Algebraic Structures and Their Applications Vol. X No. X (20XX) pp XX-XX.

Research Paper

A NEW CLASS OF DUAL NOTIONS: S-CO-r-SUBMODULES AND S-CO-n-SUBMODULES BASED ON MULTIPLICATIVELY CLOSED SUBSETS

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Abstract. Let R be a commutative ring, M be an R-module and $S \subseteq R$ be a multiplicatively closed subset of R. The purpose of this paper is to introduce and investigate the concepts of S-co-r-submodules and S-co-n-submodules by using the notion of a multiplicatively closed subset of R. A non-zero submodule N of M with $Rad(Ann(M)) \cap S = \emptyset$ is called an S-co-n-submodule, if there exists $s \in S$ such that whenever $aN \subseteq K$ and $sa \notin Rad(Ann(M))$ for some $a \in R$ and a submodule K of M, then $sN \subseteq K$. Many properties and examples are given of such submodules. Also, we state the correspondence between S-co-r-submodules and S-co-n-submodules.

1. Introduction

Throughout this paper, R is a commutative ring with non-zero identity and M is a unital R-module. The notion of prime submodules has an important place in commutative algebra

DOI: 10.22034/as.2025.23319.1803

MSC(2010): Primary: 13C05, 16P40.

Keywords: S-co-r-submodule, S-co-n-submodule, S-comultiplication module, S-multiplication module.

Received: 27 June 2025, Accepted: 06 November 2025.

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and it is frequently used to classify the modules. The concept of prime submodules is important for many research areas and numerous authors have delved into their generalizations, yielding diverse findings, for example, weakly prime submodules [2], almost prime submodules [10], semiprime submodules [9], almost semiprime submodules [7] and 2-absorbing submodules [13]. Recently, the notion of S-prime submodules and generalizations of it have been introduced and studied in [8, 12, 14, 15, 16]. Here we introduce and study the notions of S-co-r-submodules and S-co-n-submodules of a module over a commutative ring. Various properties of such submodules are considered. Consider a non-empty subset S of R. Then S is called a multiplicatively closed subset of R (briefly, m.c.s) if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $ss' \in S$ for all $s, s' \in S$ [17]. Let S be a m.c.s of R and N be a non-zero submodule of M such that $(P :_R M) \cap S = \emptyset$. Then the submodule P is called an S-prime (resp. S-primary) submodule of M, if there exists $s \in S$ such that whenever $am \in P$, then $sa \in (P :_R M)$ or $sm \in P$ ($sa \in Rad(P :_R M)$ or $sm \in P$) for each $a \in R$ and $m \in M$. Particularly, an ideal I of R is called an S-prime (resp. S-primary) ideal of R if I is an S-prime (resp. S-primary) submodule of R-module R see, [15]. Let M be an R-module. The subset $W_R(M)$ of R, the set of all cozero divisors of R (that is the dual notion of $Z_R(M)$), is defined by $\{r \in R \mid rM \neq M\}$. In [6], the author introduced and studied the concept of co-r-submodules as a dual notion of r-submodules. A non-zero submodule N of M is said to be a co-r-submodule if for $a \in R$ and submodule K of M, whenever $aN \subseteq K$ and $a \notin W_R(M)$, then $N \subseteq K$. A non-zero submodule N of M is said to be a co-n-submodule if for $a \in R$ and submodule K of M, whenever $aN \subseteq K$ and $a \notin Rad(Ann(M))$, then $N \subseteq K$. In this paper, we introduce and study the concepts of S-co-r-submodules as a generalization of co-r-submodules and S-co-n-submodules as a generalization of co-n-submodules and we derive some properties of them. For example, we show that every S-copure submodule is an S-co-r-submodule. Also, we show that every S-co-n-submodule is an S-co-r-submodule.

2. S-co-r-submodules

Definition 2.1. Let $S \subseteq R$ be a m.c.s of R and let M be an R-module with $W_R(M) \cap S = \emptyset$. A non-zero submodule N of M is called an S-co-r-submodule, if there exists $s \in S$ such that whenever $aN \subseteq K$ and $sa \notin W_R(M)$ for some $a \in R$ and a submodule K of M, then $sN \subseteq K$. This fixed element $s \in S$ is called an S-element of N.

