANT UNDER THE UNITARY GROUP $U(3,3)$

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Abstract. In this paper, we outline a method for constructing linear codes invariant under primitive permutation groups. We will demonstrate that when a group G possesses a trivial Schur multiplier, every binary linear code that admits G as a permutation group can be regarded as a submodule of the permutation module within the primitive action of G. As an illustrative example, we select the finite simple group $G = U(3,3)$ and identify the complete set of linear codes derived from its 2-representations.

In addition, we use the supports of the codewords to construct certain designs that remain invariant under the action of $U(3,3)$ and establish connections between these designs and the corresponding linear codes.

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1. Introduction

In this paper, we employ a modular representation theory-based method to construct codes from finite permutation groups. The codes we construct are submodules of the permutation modules of the group, corresponding to its primitive actions. The method is based on a result of [9], and in some cases, allows us to determine the entire set of binary linear codes of given lengths that admit the chosen group as a primitive permutation automorphism group. Particularly, it is effective for simple groups with a trivial Schur multiplier. We have adopted the unitary group $U(3,3)$, a simple group with a trivial Schur multiplier, to illustrate this method and analyse its G-invariant codes. This methodology crucially relies on MAGMA [2], a powerful computational algebra system. Other computational algebra system such as GAP [7] can be used. In [6], the authors used the same approach to find all binary codes of the Mathieu group M_{11}. In [4], [8], [10] and [11] similar research was done, using different techniques. It is worth noting that the collection of binary linear codes for $G = U(3,3)$ is already documented in [3]. However, our main focus in this paper is to study the structures and properties of these codes and their corresponding designs.

The paper is organized as follows. Section 2 introduces the terminology and notation used throughout this paper. In Section 3, we present some properties of the unitary group $U(3,3)$. Finally, in Section 5, we provide a comprehensive analysis of the linear codes and support designs constructed from $U(3,3)$.

2. Definitions and notations

Let $\mathbb{F}_q$ be a finite field of q-elements. A linear code C of length n over $\mathbb{F}_q$ is a vector subspace of $\mathbb{F}_q^n$ denoted by $C = [n,k]_q$, where n is the length of each vector and k is the dimension of the code. Every vector in C is called a codeword. The Hamming distance $d(u,v)$ of the codewords $u, v \in C$ is the number of coordinates in which u and v differ. The minimum distance of the code is the minimum of the Hamming distances of any two distinct codewords. If the minimum distance d is known, then the code C is denoted by $[n,k,d]_q$. If $q = 2$, we simply use the notation $[n,k,d]$, instead of $[n,k,d]_2$. A code $[n,1,n]_q$ is called a repetition code. Let $v = (v_1, v_2, \ldots, v_n)$ be a codeword in C. The support of v is defined by

$$\mathrm{supp(v)} = \{\mathrm{i} : \mathrm{v_i} \neq 0,\ 1 \leq \mathrm{i} \leq \mathrm{n}\} \subseteq \{1, 2, 3, \ldots, \mathrm{n}\}.$$

The weight of v is $|\mathrm{supp(v)}|$ and is denoted by $\mathrm{wt(v)}$. The minimum weight of a code C is the minimum of the weights of all non-zero codewords. If C is a linear code, then the minimum weight of the code is just the minimum distance of the code. The dual code or orthogonal code of C is defined by $C^{\perp} = \{y \in \mathbb{F}_p^n | x \cdot y = 0,\ \mathit{for\ all}\ x \in C\}$. A code is called projective if any 2 of its coordinates are linearly independent, i.e. it has a dual distance $d^{\perp} \geq 3$. A code is even if the weight of each codewords is divisible by 2 and it is doubly even if the weights are divisible by 4. A code is said to be self-dual if $C = C^{\perp}$ and it is self-orthogonal if $C \subseteq C^{\perp}$. An automorphism of a binary linear code C is an element of S_n that sends codewords to codewords. The automorphism group of C is

$$\mathrm{Aut(C)} = \{\pi \in \mathrm{S_n} | \mathrm{c}\pi \in \mathrm{C}\, \text{for all } \mathrm{c} \in \mathrm{C}\}.$$

Let A be a $k \times n$ matrix whose rows generate the linear code $C = [n, k]_q$. Then the permutation group of C is the stabilizer of C in the symmetric group S_n with respect to action on the set of the columns of A. We denote the permutation group of C by PAut(C) for binary codes Aut(C) = PAut(C).

An incidence structure is a set $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $\mathcal{P}$ is the point set, $\mathcal{B}$ is the block set and $\mathcal{I}$ is an incidence relation between $\mathcal{P}$ and $\mathcal{B}$. If the pair (x, B) is in $\mathcal{I}$ for $x \in \mathcal{P}$ and $B \in \mathcal{B}$, we say that P is incident with B, or B contains the point x. The pair (x, B) is called a flag if it is in $\mathcal{I}$ and anti-flag otherwise [1]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a t-design or a $t - (v, k, \lambda)$ design if $|\mathcal{P}| = v$, and each $B \in \mathcal{B}$ is incident with exactly k points and every t distinct points are together incident with λ blocks. A symmetric design is a design with the same number of points and blocks.

