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Research Paper

RADICAL-BASED GENERALIZATIONS OF H-SUPPLEMENTED MODULES

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ABSTRACT. In this paper, we introduce a novel extension of the well-studied class of lifting modules, which we call J-H-supplemented modules. This new class is closely tied to the concept of J-small submodules, a refinement of small submodules characterized by their interaction with the Jacobson radical. A key feature of J-H-supplemented modules is the existence, for every submodule U of W that satisfies a certain radical condition on the quotient W/U, of a direct summand T of W that reflects the relative position of U with respect to other submodules. More precisely, these modules are defined by the property that for every submodule U and for all submodules P where $U + P = W$, the equality $T + P = W$ also holds if and only if T is a direct summand of W and the quotient W/U is a radical module.

This characterization establishes a deep connection between J-H-supplemented modules and the structure of radicals in quotient modules, thereby extending the classical theory of lifting modules by incorporating radical-sensitive conditions. Our results demonstrate how the presence of such direct summands governs the decomposition properties of these modules and sheds light on new structural phenomena in module theory. This framework not only broadens the theoretical foundation of supplemented modules but also opens pathways for further exploration of module decompositions influenced by radical submodules.

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1. Introduction

All rings studied in this study are assumed to have an identity element and operate associatively. Additionally, all modules considered are assumed to be unitary right modules over S. If W and U are both S-modules, it is important to clarify that when we say $U \leq W$, we are indicating that U is a s.m(submodule) of W. Also, a direct summand will be abbreviated as d.s. A s.m U of W is considered to be "small" in W if $U + H$ does not equal W for any proper s.m H of W, and we denote this as $U \ll W$. In a previous study by Kabban and Khalid a new generalization of small s.ms called J-small s.ms was introduced. A s.m H of W is defined to be J-small (denoted as $H \ll_J W$) if $W = H + E$ with W/E being radical implies $W = E$ (see [1]). In [9], applying the concept of Jacobson small submodules, the authors introduced Jacobson Hopfian modules. A module W is said to be Jacobson Hopfian (J-Hopfian, for short) in case any surjective endomorphism of W has J-small kernel.

Lifting modules is a key concept in module theory that is closely tied to the idea of smallness. A module W is considered a lifting module if every s.m U contains a d.s T such that $U/T \ll W/T$. There have been numerous studies on lifting modules in recent years, with many authors exploring generalizations and variations of this concept.

In [5], the notion of an H-supplemented module was introduced: a module W is called H-supplemented if for every submodule U of W, there exists a direct summand T of W such that for every submodule P of W, the equality $W = U + P$ holds if and only if $W = T + P$. This concept serves as a significant generalization of lifting modules, which are themselves central objects in module theory due to their structural decomposition properties. The distinctive characteristics and structural implications of H-supplemented modules have sparked considerable interest among researchers since their initial introduction by Mohamed and Müller in [5].

Following this foundational work, numerous studies have expanded the theory, exploring deeper properties and various equivalent conditions that characterize H-supplemented modules. For instance, in [4] and [3], the authors investigated criteria under which modules can be classified as H-supplemented, providing valuable insights into their behavior and relationships to other module classes. Furthermore, Talebi and his collaborators in [11] extended this framework by examining H-supplemented modules from the perspective of preradicals, leading to the introduction of the concept of τ-Hsupplemented modules, which enriches the structural theory by connecting H-supplementedness with torsion theories.

In addition, the literature includes significant contributions such as those found in [8, 12], [10], [13], and [14], where the notion of H-supplemented modules and also some generalizations of supplemented modules have been further generalized and analyzed in various algebraic contexts. These works have helped clarify the role of H-supplemented modules within the broader environment of module theory and have established connections to other important classes of modules. Moreover, two notable generalizations H-supplemented modules were introduced and rigorously studied in [6] and [7], providing a pathway for future research and potential applications.

Collectively, these investigations highlight the richness and versatility of H-supplemented modules as a research topic in modern algebra, offering both theoretical advancements and tools for understanding module decompositions in more complex settings.

In this paper, we undertake a detailed investigation of H-supplemented modules by drawing inspiration from the framework of J-small submodules. Our primary objective is to introduce and analyze a natural generalization of H-supplemented modules, which we develop in Section 2. Specifically, we define a module W to be J-H-supplemented if for every submodule U of W, there exists a direct summand T of W such that the equality $W = U + P$ if and only if $W = T + P$, holds for all submodules P of W that satisfy the condition $Rad(W/P) = W/P$, where denotes the radical of the factor module. This additional restriction involving the radical condition effectively refines the notion of smallness in this context and allows us to capture subtle structural nuances that are not covered by the classical H-supplemented condition.