Note that if $S \cap W_R(M) \neq \emptyset$, then there exists $s \in S$ such that $sM \neq M$. Thus for any $a \in R, saM \neq M(sa \in W_R(M))$. Because if $saM = M$, then $M = saM \subseteq sM$, so $sM = M$, a contradiction.

Remark 2.2. Let N be a non-zero submodule of an R-module M. If N is a co-r-submodule of M, then clearly, N is an S-co-r-submodule for any multiplicatively closed subset S of R. However, the classes of co-r-submodules and S-co-r-submodules coincide if $S \subseteq U(R)$.

An R-module M is said to be an S-multiplication module, if for every submodule N of M, there exists $s \in S$ such that $sN \subseteq IM \subseteq N$ for some ideal I of R [1] Definition 1.

Theorem 2.3. *Let S be a m.c.s of R and M be an S-multiplication module with $W_R(M) \cap S = \emptyset$. Then every non-zero submodule N of M is an S-co-r-submodule.*

Proof. Let N be a non-zero submodule M. Then there exists $s \in S$ such that $sN \subseteq IM \subseteq N$ for some ideal I of R since M is an S-multiplication module. Assume that $aN \subseteq K$ with $saM = M$ for $a \in R$ and submodule K of M. Thus we have

$$sN \subseteq IM = I(saM) = sa(IM) \subseteq saN \subseteq sK \subseteq K.$$

Therefore, N is an S-co-r-submodule. □

Example 2.4. Consider the $\mathbb{Z}$-module $\mathbb{Z}$ and the multiplicatively closed subset $S = \mathbb{Z} - 2\mathbb{Z}$. Then every non-zero submodule of $\mathbb{Z}$ is S-co-r-submodule.

Let $S \subseteq R$ be a m.c.s of R. The saturation S^* of S is defined as $S^* = \{x \in R \mid \frac{x}{1}$ is a unit of $S^{-1}R\}$. Note that S^* is a m.c.s of R containing S.

Proposition 2.5. *Let M be an R-module and S be a m.c.s of R with $W_R(M) \cap S = \emptyset$. Then the following statements hold:*

(i) *Let $S_1 \subseteq S_2$ m.c.s of R. If N is an S_1-co-r-submodule and $W_R(M) \cap S_2 = \emptyset$, then N is an S_2-co-r-submodule.*

(ii) *A submodule N of M is an S-co-r-submodule submodule if and only if it is an S^*-co-r-submodule.*

Proof. The proofs are completely straightforward. □

Proposition 2.6. *Let M be an R-module and S be a m.c.s of R with $W_R(M) \cap S = \emptyset$. Then the following statements hold.*

(i) *M is an S-co-r-submodule of M.*

(ii) *If $\{N_i\}_{i=1}^n$ is a family of S-co-r-submodules of M, then $\sum_{i=1}^n N_i$ is an S-co-r-submodule of M.*

Proof. (i) It is clear.

(ii) Let N_i be an S-co-r-submodule of M with S-element s_i for each $i = 1, 2, \ldots, n$. We take $s = s_1 \cdots s_n$. Assume that $a \sum_{i=1}^{n} N_i \subseteq K$ and $as \notin W_R(M)$ for some $a \in R$ and a submodule K of M. Thus for each i, $aN_i \subseteq K$ and $s_i a \notin W_R(M)$. Since N_i is an S-co-r-submodule, $s_i N_i \subseteq K$ and so $sN_i \subseteq K$ for each $i = 1, \ldots, n$. Therefore, $s \sum_{i=1}^{n} N_i \subseteq K$, as required. □

Note that the intersection of two S-co-r-submodules need not be an S-co-r-submodule.