3. Structure of the unitary group $U(3,3)$

In this section, we consider the unitary group $G = U(3,3)$, a classical simple gr oup of Lie type[3]. We first review some well-known properties of this group. According to [5], the Schur multiplier of G is trivial (see Definition 4.3 below). Moreover by [5], $U(3,3)$ has 4 maximal subgroups up to conjugation listed in Table 1.

Table 1. Maximal subgroups of $U(3,3)$.

Maximal Subgroup	**Group Structure**	**Degree**
M_1	$3_+^{1+2} : 8$	28
M_2	$L_2(7)$	36
M_3	$4.S_4$	63
M_4	$4^2 : S_3$	63

For each maximal subgroup $M_k,\ k = 1, 2, 3, 4$, the group G acts by conjugation on the set of conjugates of M_k in G. The degree of this action is $|G{:}M_k| \in \{28, 36, 63\}$. Moreover, the action induces a permutation module $P_k(q)$ over $\mathbb{F}_q$. Our aim is to find all binary codes of length $|G{:}M_k|$ that contain G in their permutation automorphism groups. According to Theorem 4.5 below, every binary code of length $|G{:}M_k|$ that admits G as a primitive permutation group is a submodule of the permutation group of G with respect to the action of G. If $q > 2$, then the assertion holds only if $(|M_k{:}M_k'|, q-1) = 1$, where M_k' is the derived subgroup of M_k. According to Table 1, $M_2 \cong L_2(7)$ is perfect. Therefore, for every prime power q dividing the order of G, we can find all linear codes of length 36 in $\mathbb{F}_q$. However, for the rest of the maximal subgroups M_k of G, the value of $|M_k{:}M'|$ happens to be 2, 4 or 8. So in these cases, submodules of $P_k(q)$ may not give us all G-invariant linear codes over $\mathbb{F}_q$. Using ATLAS [5] or MAGMA, the unitary group $U(3,3)$ has five irreducible subgroups in $\mathbb{F}_2$ with dimensions $1, 6, 14, 32, 32$ and they are all absolutely irreducible.

4. Construction of G invariant code

In this section we discuss the methods used in this paper to construct codes and designs. For these methods we also use MAGMA to develop algorithm that used in our constructions. The construction used in this paper is based on Proposition 4.4 and Theorem 4.5.

Definition 4.1. Let $G \cong E/K$, where $K \rhd E$. Then E is called an extension of K by G. The set of all isomorphism classes of groups E that are extensions of K by G is denoted by $Ext(K, G)$.

Definition 4.2. An extension E of K by G is called a central extension if $K \leq Z(E)$. If $K \leq E' \cap Z(E)$, then E is called a stem extension. A maximal stem extension of K by G is called a stem cover.

Definition 4.3. Let $G = F/K$, where F is a free group. Then the Schur multiplier $M(G)$ of G is defined to be $\dfrac{F' \cap K}{[F, K]}$. It is well-known that the Schur multiplier of a group G is trivial if and only if G does not have a central extension apart from itself.

Proposition 4.4. *Assume (G, X) is a primitive permutation group and $\mathbb{F}$ a field such that $Ext(G/G', \mathbb{F}^*) = 0$. Let E be a stem cover of G and E_0 the inverse in E of the stabilizer of G induced up to E all 1-dimensional $\mathbb{F}E_0$-module. Then the submodules of the resulting $\mathbb{F}E$-modules provide for a complete list of codes over $\mathbb{F}$ admitting (G, X) as a permutation group.*

Proof. See [9, Corollary 3.2]. □

Theorem 4.5. *Let G be a finite simple group with a maximal subgroup M. Let P be the permutation $\mathbb{F}_n$-module corresponding to the primitive action of G on M, where $\mathbb{F}$ is a finite field. Also assume that the Schur multiplier of G is trivial and $(|M/M'|, |\mathbb{F}^*|) = 1$. Then the set of linear codes of length m over $\mathbb{F}$ equals the set of all submodules of P.*

Proof. See [6, Theorem 2]. □

For a positive integer m, we define: $W_m(C) = \{v \in C : wt(v) = m\}$. If there is no ambiguity, we may simply write W_m. Since the automorphisms of C preserve the weight of codewords, we deduce that A acts on W_m for every integer m with $W_m \neq \emptyset$. The stabilizer of this action is of interest. If $v \in W_m$, then the stabilizer of v in A is the set of all $g \in A$ with $vg = v$. So if the code is binary, the stabilizer of v in A is isomorphic to the stabilizer of the support of v in A. Some 1-designs may be constructed from the codes, using this action.