Moreover, we explore an equivalent characterization that underscores the relationship between J-H-supplemented modules and the concept of J-small submodules, thereby illuminating how these generalized modules interact with the radical layers of their factor modules. This connection not only broadens our understanding of module supplements but also provides new avenues for further research in the decomposition theory of modules with radical-sensitive conditions.

Through this framework, we aim to deepen the theory of H-supplemented modules by incorporating the influence of Jacobson radicals, thereby enriching the classification and structural analysis of modules in this generalized setting.

2. J-small submodules and J-H-supplemented modules

In the work by Kabban and Khalid, a novel type of s.ms called J-small s.ms was introduced as a generalization of small s.m ([1]). A s.m U of a module W is considered to be J-small in W (denoted as $U \ll_J W$) if $W = U + H$ and whenever $Rad(W/H) = W/H$, it implies $W = H$.

Additional properties of J-small s.ms were explored in the literature, but some aspects were not covered in the original work and are discussed here for completeness.

It is clear that every small s.m of a module is J-small in that module.

We list some properties of J-small s.m that are similar to those for small case.

Proposition 2.1. ([1, Propositions 2.6, 2.7, 2.8]) *Let W be an S-module. Then the following statements hold.*

(1) *Let* $A \leq B \leq W$*. Then* $B \ll_J W$ *iff* $A \ll_J W$ *and* $\frac{B}{A} \ll_J \frac{W}{A}$*.*

(2) *Let* A*,* B *be s.ms of* W *with* $A \leq B$*. If* $A \ll_J B$*, then* $A \ll_J W$*.*

(3) *Let* $\varpi : W \to W'$ *be an epimorphism such that* $A \ll_J W$*, then* $\varpi(A) \ll_J W'$*.*

(4) *Let* $M = W_1 \oplus W_2$ *be an* S*-module and let* $A_1 \leq W_1$ *and* $A_2 \leq W_2$*. Then* $A_1 \oplus A_2 \ll_J W_1 \oplus W_2$ *iff* $A_1 \ll_J W_1$ *and* $A_2 \ll_J W_2$*.*

The following provides a characterization of a module W such that every s.m of W is J-small in W.

Proposition 2.2. *Let* W *be a module. The following are equivalent:*

(1) $W \ll_J W$*;*

(2) *Each s.m of* W *is* J*-small in* W*;*

(3) *None of nonzero homomorphic images of* W *is radical.*

Proof. (1) $\Rightarrow$ (2) It follows from [1, Proposition 2.6(1)].

(2) $\Rightarrow$ (3) Suppose that every s.m of W is J-small in W. Consider a s.m P of W such that W/P is radical. Since $W = W + P$ and $W \ll_J W$, then $W = P$.

(3) $\Rightarrow$ (1) Let U be a s.m of W such that W/U is radical. By assumption $W = U$ which shows that $W \ll_J W$. □

It is known that if $W \ll W$, then $W = 0$. We see that there are some non-zero modules W such that $W \ll_J W$.

Example 2.3. Let W be a finitely generated module with $Rad(W) = 0$. Then by Proposition 2.2, W is a J-small s.m of itself while W can not be a small s.m of W. To ensure, consider $\mathbb{Z}$ as a module over itself or W can be any nonzero semisimple module.

It is clear that every small s.m of a module is J-small. We provide some examples to indicate that the converse may not hold.

Example 2.4. (1) Let $W = \mathbb{Z}$ as a module over itself. Since every homomorphic image of W is a finitely generated module, every s.m of W is J-small in W by Proposition 2.2. Note that none of nonzero s.ms of W is small in W.

(2) Let W be a semisimple module. Since $Rad(W) = 0$, every s.m of W is J-small in W by Proposition 2.2, while the only small s.m of W is the zero s.m.

The following presents some conditions which under two concepts small and J-small coincide.

Proposition 2.5. *Let* W *be a module and* $U \leq W$*. Then in each of the following cases* $U \ll W$ *iff* $U \ll_J W$*:*

(1) U *is radical.*

(2) $U/T \ll W/T$ *where* T *is a radical d.s of* W*.*

Proof. (1) Suppose U is a radical module and $U \ll_J W$. Assume $U + H = W$ for a s.m H of W. As U is radical, so $U/U \cap H \cong W/H$ is radical. It follows that $W = H$, as required.