Example 2.7. Consider the $\mathbb{Z}$-module $\mathbb{Z}_6$ and the multiplicatively closed subset $S = \{5^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. As $\mathbb{Z}_6$ is an S-multiplication module, then by Theorem 2.3, $\langle \bar{2} \rangle$ and $\langle \bar{3} \rangle$ are S-co-r-submodules of $\mathbb{Z}_6$. But $\langle \bar{2} \rangle \cap \langle \bar{3} \rangle = 0$.

Proposition 2.8. *Let M be an R-module and S be a m.c.s of R with $W_R(M) \cap S = \emptyset$. Then if N is an S-co-r-submodule of M and A be a non-empty subset of R with $A \nsubseteq Ann_R(N)$, then AN is an S-co-r-submodule of M. In particular, AM is an S-co-r-submodule if $A \nsubseteq Ann(M)$.*

Proof. Suppose that N is an S-co-r-submodule of M with S-element $s \in S$. Let $rAN \subseteq K$ and $srM = M$ for $r \in R$ and a submodule K of M. Then $raN \subseteq K$ for every $a \in A$, so $rN \subseteq (K :_M a)$. Since N is an S-co-r-submodule, we conclude $sN \subseteq (K :_M a)$ and so $saN \subseteq K$ for every $a \in A$. Thus $sAN \subseteq N$, so AN is an S-co-r-submodule of M. Moreover, since $A \nsubseteq Ann(M)$, by the first part, AM is an S-co-r-submodule. □

Corollary 2.9. *Let S be a m.c.s of R and M be an R-module M. If $a \in R \setminus Ann(M)$, then aM is an S-co-r-submodule.*

Proof. It follows from Proposition 2.8. □

Proposition 2.10. *Let N be a non-zero submodule of an R-module M and S be a m.c.s of R with $W_R(M) \cap S = \emptyset$. Then the following are equivalent:*

(i) *N is an S-co-r-submodule of M.*

(ii) *There exists $s \in S$; $sN \subseteq aN$ for $sa \in R \setminus W_R(M)$.*

(iii) *There exists $s \in S$; $s(N :_M a) \subseteq N + (0 :_M a)$ for $sa \in R \setminus W_R(M)$.*

Proof. $(i) \Rightarrow (ii)$ Let N be an S-co-r-submodule of M. Since $aN \subseteq aN$ and $saM = M$, so $sN \subseteq aN$.

$(ii) \Rightarrow (i)$ Let $aN \subseteq K$ and $saM = M$ for $a \in R$ and a submodule K of M. Thus by (ii), $sN \subseteq aN \subseteq K$ and so N is an S-co-r-submodule.

$(ii) \Rightarrow (iii)$ Let $sm \in s(N :_M a)$ for some $m \in (N :_M a)$. Then $sam \in sN \subseteq aN$ and so $sam = an$ for some $n \in N$. Hence $sm - n \in (0 :_M a)$ and $sm = n + sm - n \in N + (0 :_M a)$. Therefore $s(N :_M a) \subseteq N + (0 :_M a)$.

$(iii) \Rightarrow (ii)$ Let $y \in sN$. Then $y = sx$ for some $x \in N$. Since $saM = M$, so $x = sam$ for some $m \in M$. Hence $sm \in (N :_M a)$ and so $s^2m \in s(N :_M a) \subseteq N + (0 :_M a)$. Then $s^2m = n + m'$ for some $n \in N$ and $m' \in (0 :_M a)$. Thus $sx = s^2am = an + am' = an \in aN$. Therefore $y \in aN$ and so $sN \subseteq aN$, as desired. □

A submodule N of an R-module M is an S-copure, if there exists $s \in S$ such that $s(N :_M I) \subseteq N + (0 :_M I)$ for every ideal I of R [5], Definition 1. By Proposition 2.10, we have the following corollary:

Corollary 2.11. *Every S-copure submodule of an R-module M is an S-co-r-submodule.*

The converse of Corollary 2.11, is not true in general. See the following example.

Example 2.12. Consider the $\mathbb{Z}$-module $\mathbb{Z}_8$ and the multiplicatively closed subset $S = \{3^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then $\langle \bar{2} \rangle$ is an S-co-r-submodules of $\mathbb{Z}_8$. But $\langle \bar{2} \rangle$ is not an S-copure submodule of $\mathbb{Z}_8$.