Proposition 4.6. *Let $C = [n, k, d]$ be a binary linear code admitting G as a permutation automorphism group and $W_m(C) \neq \emptyset$. If S is an orbit of the action of G on W_m, then we have a $1-(n, m, m|S|/n)$ design with block set $B = \{Supp(w) : w \in S\}$.*

Proof. See [6, Proposition 1]. □

Lemma 4.7. *A binary code is even if and only if it is contained in the dual of a repetition code.*

Proof. See [6, Corollary 1]. □

Theorem 4.8. *Let C be a code over $\mathbb{F}_3^n$. Then every codeword c has a weight divisible by 3 if and only if C is self-orthogonal.*

Proof. See [11, Theorem 1.4.8]. □

5. Linear codes invariant under $U(3, 3)$

In this section we find codes of length $|G : M_k|$ that contain G in their permutation automorphism group. Denote by $P_i(q)$ the permutation module over $G\mathbb{F}(q)$ with respect to the primitive action of $G = U(3, 3)$ on the set of the conjugates of M_k in G. According to Theorem 4.5, the codes are of type $c := [n, d, m]_2$, where $n \in \{28, 36, 63\}$. If $d = 0$ or $d = n$, then we consider the code as trivial. So the trivial codes are of type $c := [n, 0, m]_2$ and $c := [n, n, m]$. Also if $d = 1$ then C is a repitition code therefore repitition codes are of type $C := [n, 1, m]$.

5.1. **Representations of degree 28.**

For a permutation group acting on a set Ω of degree 28, we find a 28-dimensional permutation module invariant under G. We take the permutation module to be our working module and find all submodules. That permutation module breaks into submodules of dimensions $0, 1, 7, 21$ and 27, with the composition series $V = 28 \supset 27 \supset 21 \supset 7 \supset 1 \supset 0$. This shows that the only irreducible submodule is of dimension 1. The non-trivial submodules are of dimensions $1, 7, 21$ and 27. Therefore, according to Theorem 4.5, it suffices to find codes of dimensions

$1, 7, 21$ and 27. The list of all binary codes invariant under $U(3, 3)$ are mentioned in [3]. Here we focus on their properties, automorphism groups and support designs.

Proposition 5.1. *Let G be a primitive subgroup of degree 28 of the unitary group $U(3, 3)$, then,*

(1) *The binary linear code $[28, 7, 12]$ is doubly even.*
(2) *The binary linear code $[28, 21, 4]$ of dimension 21 is even and it is self-orthogonal.*

Proof. Using MAGMA, we have that the weight distribution of the code $[28, 7, 12]$ is $\{0^1, 12^{63}, 16^{63}, 28^1\}$. Since all the weights of this code are divisible by 4, therefore it is doubly even. Now we consider the code $[28, 21, 4]$. The weight distribution of this code is as follows:

$$\{0^1, 4^{315}, 6^{6048}, 8^{47817}, 10^{206976}, 12^{472059}, 14^{630720}, 16^{472059}, 18^{206976}, 20^{47817}, 22^{6048}, 24^{315}, 28^1\}.$$

Moreover, all the weights of this code are even which implies the code is even. Furthermore, its dual is a code with parameters $[25, 7, 12]$ and according to the composition series of the permutation module it contains its dual, whence it is self-orthogonal. □

Proposition 5.2. *All 21 dimensional binary codes accepting $U(3, 3)$ as primitive permutation group are even.*

Proof. Using the composition series of the permutation module we note that all the non-trivial submodules are contained in the 27 dimensional submodule. This implies that all the codes are contained in the dual of the repetition code. Hence by Lemma 4.7, all the codes are even.
□

Table 2 shows some of the properties of the non-trivial codes we constructed. The computations are done using MAGMA.

TABLE 2. non-trivial codes of length 28.

Parameters	(C)	**Primitivity of C**
$[28, 7, 12]$	$S_6(2)$	yes
$[28, 21, 4]$	$S_6(2)$	yes

Support designs of length 28.

Suppose that w_n is a codeword of nonzero weight m in a non-trivial code C. In this section we determine the structure of $(\mathrm{Aut(C)})_{\mathrm{w_m}}$. That is, the stabilizer of w_m in Aut(C), where $\mathrm{w_m} = \{\mathrm{c} \in \mathrm{C} : \mathrm{w(c)} = \mathrm{m}\}$. We now examine the action of Aut(C) on the set w_m. In addition, we look at the structure of the stabilizer $(\mathrm{Aut(C)})\mathrm{w_m}$ and construct the support 1-designs using Proposition 4.6.