(2) Assume that $U \ll_J W$. We are going to show that $U \ll W$. To show this assertion, suppose $U + H = W$. Note also that $T \oplus T' = W$ for $T' \leq W$. Now, $U + T' = W$. Being T a radical module, implies W/T' is radical. As $U \ll_J W$, we have $W = T'$ which yields $T = 0$. Hence, $U \ll W$. □

Recall that a ring S is a right V-ring provided every simple module is injective. It is know that, the radical of each right S-module is zero. So each s.m of an arbitrary S-module is J-small. For instance, $S = \prod_{i=1}^{\infty} K_i$ where $K_i = K$ is a field, is a von Neumann regular V-ring, hence each s.m of an S-module is J-small in that S-module while the only small S-module is $\{0\}$.

Unlike small s.ms, we may have nonzero J-small d.ss of a module. In fact each s.m (d.s) of a nonzero semisimple module is J-small.

Recall that a module W is called H-supplemented in case for every s.m U of W, there is a d.s T of W such that $W = U + P$ iff $W = T + P$ for every s.m P of W. Based on this definition, we introduce a generalization of H-supplemented modules applying J-small s.ms.

Before starting with the key definition, we may recall the definition of a J-lifting module. Following [2], a module W is called J-lifting (stands for Jacobson lifting) in case for each s.m U of W, there exists a decomposition $W = Z \oplus Z'$ such that $Z \subseteq U$ and $U \cap Z' \ll_J Z'$. Equivalently, a module W is J-lifting iff each s.m U contains a d.s T of W such that $U/T \ll_J W/T$.

Also Jacobson version of supplemented modules was defined in [1]. A module W is called J-supplemented in case for each s.m A of W, there is a J-supplement B i.e. $W = A + B$ and $A \cap B \ll_J B$.

Definition 2.6. Let W be a module. Then W is said to be J-H-supplemented, provided for every s.m U of W there is a d.s T of W such that $U + P = W$ iff $T + P = W$ for each s.m P of W with $Rad(W/P) = W/P$.

Lemma 2.7. *Every J-lifting module is J-H-supplemented. In particular, any J-hollow module is J-H-supplemented.*

Proof. Assume W is J-lifting and $U \leq W$. Then $W = Z \oplus Z'$ such that $Z \subseteq U$ and $Z' \cap U \ll_J Z'$. We shall prove $U + P = W$ iff $Z + P = W$ for each s.m P of W with $Rad(W/P) = W/P$. Suppose $U + P = W$ with W/P radical. Now, $U/Z + (P + Z)/Z = W/Z$. Note that $\frac{W/Z}{(P+Z)/Z}$ is a radical module as a homomorphic image of W/P. As $U/Z \ll_J W/Z$, we conclude that $(P + Z)/Z = W/Z$. Hence $Z + P = W$. Other implication is clear as Z is contained in U. □

Above-mentioned results and comments yield the implications: J-hollow module $\Rightarrow$ J-lifting $\Rightarrow$ J-H-supplemented and J-lifting $\Rightarrow$ J-supplemented.

In the following, we show that the class of H-supplemented modules is contained properly in the class of J-H-supplemented modules.

Example 2.8. (1) It is easy to check that, for radical modules (a module W is radical in case $Rad(W) = W$), two concepts small s.ms and J-small s.ms coincide. In fact, for a radical module W, we have W is J-H-supplemented iff W is H-supplemented.

(2) Since each injective module over a Dedekind domain is radical, so every injective non-supplemented module over a Dedekind domain is neither J-H-supplemented nor H-supplemented. For example, as an $\mathbb{Z}$-module $\mathbb{Q}$ is not J-H-supplemented.

(3) A J-H-supplemented may not be H-supplemented. Note that being H-supplemented implies J-H-supplemented as every small s.m of a module is J-small in that module. Suppose that W is a module with zero radical such that none of its homomorphic images is a radical module. Then each nonzero s.m of W is J-small while none of them is small in W. In fact W is a J-hollow module and hence a J-H-supplemented module. For instance, we may choose $\mathbb{Z}$ as an $\mathbb{Z}$-module.