Definition 2.13. Let $S \subseteq R$ be a m.c.s of R and let M be an R-module. We say that an S-co-r-submodule N of M is a minimal S-co-r-submodule of M, if there does not exist an S-co-r-submodule K of M such that $K \subsetneq N$.

A non-zero submodule N of an R-module M with $S \cap Ann(N) = \emptyset$ is said to be S-second, if there exists $s \in S$ such that for each $a \in R$, $saN = sN$ or $saN = 0$.

Proposition 2.14. *Let M be an R-module and S be a m.c.s of R and N be a submodules of M with $S \cap Ann(N) = \emptyset$. Then N is an S-second submodule of M if and only if there exists an $s \in S$ and whenever $aN \subseteq K$, where $a \in R$ and K is a submodule of M, implies that either $saN = 0$ or $sN \subseteq K$.*

Proof. Let $aN \subseteq K$ for some $a \in R$ and a submodule K of M. Since N is an S-second submodule, there exists $s \in S$ such that $saN = sN$ or $saN = 0$. Thus $sN = saN \subseteq sK \subseteq K$ or $saN = 0$, as needed. The converse is evidently. □

Proposition 2.15. *If N is a minimal S-co-r-submodule of an R-module M, then N is an S-second submodule.*

Proof. Let N be a minimal S-co-r-submodule with S-element $s \in S$. Suppose that $aN \subseteq K$ and $sN \nsubseteq K$, we show that $sa \in Ann(N)$. Let $sa \notin Ann(N)$. Then by Proposition 2.8, aN is an S-co-r-submodule of M. Since N is a minimal S-co-r-submodule, we have $aN = N \subseteq K$ and so $sN \subseteq K$ which is a contradiction. Therefore, we get $sa \in Ann(N)$, as required. □

Let M_i be an R_i-module for each $i = 1, 2, \ldots, n$ and $n \in \mathbb{N}$. Assume that $M = M_1 \times M_2 \times \cdots \times M_n$ and $R = R_1 \times R_2 \times \cdots \times R_n$. Then M is clearly an R-module with component wise addition and multiplication. Also, if S_i is a m.c.s of R_i for each $i = 1, 2, \ldots, n$, then $S = S_1 \times S_2 \times \cdots \times S_n$ is a m.c.s of R. Furthermore, each submodule of M is of the form $N = N_1 \times N_2 \times \cdots \times N_n$ where N_i is a submodule of M_i.

Theorem 2.16. *Let $M = M_1 \times M_2$ and $R = R_1 \times R_2$ where M_i is an R_i-module for $i = 1, 2$. Then if $S = S_1 \times S_2$ is a m.c.s of R and $N = N_1 \times N_2$ is a submodule of M, then the following statements are equivalent:*

(i) *N is an S-co-r-submodule of M.*

(ii) *$N_1 = 0$ and N_2 is an S_2-co-r-submodule of M_2 or N_1 is an S_1-co-r-submodule of M_1 and $N_2 = 0$ or N_1 is an S_1-co-r-submodules of M_1 and N_2 is an S_2-co-r-submodules of M_2, respectively.*