Proposition 5.3. *Let C be a G-invariant code of length 28. Suppose w is a codeword of the code C of weight m and* $\mathrm{A} = \mathrm{Aut(C)}$. *If* $\mathrm{w_m} \neq \emptyset$*, then the action of A on* $\mathrm{w_m(C)}$ *is transitive. The stabilizer* $\mathrm{w} = \mathrm{w_m(C)}$ *in A is a maximal subgroup of $U(3,3)$ and the support designs constructed from these codes are shown in Table 3 and Table 4.*

Proof. The values of m and s are given in the weight distribution of the codes. Using Theorem 4.6, the designs from the supports of these codes are of form $1-(n,m,\lambda)$, where n is the length of the code and $\lambda = ms/n$, where s is the size of the orbits of $\mathrm{w_m}$. We checked for transitivity using MAGMA. The results follows. □

Table 3. Stabilizer and support designs from $[28, 7, 12]$.

m	$\mathrm{s} = \lvert\mathrm{w_m}\rvert$	**Stabilizer**	**Maximal in** A	**Design**
12	63	$2^5 : S_6$	yes	$1-(28,12,27)$
16	63	$2^5 : S_6$	yes	$1-(28,16,36)$

Remark 5.4. We notice that the number of designs rely on the number of orbits of $\mathrm{w_m}$ and orbits' size is the number of blocks in the design. All support designs under each m_i have the same stabilizer so we can add all orbits' size and get $\lvert\mathrm{w_m}\rvert$ and use it to find one design with a large parameter λ. For example if $m_2 = 6$, we can have $\lvert\mathrm{w_m}\rvert = 1008 + 5040 = 6048$. So we can construct a $1-(28,6,1296)$ design with 6048 blocks and stabilizer $A_6.2^2$.

5.2. **Representations of degree 36.**

We construct a 36-dimensional permutation module invariant under permutation group $G = U(3,3)$ acting on set Ω of degree 36. Let the permutation module be our working module and recursively find all submodules. We find that the permutation module breaks into 11 submodules of dimension $0,1,7,8,14,15,21,22,28,29$ and 35. The submodule lattice of this representation is shown in Figure 5.2, showing that only the code of dimension one is irreducible.

According to Theorem 4.5 there exist codes of length 36 and dimension k where

$$k \in \{1,7,8,14,15,21,22,28,29,35\}.$$

Table 4. Stabilizer and support designs from $[28, 21, 4]$.

m	Orbits' size	Stabilizer	Maximal in A	Design
4	315	$2.[2^6]:(S_3 \times S_3)$	yes	$1-(28,4,45)$
6	1008	$A_6.2^2$	no	$1-(28,6,216)$
	5040	$A_6.2^2$	no	$1-(28,6,1080)$
8	945	$(4^2 \times 2).2^3.S_3$	no	$1-(28,8,215)$
	22680	$(4^2 \times 2).2^3.S_3$	no	$1-(28,8,6480)$
	24192	$(4^2 \times 2).2^3.S_3$	no	$1-(28,8,6912)$
18	336	$S_3 \times S_6$	yes	$1-(28,18,216)$
	5040	$S_3 \times S_6$	yes	$1-(28,18,3240)$
	15120	$S_3 \times S_6$	yes	$1-(28,18,9720)$
22	1008	$A_6.2^2$	no	$1-(28,22,792)$
	5040	$A_6.2^2$	no	$1-(28,22,3690)$
24	315	$2.[2^6]:(S_3 \times S_3)$	yes	$1-(28,24,270)$

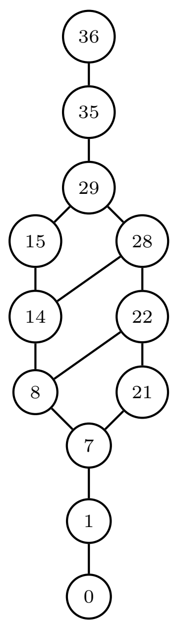

Figure 1. Submodule lattice for a 36-dimensional representation.

Proposition 5.5. *Let $G = U(3,3)$ and $\mathbb{F} = \mathbb{F}_2$. Then all linear codes of length 36 over $\mathbb{F}$ admitting G as primitive permutation group are even and not self-dual.*

Proof. Using the lattice diagram, we can see that all codes are contained in the dual of the repetition code. So by Lemma 4.7, all codes are even. Moreover, let C be the self-dual code of dimension k. Then, $36 - k = k$, while there is no submodule of $P_1(2)$ that has a dimension 18. Therefore, such a code does not exist and the result follows. □

Proposition 5.6. (1) *The code* $C = [36, 7, 16]$ *is doubly even and self-orthogonal.*
(2) $\mathrm{Hull(C)} = [36, 8, 14]$.