(4) By [4, Theorem 2.15], each projective module over a right perfect ring is H-supplemented and hence J-H-supplemented. In particular, for each natural number n, the $\mathbb{Z}_n$-module $\mathbb{Z}_n$ is J-H-supplemented. Note that every Artinian ring is perfect, hence $\mathbb{Z}_n$ is perfect.

A useful characterization of J-H-supplemented modules can be found in the next result.

Proposition 2.9. *Let W be a module. Then W is J-H-supplemented iff for every s.m U of W there exists a d.s T of W such that $(U+T)/U \ll_J W/U$ and $(U+T)/T \ll_J W/T$.*

Proof. Let W be J-H-supplemented and $U \leq W$. Then there is a d.s T of W such that $W = U + P$ iff $W = T + P$, for every s.m P of W such that W/U is radical. Suppose that $(U+T)/U + P/U = W/U$ for a s.m P of W containing U with $\frac{W/U}{P/U}$ radical. Then $T+P = W$. Note that W/P is radical as well as $\frac{W/U}{P/U}$. Now, by assumption $U + P = W$, which implies that $P = W$ as required. To verify the second J-small case, let $(U+T)/T + B/T = W/T$ where $\frac{W/T}{B/T}$ is radical. Then $U + B = W$. Being W a J-H-supplemented module implies $T + B = W$. Therefore, $B = W$. Conversely, let $U + A = W$ with W/A radical. Then $(U+T)/T + (A+T)/T = W/T$. Being $W/(A+T)$ a homomorphic image of W/A implies, $W/(A+T)$ is radical. Hence $(A+T)/T = W/T$ and so $W = A+T$, since $(U+T)/T \ll_J W/T$. Now, suppose that $T + Z = W$ for a s.m Z of W such that $Rad(W/Z) = W/Z$. Then $(U+T)/U + (U+Z)/U = W/U$ and $W/(U+Z)$ as a homomorphic image of W/Z is radical. Being $(U+T)/U$ a J-small s.m of W/U combining with last equality implies $U + Z = W$. □

Proposition 2.10. *Let W be an indecomposable module. Then W is J-H-supplemented iff for every proper s.m U of W, we have $U \ll_J W$ or $W/U \ll_J W/U$.*

Proof. Let W be indecomposable and J-H-supplemented. Consider an arbitrary proper s.m U of W. Then there is a d.s T of W such that $(U+T)/U \ll_J W/U$ and $(U+T)/T \ll_J W/T$. If $T = 0$, then clearly $U \ll_J W$. In other words, $T = W$ implies $W/U \ll_J W/U$. Conversely, let $U < W$. If $U \ll_J W$, then $(U+0)/U \ll_J W/U$ and $(U+0)/0 \ll_J W/0$. Otherwise, $(U+W)/W \ll_J W/W$ and $(U+W)/U \ll_J W/U$. □

Proposition 2.11. *Let W be J-H-supplemented and K a s.m of W such that for each d.s T of W, the s.m $(T+K)/K$ is a d.s of W/K. Then W/K is J-H-supplemented.*

Proof. Consider an arbitrary s.m U/K of W/K. As W is J-H-supplemented, there exists a d.s T of W such that $W = U+P$ iff $W = T+P$ for each s.m P of W with $Rad(W/P) = W/P$. Note that $(T+K)/K$ is a d.s of W/K. We are going to show $W/K = U/K + B/K$ iff $W/K = (T+K)/K + B/K$ for each s.m B/K of W/K such that W/B is radical. Assume $W/K = U/K + B/K$ for $B/K \leq W/K$ such that $\frac{W/K}{B/K} \cong W/B$ is radical. Then $W = U+B$. As W/B is radical, above assumption implies $W = T+P$ which yields $W/K = (T+K)/K+B/K$. For the other side, suppose that $W/K = (T+K)/K + E/K$ for $E/K \leq W/K$ such that $\frac{W/K}{E/K} \cong W/E$ is radical. Now, $W = T+E$ combining with W/E is a radical modules yield that $W = U+E$. Therefore, $W/K = U/K + E/K$, which we desired. □

A s.m U of a module W is considered (projection invariant) fully invariant if for every (idempotent) endomorphism ϖ of W, the image of U under ϖ is contained in U. It is important to note that every fully invariant s.m is also projection invariant. A module is classified as (weak) duo if every (d.s) s.m is fully invariant.