Proof. $(i) \Rightarrow (ii)$ Suppose that N is an S-co-r-submodule of M with S-element $s = (s_1, s_2)$ and $N_1 = 0$. Then $N_2 \neq 0$ because N is an S-co-r-submodule of M. Let $r_2N_2 \subseteq K_2$ and $s_2r_2 \notin W_{R_2}(M_2)$. Thus $(0, r_2)(N_1 \times N_2) \subseteq M_1 \times K_2$ and $(s_1, s_2)(0, r_2) \notin W_R(M)$. Hence $(s_1, s_2)(N_1 \times N_2) \subseteq M_1 \times K_2$, so $s_2N_2 \subseteq K_2$, as needed. In other cases, a similar argument shows that $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$ Let N_1 be an S_1-co-r-submodule of M_1 with S_1-element $s_1 \in S_1$ and N_2 be S_2-co-r-submodules of M_2 with S_2-element $s_2 \in S_2$. Assume that $(r_1, r_2)(N_1 \times N_2) \subseteq K_1 \times K_2$ and $(s_1, s_2)(r_1, r_2) \notin W_R(M)$. Then $r_1N_1 \subseteq K_1$ and $s_1r_1 \notin W_{R_1}(M_1)$ and also, $r_2N_2 \subseteq K_2$ and $s_2r_2 \notin W_{R_2}(M_2)$. Since N_1 and N_2 are S-co-r-submodules of M_1 and M_2, respectively, we conclude $s_1N_1 \subseteq K_1$ and $s_2N_2 \subseteq K_2$. Hence $(s_1, s_2)(N_1 \times N_2) \subseteq K_1 \times K_2$. Therefore, $N = N_1 \times N_2$ is an S-co-r-submodule of M. □

Corollary 2.17. *Let $M = M_1 \times M_2 \times \ldots M_t$ and $R = R_1 \times R_2 \times \ldots R_t$ where M_i is an R_i-module for $i = 1, 2, \ldots, t$. Then if $S = S_1 \times S_2 \times \ldots S_t$ is a m.c.s of R and $N = N_1 \times N_2 \times \ldots N_t$ is a submodule of M, then the following statements are equivalent:*

(i) *N is an S-co-r-submodule of M.*

(ii) *$N_i = 0$ for $i \in \{k_1, k_2, \ldots, k_m; m < t\} \subseteq \{1, 2, \ldots, t\}$ and N_i is an S_i-co-r-submodule of M_i for $i \in \{1, 2, \ldots, t\} \setminus \{k_1, k_2, \ldots, k_m; m < t\}$.*

Theorem 2.18. *Let $S \subseteq R$ be a m.c.s of R and N be a non-zero submodule of an R-module M. Then the following assertions hold:*

(i) *N is an S-co-r-submodule of M with S-element $s \in S$ if and only if whenever I is an ideal of R such that $sI \cap (R \setminus W_R(R)) \neq \emptyset$ and K is a submodule of M with $IN \subseteq K$, then $sN \subseteq K$.*

(ii) *If $Ann(N) \subseteq W_R(M)$ and N is not an S-co-r-submodule of M, then there exists an ideal I of R such that $sI \cap (R \setminus W_R(R)) \neq \emptyset$, $K \subset N$, $Ann(N) \subseteq I$ and $IN \subseteq K$.*

Proof. (i) Let N be an S-co-r-submodule of M with S-element $s \in S$. Let $IN \subseteq K$ for some ideal I of R with $sI \cap (R \setminus W_R(R)) \neq \emptyset$ and submodule K of M. There exists $a \in I$ such that $sa \in I$ and $saM = M$. Thus $sN \subseteq K$ since N is an S-co-r-submodule.

Conversely, let $aN \subseteq K$ and $saM = M$ for some $a \in R$ and submodule K of M. Take $I = \langle a \rangle$. Hence $sI \cap (R \setminus W_R(R)) \neq \emptyset$. Then by assumption, $sN \subseteq K$. Thus N is an S-co-r-submodule of M.

(ii) Since N is not an S-co-r-submodule of M, so for any $s \in S$ there exists $a \in R$ and a submodule K of M such that $aN \subseteq K$ with $saM \neq M$ and $sN \nsubseteq K$. Let $I = (K : N)$. Note that $a \in I$ and $sa \notin Ann(N)$ (since $saM \neq M$). Thus $Ann(N) \subset I$. Now, we take $K = IN$. Therefore $K \subset N$ ($sN \nsubseteq K$), $sAnn(N) \subset I$ and $IN = (IN :_M I) \subseteq K$. □

Proposition 2.19. *Let M be an R-module and S be a m.c.s of R and $K \subseteq N$ be submodules of M. Then if N/K is an S-co-r-submodule of R-module M/K and K is a co-r-submodule of M, then N is an S-co-r-submodule of M.*