Proof. (1) Using MAGMA, we find that the weight distribution of $[36, 7, 16]$ is $\{0^1, 16^{63}, 20^{63}, 36^1\}$. Note all weights of above mentioned codes are divisible by 4. Moreover, the dual of $[36, 7, 16]$ is $[36, 29, 4]$ and looking at the submodule lattice, we have $[36, 29, 4] \subseteq [36, 29, 4]$ hence it is self-dual.
(2) $\mathrm{Hull(C)} = \mathrm{C} \cap \mathrm{C}^{\perp}$ where the dual $C^{\perp} = [36, 14, 8]$. Looking at the lattice diagram, the biggest code contained in both the codes is $[36, 8, 14]$.
Whence $\mathrm{Hull(C)} = [36, 8, 14]$.

This proves the proposition □

All other non-trivial codes and their automorphism groups are shown in Table 5 and the weight distributions are found in [3].

TABLE 5. non-trivial codes of length 36.

Parameters	$\mathrm{Aut}(C_i)$	**Primitivity of** $\mathrm{Aut}(C_i)$
$[36, 7, 16]$	$S_2(6)$	yes
$[36, 8, 14]$	$G_2(2)$	yes
$[36, 14, 8]$	$G_2(2)$	yes
$[36, 15, 8]$	$S_2(6)$	yes
$[36, 21, 6]$	$S_2(6)$	yes
$[36, 22, 6]$	$G_2(2)$	yes
$[36, 28, 4]$	$G_2(2)$	yes
$[36, 29, 4]$	$S_2(2)$	yes

Support designs of length 36.

Suppose that $\mathrm{w_n}$ is a codeword of nonzero weight m in a non-trivial code C. In this section, we determine the structure of $(\mathrm{Aut(C)})_{\mathrm{w_m}}$, that is, the stabilizer of w_m in Aut(C), where $\mathrm{w_m} = \{\mathrm{c} \in : \mathrm{w(c) = m}\}$.

Proposition 5.7. *Let w be a codeword of the code C of weight m and* $\mathrm{A} = \mathrm{Aut(C)}$. *The stabilizer of* $\mathrm{w} = \mathrm{W_m(C)}$ *in A is a maximal subgroup of $U(3,3)$ and the support designs constructed from these codes are shown in Table 6 and Table 7.*

Table 6. Stabilizer and support designs from $[36, 8, 14]$.

m	s := $\|w_m\|$	Stabilizer	Maximal in A	Design
14	36	$L_3(2):2$	yes	$1-(36,14,14)$
16	63	$M_8.S_4$	no	$1-(36,16,28)$
18	56	$3_+^{1+2}:8$	yes	$1-(36,18,28)$
20	63	$M_8.S_4$	no	$1-(36,20,35)$
22	36	$L_3(2):2$	no	$1-(36,22,22)$

Table 7. Some stabilizers and support designs from $[36, 14, 8]$.

m	s := $\|w_m\|$	Stabilizer	Maximal in A	Design
8	63	$M_8.S_4$	yes	$1-(36,8,14)$
12	441	$M_8.S_4$	yes	$1-(36,12,147)$
14	2304	$L_3(2):2$	no	$1-(36,14,896)$
16	3591	$M_8.S_4$	yes	$1-(36,16,1596)$
18	3584	$3_+^{1+2}:8$	no	$1-(36,18,1792)$
28	63	$M_8.S_4$	yes	$1-(36,28,49)$

Proof. Values of m and s are given in the weight distribution of the codes. Using Theorem 4.6, the designs from the supports of these codes are of the form $1-(n,m,\lambda)$, where n is the length of the code and $\lambda = ms/n$. The result now follows. □

Non-binary codes from representations of degree 36.

As discussed in Section 3, $M_3 \cong L(2,7)$ is perfect, therefore we can find linear codes of length 36 in $\mathbb{F}_q$, where q the prime factor of $|G|$. The method presented here can be used to find non-binary codes. In this section we find codes of length 36 in $\mathbb{F}_3$ and $\mathbb{F}_7$. Codes in $\mathbb{F}_3$ are called ternary codes. In the field $\mathbb{F}_3$ the permutation module of degree 36 breaks into submodules of dimensions $0, 1, 14^4, 15, 21, 22^4, 23, 35$ and 36.

The submodule lattice is shown in Figure 5.2, which also shows that the only irreducible code is of dimension 1.

According to Theorem 4.5 it suffices to find codes with those dimensions. Therefore, we have 11 non-trivial ternary codes admitting $U(3,3)$ as primitive permutation group. We have the length and the dimensions of the ternary codes we left with determining minimum distances to have the complete codes. Using MAGMA, we find that the non-trivial codes are as follows.

(1) $[36, 13, 12]_3$;
(2) $[36, 14, 8]_3$ (three codes);
(3) $[36, 14, 12]_3$;
(4) $[36, 15, 8]_3$;
(5) $[36, 21, 6]_3$;
(6) $[36, 22, 6]_3$ (four codes).