Proposition 2.12. *Let W be a module and U a projection invariant s.m of W. If W is J-H-supplemented, then W/U is also J-H-supplemented.*

Proof. Let Z/U be an arbitrary s.m of W/U. Then there exists a direct summand T of W such that $W = Z+P$ iff $W = T+P$ for every s.m P of W such that W/P is radical. Set $W = T \oplus T'$. As U is a projection invariant s.m of W, we conclude that $(U+T)/T \oplus (U+T')/U = W/U$. Now, suppose $Z/U + A/U = W/U$ for a s.m A/U of W/U with $\frac{W/U}{A/U}$ radical. It follows that W/A is a radical module. Now, the equality $Z+A = W$ combining our assumption, imply $W = T+A$. Clearly $W/U = (T+U)/U + A/U$. Now for the other implication, let $W/U = (T+U)/U + B/U$ with $\frac{W/U}{B/U}$ radical. Hence $W = T+B$ and note also that W/B is radical. Hence $W = Z+B$. Obviously $W/U = Z/U + B/U$. □

It is known that a module W is said to be *distributive* in case the lattice of s.ms of W is distributive, i.e. for each s.ms U, Z, H of W the equalities $(U \cap H) + (U \cap Z) = U \cap (Z + H)$ and $U + (Z \cap H) = (U + Z) \cap (U + H)$ hold.

Corollary 2.13. (1) *Every homomorphic image of a distributive J-H-supplemented module is J-H-supplemented.*

(2) *Every d.s of a weak duo J-H-supplemented module is J-H-supplemented.*

Theorem 2.14. *Let $M = W_1 \oplus W_2$ be a distributive module. Then W is J-H-supplemented module iff W_1 and W_2 are J-H-supplemented.*

Proof. Let W_1 and W_2 be J-H-supplemented and $U \leq W$. Set $U_1 = U \cap W_1$ and $U_2 = U \cap W_2$. Then $U = U_1 + U_2$. Now, there exist d.ss T_i of W_i for $i = 1, 2$, such that $(U_i + T_i)/U_i \ll_J W_i/U_i$ and $(U_i + T_i)/T_i \ll_J W_i/T_i$. We shall prove that $(U + T)/U \ll_J W/U$ and $(U + T)/T \ll_J W/T$ where $T = T_1 \oplus T_2$ which is a d.s of W. Suppose that $(U + T)/U + P/U = W/U$ for a s.m P of W containing U with $\frac{W/U}{P/U} \cong W/P$ radical. Then $T + P = W$. It follows that $T_1 + (P \cap W_1) = W_1$. Now $(U_1 + T_1)/U_1 + (P \cap W_1)/U_1 = W_1/U_1$ and $W_1/(P \cap W_1) \cong T_1/(P \cap T_1)$ as a d.s of $T/(P \cap T) \cong W/P$ is a radical module. Therefore, $P \cap W_1 = W_1$ which implies that W_1 is contained in P. Now consider again the equality $T + P = W$. So $T_2 + (P \cap W_2) = W_2$. As $(U_2 + T_2)/U_2 + (P \cap W_2)/U_2 = W_2/U_2$ and $(U_2 + T_2)/U_2 \ll_J W_2/U_2$ and also $W_2/P \cap W_2 \cong T_2/(P \cap D_2)$ as a d.s of $P/(P \cap T) \cong W/P$ is radical, we conclude that $P \cap W_2 = W_2$. So that W_2 is contained in P which implies that $P = W$. For the other J-small case, let $(U + T)/T + B/T = W/T$ where $B/T \leq W/T$ and $\frac{W/T}{B/T} \cong W/B$ is radical. Now $U + B = W$ and hence $U_1 + (B \cap W_1) = W_1$. Being $(U_1 + T_1)/T_1$ a J-small s.m of W_1/T_1 combining with the fact that $W_1/(B \cap W_1) \cong U_1/(U_1 \cap B)$ as a d.s of $U/(U \cap B) \cong W/B$ is radical and the last equality imply that $B \cap W_1 = W_1$ and therefore $W_1 \subseteq B$. By a same process, B will contain W_2. Hence $B = W$ as required. It follows now that W is J-H-supplemented. The converse follows from Proposition 2.12. □

3. Some problems

(1) When is a Jacobson Hopfian module, J-H-supplemented?

(2) Assume that $f : W \to A$ is an epimorphism with W a J-H-supplemented module. What can we say about $Kerf$? Otherwise, suppose $g \in End(W)$ is an epimorphism and W a J-H-supplemented module. Does g has a J-small kernel?

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