Proof. Assume that N/K is an S-co-r-submodule of M/K with S-element $s \in S$ and let $aN \subseteq T$ and $sa \notin W_R(M)$ for some $a \in R$ and submodule T of M. We have $aK \subseteq aN \subseteq T$ and $a \notin W_R(M)$ ($sa \notin W_R(M)$). Thus $K \subseteq T$ since K is a co-r-submodule of M. Thus $a(N/K) = (aN + K)/K \subseteq T/K$ and clearly $sa \notin W_R(M/K)$, so $s(N/K) \subseteq T/K$. Thus $sN \subseteq T$ and so N is an S-co-r-submodule of M. □

Corollary 2.20. *Let $g : M \to M'$ be an epimorphism of R-modules, $S \subseteq R$ be a m.c.s of R and $Ker(g)$ be a co-r-submodule of M. Then if T is an S-co-r-submodule of M', then $g^{-1}(T)$ is an S-co-r-submodule of M.*

Proof. It follows from Proposition 2.19. □

3. S-co-n-submodules

Definition 3.1. Let $S \subseteq R$ be a m.c.s of R and let M be an R-module. Then a non-zero submodule N of M with $Rad(Ann(M)) \cap S = \emptyset$ is called an S-co-n-submodule, if there exists $s \in S$ such that whenever $aN \subseteq K$ and $sa \notin Rad(Ann(M))$ for some $a \in R$ and a submodule K of M, then $sN \subseteq K$.

Proposition 3.2. *Let M be an R-module and S be a m.c.s of R and N be a submodules of M with $S \cap Rad(Ann(N)) = \emptyset$. Then N is an S-secondary submodule of M if and only if there exists $s \in S$ and whenever $aN \subseteq K$, where $a \in R$ and K is a submodule of M, implies that either $(sa)^n N = 0$ for some $n \in \mathbb{N}$ or $sN \subseteq K$.*

Proof. By Proposition 3.2, the proof is hold. □

By Proposition 3.2, we have if N is an S-co-n-submodule, then N is an S-secondary submodule, because $Rad(Ann(M)) \subseteq Rad(Ann(N))$. While, the following example shows that the concepts of S-second submodules and S-co-n-submodules are different in general.

Example 3.3. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{12}$, the multiplicatively closed subset $S = \{5^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then

(i) The submodule $N = \langle \bar{3} \rangle$ is an S-co-n-submodule of $\mathbb{Z}_{12}$. While N is not an S-second submodule of $\mathbb{Z}_{12}$. Because, $3N \subseteq \langle \bar{9} \rangle$, but for any $s \in S$, $(s \times 3)N \neq 0$ and $sN \nsubseteq \langle \bar{9} \rangle$.

(ii) The submodule $N = \langle \bar{4} \rangle$ is an S-second submodule of $\mathbb{Z}_{12}$. While $N = \langle \bar{4} \rangle$ is not an S-co-n-submodule of $\mathbb{Z}_{12}$. Because, $3N = 0$, but for any $s \in S$, $sN \neq 0$ and $(s \times 3) \notin Rad(Ann(\mathbb{Z}_{12}))$.

By Example 3.3(ii), since every S-second submodule is an S-secondary submodule, we conclude that every S-secondary submodule need not be an S-co-n-submodule.

Let R be a commutative ring, $S \subseteq R$ be a m.c.s of R and I be an ideal of R disjoint with S. We say that ideal I is an S-n-ideal of R if there exists $s \in S$ such that for all $a, b \in R$, $ab \in I$ and $sa \notin Rad(0)$, then $sb \in I$ [11], Definition 1.