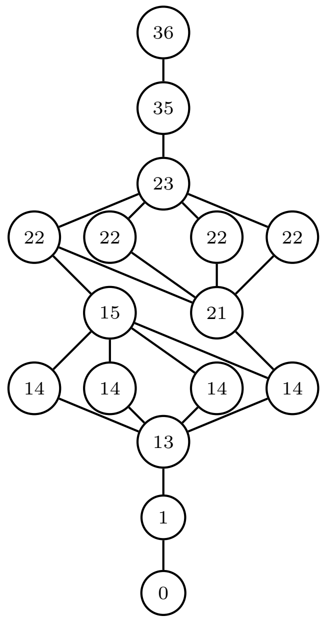

Figure 2. Submodule lattice for a 36-dimensional representation in $\mathbb{F}_3$.

Looking at the weight distributions computed by MAGMA, we notice that 2 of the $[36, 14, 8]_3$ codes are isomorphic, but their duals $[36, 22, 4]_3$ are not isomorphic, this means that if the codes are isomorphic it does not imply that their dual codes are also isomorphic.

We also notice that all non-trivial codes have the minimum distance $d \geq 3$. Therefore from definition of projective codes, all the non-trivial ternary codes under $U(3, 3)$ are projective. Table 8 shows Aut(C) for all non-trivial ternary codes.

Table 8. Automorphism and permutation automorphism groups of non-trivial ternary codes.

Parameters	Aut(C)	PAut(C)	**Primitivity of** Aut(C)
$[36, 13, 12]_3$	$2 \times U(3, 3) : 2$	$U(3, 3)$	yes
$[36, 14, 8]_3$	$W(E_7)$	$U(3, 3)$	yes
$[36, 14, 12]_3$	$U(3, 3) : 2$	$U(3, 3)$	yes
$[36, 15, 8]_3$	$W(E_7)$	$U(3, 3)$	yes
$[36, 21, 6]_3$	$W(E_7)$	$U(3, 3)$	yes
$[36, 22, 6]_3$	$U(3, 3) : 2$	$U(3, 3)$	yes
$[36, 23, 6]_3$	$2 \times U(3, 3) : 2$	$U(3, 3)$	yes

Proposition 5.8. *Let $G = U(3, 3)$. then, the only self-orthogonal ternary codes accepting G as the primitive permutation group are* $[36, 13, 12]_3$ *and* $[36, 14, 12]_3$.

Proof. The weight distributions of $[36, 13, 12]_3$ and $[36, 14, 12]_3$ are

$$\{0^1, 12^{882}, 15^{3024}, 18^{77196}, 21^{381528}, 24^{648270}, 27^{421568}, 30^{58716}, 33^{3024}, 36^{114}\},$$

and

$$\{0^1, 12^{2520}, 15^{8640}, 18^{237720}, 21^{1125504}, 24^{1973160}, 27^{1242080}, 30^{185472}, 33^{7560}, 36^{312}\},$$

respectively. The weight distribution shows that all codewords have weights divisible by 3. So by Theorem 4.8, these codes are self-orthogonal and they are the only ternary codes with all the codewords having weight divisible by 3. Therefore, they are the only self-dual ternary codes. □

In the field $\mathbb{F}_7$ the 36 dimensional representation split into 5 non-trivial submodules of length $14, 15, 21, 22$ and 35. So we have 5 non-trivial codes

$$[36, 14, 8], [36, 15, 8], [36, 21, 6], [36, 22, 6], [36, 35, 2],$$

which are all even and projective.

5.3. **Representations of degree 63.**

We have 2 non-equivalent representations of degree 63 under $U(3, 3)$ and we shall name them $63a$ and $63b$. We construct a 63-dimensional permutation module invariant under $U(3, 3)$, acting on a set of degree 36, by letting the permutation module being our working module.

Representation 63a. We look at the first permutation module of degree 63, and let it be our working module to find all its submodules. The module splits into 44 submodules of dimension:

$$\{0, 1, 6, 7^3, 8, 13, 14^3, 15, 20, 21^3, 22, 27, 28^3, 29, 34, 35^3, 36, 41, 42^3, 43, 48, 49^3, 50, 55, 56^3, 57, 62, 63\}.$$

The submodule lattice of this representation as shown in Figure 5.3.

According to Theorem 4.5, there are 44 codes of length 63 and dimensions given above. It is left to find the minimum distances of those linear codes, so that the parameters will be complete. Codes under this representation are of the form $[63, n, d]$ where n is the dimension of the submodule and d is the minimum weight. Since we have dimensions of all the codes, we are left with finding the minimum weights.

Proposition 5.9. *There exists a code* $[63, 7, 31]$*, with weight distribution* $\{0^1, 31^{63}, 32^{63}, 63^1\}$.

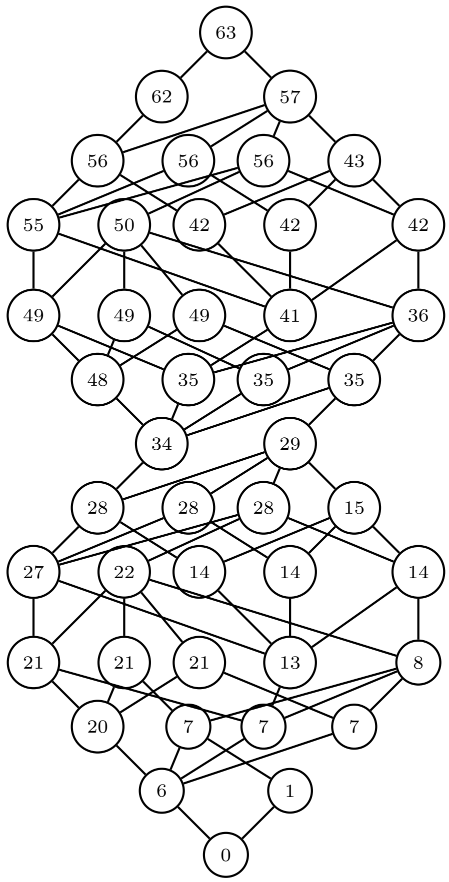

FIGURE 3. Submodule lattice for a 63-dimensional representation.

Proof. Having that from MAGMA, the weight distribution of $[63, 6, 32]$ is $\{0^1, 32^{63}\}$ and $[63, 7, d] = [63, 6, 32] \oplus [63, 1, 63]$, where $[63, 1, 63]$ is an all-one code. We can find the weight distribution of $[63, 7, d]$ by adjoining the ones-vectors of $[63, 6, 32]$ to get $\{0^1, 31^{63}, 32^{63}, 63^1\}$. We can see that $d = 31$ and the code is not even. □

In a similar way we can find the weight distributions of $[63, 8.27] = [63, 7, 28] \oplus [63, 1, 63]$ and $[63, 14, 20] = [63, 13, 20] \oplus [63, 1, 63]$ and all other codes that can be written as the direct sum of another code and the repetition code.

Remark 5.10. By looking at the two codes with parameters $[63, 14, 20]$, We notice that codes may have the same parameters but have different structures and hence not isomorphic. In fact, they have the different weight distributions.

Proposition 5.11. *The codes* $[63, 6, 32]$ *and* $[63, 7, 28]$ *are doubly even and self-orthogonal.*

Proof. Using MAGMA to compute the weight distributions, we notice that the weight distributions of the above codes are $\{0^1, 32^{63}\}$, and $\{0^1, 28^{36}, 32^{63}, 36^{28}\}$, respectively. We note that all weights are divisible by four, therefore the codes are doubly even. Moreover, we note that their dual codes are $[63, 57, 3]$ and $[63, 50, 3]$, respectively. Using the lattice diagram we note that those codes are subsets of their dual codes, therefore the codes are self-orthogonal. □

Table 9 shows the properties of some non-trivial codes of length 63. We choose codes with different automorphism groups, all other codes have the same group as their automorphism groups. The computations were based on MAGMA.

TABLE 9. non-trivial codes of length 63.

parameters	$\mathrm{Aut}(C_i)$	**Primitivity of** C_i
$[63, 6, 32]$	$L_6(2)$	yes
$[63, 7, 28]$	$S_6(2)$	yes
$[63, 13, 20]$	$U(3, 3)$	yes

Support designs of length 63.

Using Proposition 4.6 and the methods described in representations of degree 28 and 23, we determine the support designs constructed from codes of length 63, and are shown in Table 10 and Table 11.

TABLE 10. Stabilizers and support designs from $[63, 6, 32]$.

m	$\mathrm{s} := \lvert \mathrm{w_m} \rvert$	**Stabilizer**	**Maximal in** A	**Design**
32	63	$2^5 : L_5(2)$	yes	$1-(63, 32, 32)$

TABLE 11. Stabilizers and support designs from $[63, 7, 28]$.

m	$\mathrm{s} := \lvert \mathrm{w_m} \rvert$	**Stabilizer**	**Maximal in** A	**Design**
28	36	S_8	yes	$1-(63, 28, 16)$
32	63	$2^2 : S_6$	yes	$1-(63, 32, 32)$
36	28	$U(4, 2) : 2$	yes	$1-(63, 36, 16)$

Representation 63b.