Proposition 3.4. *Let N be a submodule of an R-module M. Then*

(i) *If N is an S-co-n-submodule of M such that $Rad(0) = Rad(Ann(M))$, then $Ann(N)$ is an S-n-ideal of R.*

(ii) *If M is an S-comultiplication R-module and $Ann(N)$ is an S-n-ideal of R, then N is an S-co-n-submodule of M.*

Proof. (i) Assume that N is an S-co-n-submodule of M with S-element s and let $ab \in Ann(N)$ and $sa \notin \mathfrak{n}(R)$. We show that $sb \in Ann(N)$. Since $sa \notin \mathfrak{n}(R)$, we conclude $sa \notin Rad(0)$ and so $sa \notin Rad(Ann(M))$ by assumption. As $aN \subseteq aN$ and $sa \notin Rad(Ann(M))$, we have

$sN \subseteq aN$ because N is an S-co-n-submodule of M. Thus $sbN \subseteq abN = 0$, hence $sbN = 0$ and $sb \in Ann(N)$.

(ii) Let $Ann(N)$ be an S-n-ideal of R with S-element $s \in S$. Assume that $aN \subseteq K$ with $sa \notin Rad(Ann(M))$ for some $a \in R$ and a submodule K of M. Thus $aNAnn(K) \subseteq KAnn(K) = 0$, so $aNAnn(K) = 0$. Hence $aAnn(K) \subseteq Ann(N)$, since $sa \notin \mathfrak{n}(R)$ and $Ann(N)$ is an S-n-ideal of R, we conclude $sAnn(K) \subseteq Ann(N)$. Therefore $(0 :_M Ann(N)) \subseteq (0 :_R sAnn(K))$. Since M is an S-comultiplication R-module, $sN \subseteq (0 :_M Ann(N)) \subseteq (0 :_R sAnn(K)) \subseteq sK \subseteq K$ and so $sN \subseteq K$, as needed. □

Theorem 3.5. *Let $\varphi : M_1 \to M_2$ be a monomorphism of R-modules and $S \subseteq R$ be a m.c.s of R. Then the following assertions hold:*

(i) *If N_1 is an S-co-n-submodule of M_1, then $\varphi(N_1)$ is an S-co-n-submodule of $\varphi(M_1)$.*

(ii) *If N_2 is an S-co-n-submodule of M_2 and $N_2 \subseteq \varphi(M_1)$, then $\varphi^{-1}(N_2)$ is an S-co-n-submodule of M_1.*

Proof. (i) Since φ is monomorphism and $N_1 \neq 0$, we have $\varphi(N_1) \neq 0$. Let $a \in R$ and K_2 be a submodule of M_2 such that $a\varphi(N_1) \subseteq K_2$. Then $aN_1 \subseteq \varphi^{-1}(K_2)$. Since N_1 is an S-co-n-submodule of M_1, there exists $s \in S$ such that $sN_1 \subseteq \varphi^{-1}(K_2)$ or $sa \in Rad(Ann(M))$. Thus $s\varphi(N_1) \subseteq \varphi(\varphi^{-1}(K_2)) = \varphi(M) \cap K_2 \subseteq K_2$ or $sa \in Rad(Ann(\varphi(M)))$ since φ is a monomorphism, so $Rad(Ann(M_1)) = Rad(Ann(\varphi(M_1))))$. Therefore, $\varphi(N_1)$ is an S-co-n-submodule of M_2.

(ii) It is clear that $\varphi^{-1}(N_2) \neq 0$. Let $a \in R$ and K_1 be a submodule of M_1 such that $a\varphi^{-1}(N_2) \subseteq K_1$. Then $aN_2 = a(\varphi(M_1) \cap N_2) = a\varphi(\varphi^{-1}(N_2)) \subseteq \varphi(K_1)$. Since N_2 is an S-co-n-submodule of M_2, there exists $s \in S$ such that $sN_2 \subseteq \varphi(K_1)$ or $sa \in Rad(Ann(M_2))$. Thus $\varphi^{-1}(sN_2) \subseteq \varphi^{-1}(\varphi(K_1)) = K_1$, so $s\varphi^{-1}(N_2) \subseteq K_1$ or $sa \in Rad(Ann(M_1))$, as required.

□

By the previous theorem, we have the following corollary.