We now look at the second permutation module of degree 63, and letting it be our working module and find all its submodules. The module splits into 28 submodules of dimension

$$0, 1, 14, 15, 20, 21^3, 22, 27, 28^3, 29, 34, 35^3, 41, 42^3, 43, 48, 49, 62, 63.$$

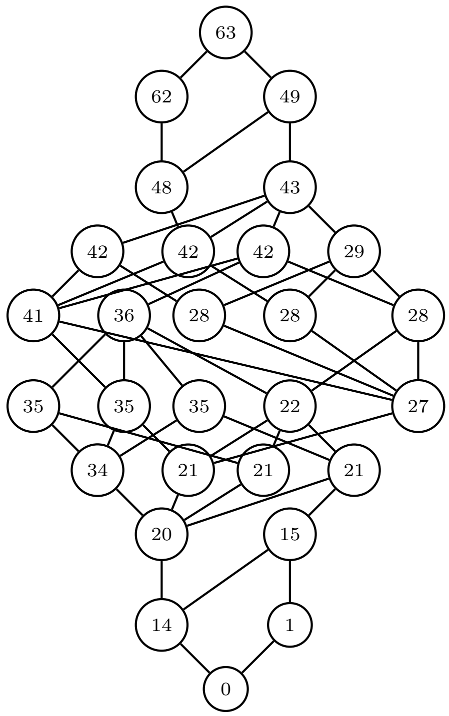

Figure 4. Submodule lattice for a 63-dimensional representation.

According to Theorem 4.5, there are 28 codes of length 63 and dimensions given above. So, it is left to find the minimum distance of those linear codes so that the parameters will be complete. The submodule lattice of this representation as shown in Figure 5.3.

Lemma 4.7 implies that the codes are even if and only if they are contained in the dual of repetition code. From the lattice diagram we can see which submodule are contained in the submodule of dimension 63, hence they are even.

Proposition 5.12. *Codes* $[63, 14, 16]$ *and* $[63, 20, 16]$ *are doubly even.*

Proof. The codes have the weight distributions

$$\{0^1, 16^{126}, 24^{1596}, 28^{2880}, 32^{7497}, 36^{252}\}$$

and

$$\{0^1, 16^{693}, 20^{3024}, 24^{78456}, 28^{278064}, 32^{420651}, 36^{222768}, 40^{41832}, 44^{3024}, 48^{63}\},$$

respectively. We note that all the weights are divisible by 4, therefore the codes are doubly even. □

Proposition 5.13. *There is a code with parameters* $[63, 15, 16]$ *which is self-orthogonal, projective and not even.*

Proof. Now having the code $[63, 14, 16]$ and its weight distribution, and from the lattice diagram, we note we have a code $[63, 15, d] = [63, 14, 16] + \langle J \rangle$, where $\langle J \rangle$ is the all-one code. The weight distribution of $[63, 15, d]$ can be determined by adjoining the one-vectors to get

$$\{0^1, 16^{126}, 23^{252}, 24^{1596}, 27^{4032}, 28^{2880}, 31^{7497}, 32^{7497}, 35^{2880}, 36^{4032}, 39^{1593}, 40^{252}, 47^{126}, 63^1\}.$$

Hence we can get the minimum distance 16 from the weight distribution. We have the code $[63, 15, 16]$ which is not even and is contained in its dual code $C^{\perp} = [63, 44, 3]$. Therefore the code $[63, 44, 3]$ is self-orthogonal. Also both codes have $d \geq 3$, whence they are both projective.

□

Table 12 shows properties of some non-trivial codes of length 63. The computations were based on MAGMA the last column tell whether Aut(C) is primitive or not.

TABLE 12. Non-trivial codes of length 63.

Parameters	Aut(C)	**Primitivity of** $Aut(C)$
$[63, 14, 16]$	$U(3, 3) : 2$	yes
$[63, 15, 16]$	$U(3, 3) : 2$	yes
$[63, 20, 16]$	$U(3, 3) : 2$	yes

We notice that all non-trivial codes under representation $63b$ have $U(3, 3){:}2$ as their automorphism group.

Support designs of length 63.

Proposition 5.14. *Designs held by support of the code* $[63, 14, 16]$ *with weight distribution* $\{0^1, 16^{126}, 24^{1596}, 28^{2880}, 32^{7497}, 36^{252}\}$ *are shown in Table 13 .*

TABLE 13. Stabilizers and support designs from $[63, 14, 16]$.

m	$s := \|w_m\|$	***Stabilizer***	***Maximal in*** A	***Design***
16	*126*	S_8	*no*	$1-(63, 16, 32)$
24	*1596*	$2^2 : S_6$	*no*	$1-(63, 24, 608)$
28	*2880*	$U(4, 2) : 2$	*no*	$1-(63, 28, 1280)$
32	*7497*	S_3	*no*	$1-(63, 32, 2304)$
36	*252*	$M_8 : S_4$	*no*	$1-(63, 36, 160)$

Proof. Values of m and s are given in the weight distribution of the codes. Using Proposition 4.6, the designs from the supports of these codes are of the form $1-(n,m,\lambda)$ where n is the length of the code and $\lambda = ms/n$ and the result follows. □

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