Corollary 3.6. *Let M be an R-module and $N \subseteq K$ be submodules of M. Then N is an S-co-n-submodule of K if and only if N is an S-co-n-submodule of M.*

Proposition 3.7. *Let M be an R-module. Then we have the following statements:*

(i) *If M is an S-secondary module, then M is an S-n-submodule of M.*

(ii) *The sum of an arbitrary non-empty set $\{N_i\}_{i=1}^{n}$ of S-co-n-submodules is an S-co-n-submodule of M.*

Proof. (i) The proof is obvious.

(ii) Let $a\sum_{i=1}^{n} N_i \subseteq K$ for some $a \in R$ and a submodule K of M. Then $aN_i \subseteq K$ for every i $(1 \leq i \leq n)$. Thus for each i, there exists $s_i \in S$ such that $s_i a \in Rad(Ann(M))$ or $s_i N_i \subseteq K$. Set $s = s_1 s_2 \ldots s_n$. Therefore, $sa \in Rad(Ann(M))$ or $sN \subseteq K$, as needed. □

Theorem 3.8. *Let M be an R-module and $S \subseteq R$ be a m.c.s of R. Every proper submodule of M is an S-n-submodule of M if and only if every non-zero submodule of M is an S-co-n-submodule of M.*

Proof. $(\Rightarrow)$ Let N be a non-zero submodule of M and $aN \subseteq K$ for some $a \in R$ and a submodule K of M. Thus if $K = M$, then we are done. Let K be a proper submodule of M. Then K is an S-n-submodule of M. Assume that $m \in N$. Hence $am \in K$, so there exists $s \in S$ such that $sm \in K$ or $sa \in Rad(Ann(M))$. Therefore, $sN \subseteq K$ or $sa \in Rad(Ann(M))$, as required.

$(\Leftarrow)$ Let N be a proper submodule of M and $am \in N$ for some $a \in R$ and $m \in M$. If $m = 0$, we are done. If $m \neq 0$, then $Rm \neq 0$ is an S-co-n-submodule. As $aRm \subseteq N$, there exists $s \in S$ such that $sa \in Rad(Ann(M))$ or $sRm \subseteq N$ $(sm \in N)$. Thus N is an S-n-submodule of M. □

The following theorem states the corresponding between the S-co-n-submodules and the S-co-r-submodules.

Theorem 3.9. *Let M be an R-module and $S \subseteq R$ be a m.c.s of R. Then if N is an S-co-n-submodule of M, then N is an S-co-r-submodule of M.*

Proof. Assume that N is an S-co-n-submodule of M with S-element $s \in S$. Let $aN \subseteq K$ with $saM = M$. If $sa \in Rad(Ann(M))$, then there exists $n \in \mathbb{N}$ such that $(sa)^n M = 0$ and $(sa)^{n-1}M \neq 0$. Now, $saM = M$, then $0 = (sa)^n M = (sa)^{n-1}M$, which is a contradiction. Therefore, $sa \notin Rad(Ann(M))$. As N is an S-co-n-submodule, $sN \subseteq K$, as required. □

The following example shows that the converse of above theorem does not hold in general.

Example 3.10. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{12}$ and the multiplicatively closed subset $S = \{7^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then the submodule $N = \langle\bar{3}\rangle$ is an S-co-r-submodule, but it is not an S-co-n-submodule of $\mathbb{Z}_1 2$. Because $2N \subseteq \langle\bar{6}\rangle$, but for any $s \in S$, neither $s \times 2 \in Rad(Ann(\mathbb{Z}_{12}))$ nor $sN \subseteq \langle\bar{3}\rangle$.

4. Conclusions

In this paper, we study a new class of the dual notions of r-submodules and n-submodules. In fact, we introduced the concepts of S-co-r-submodules and S-co-n-submodules of a module over a commutative ring. Several properties, examples and characterizations of such submodules, especially in S-multiplication modules and S-comultiplication modules have been investigated. Moreover, we explored the behaviour of these submodules under module homomorphisms, quotient modules, Cartesian product. Finally, we stated the relation between two these concepts.

5. Acknowledgement

The authors would like to thank the anonymous referee for their careful reading and comments, which helped to significantly improve the paper.

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