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Research Paper

ON PRIME IDEALS ON A SEMI-RING ASSOCIATED WITH A NEXUS

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Abstract. In this study, we explore prime ideals and prime elements within a semi-ring constructed over a nexus. We characterize these elements using panels and quasi-panels. Furthermore, we establish conditions under which a semi-ring associated with a nexus N becomes unitary. The concept of homomorphism for these semi-rings is introduced, and several of their properties are examined. Additionally, by analyzing their characteristics, we demonstrate that a quotient semi-ring can be induced by an ideal of a semi-ring over a nexus, and localization is successfully defined. To illustrate these concepts, we provide specific examples.

1. Introduction

In 1980, Haristchain [14] introduced a sophisticated database structure called a *plenix* to efficiently manage the diverse data defining spatial structures (see also [5, 16]). The concepts of *formex* (plural: formices) and plenix (plural: plenices) trace their origins to the 1970s, when

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H. Nooshin led an extensive research program at the Space Structures Research Center of the University of Surrey. This program culminated in the development of *formex algebra* [7, 11, 19, 20], which laid the groundwork for further algebraic explorations. Many classical algebraic concepts have since been deeply studied in the context of *nexus algebra* [1, 6, 8, 12, 13, 21].

In 1984, Nooshin [19] formally defined a *nexus* as a mathematical object representing the structure of a plenix, using the concept of an *address set* for its axiomatic construction. Later, in 2009, Bolourian [5] introduced *nexus algebras* as an abstract algebraic structure and investigated their properties. This was followed by Afkhami et al. [1] in 2012, who explored *soft nexuses*. Over the years, numerous authors have contributed to the study of nexuses and subnexuses. For instance, Norouzi [22] investigated subnexuses in the context of N-structures in 2018, and Norouzi et al. [21] extended this work in 2020. In 2019, Bolourian et al. [6] constructed a *moduloid* over a nexus, and in 2020, Kamrani et al. [12, 13] further developed this by introducing *submoduloids*, *finitely generated submoduloids*, and *prime submoduloids* on a nexus.

In 2024, the authors of this paper constructed a semi-ring over a nexus and explored its fuzzifications ([18]). For further preliminary and applications of semi-rings and rings, see [3, 4, 9, 10, 15, 17, 23, 24]. In this work, we begin by presenting preliminary results on semi-rings constructed over a nexus. We then establish conditions under which such a semi-ring is unitary or a principal ideal domain (PID). For a semi-ring N, we define $st(N)$ and investigate prime ideals and prime elements, characterizing them using panels and quasi-panels. Additionally, we introduce the notion of homomorphism for these semi-rings and examine their properties. Finally, we demonstrate that a quotient semi-ring can be induced by an ideal of a semi-ring over a nexus, and we establish the concept of localization.

2. Preliminaries

Now, we review the basic definitions and some elementary aspects that are necessary for this paper.

An address is a sequence of $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ such that $a_k = 0$ implies that $a_i = 0$, for all $i \geqslant k$. The sequence of zero is called the empty address and denoted by $()$. In other words, every non-empty address is of the form $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$, where a_i and $n \in \mathbb{N}$, and it is denoted by $(a_1, a_2, \ldots, a_n)$.

Definition 2.1. ([1]) A set N of addresses is called a nexus if

(I) $(a_1, a_2, \ldots, a_{n-1}, a_n) \in N$ implies $(a_1, a_2, \ldots, a_{n-1}, t) \in N, \forall\ 0 \leqslant t \leqslant a_n$,

(II) $\{a_i\}_{i=1}^{\infty} \in N, a_i \in \mathbb{N}$ implies $\forall n \in \mathbb{N}, \forall\ 0 \leqslant t \leqslant a_n, (a_1, \ldots, a_n - t) \in N$.

Let $a \in N$. The level of a is said to be:

(I) n, if $a = (a_1, a_2, \ldots, a_n)$, for some $a_k \in \mathbb{N}$,

(II) ∞, if a is an infinite sequence of N,

(III) 0, if $a = ()$.

The level of a is denoted by $l(a)$ and $stem\, a = a_1$. We put $st(N) = sup\{i \in \mathbb{N} : (i) \in N\}$.

Let $a = \{a_i\}$ and $b = \{b_i\}$, $i \in \mathbb{N}$, be two addresses. Then $a \leqslant b$, if $l(a) = 0$ or if one of the following cases is satisfied:

(I) if $l(a) = 1$, that is $a = (a_1)$, for some $a_1 \in N$ and $a_1 \leqslant b_1$,

(II) if $1 < l(a) < \infty$, then $l(a) \leqslant l(b)$ and $a_{l(a)} \leqslant b_{l(a)}$ and for any $1 \leqslant i < l(a)$, $a_i = b_i$,

(III) if $l(a) = \infty$, then $a = b$.

A subset S of N is called a sub-nexus of N provided that S itself is a nexus. Let $\emptyset \neq A \subseteq N$. Then the smallest sub-nexus of N containing A is called the sub-nexus of N generated by A and is denoted by $\langle A \rangle$. If $A = \{a_1, a_2, \ldots, a_n\}$, then instead of $\langle A \rangle$ one can write $\langle a_1, a_2, \ldots, a_n \rangle$. If A has only one element a, then the sub-nexus $\langle a \rangle$, is called a cyclic sub-nexus of N. It is clear that $()$ and N are trivial sub-nexuses of the nexus N.

Definition 2.2. ([1]) Let N be a nexus and let $a = (a_1, a_2, \ldots, a_k)$ be an address of N. The set $\{(a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n) \in N : \ a_{k+i} \in \mathbb{N},\ \text{for } i = 1, 2, \ldots, n-k\}$ is called the panel of a and is denoted by q_a. In other words, if $a = (a_1, a_2, \ldots, a_k)$, then every address b of N is an address in q_a, provided that the first k terms of b are the same as the corresponding terms of a.

Notice that the panel of a does not include a. Also, $q_{()}$ include all the addresses of N except for the empty address itself. We denoted $M_a = q_a \bigcup \{a\}$.

Definition 2.3. ([1]) Let N be a nexus and a be an address of N. The set $\{b \in N : a \leqslant b\}$ is called the quasi panel of a and is denoted by Q_a. Let $A \subseteq N$.
Also, we define $Q_A = \{b \in N : \exists a \in A, a \leqslant b\} = \bigcup_{a \in A} Q_a$. Clearly, $A \subseteq Q_A$.

Example 2.4. ([6]) Consider a nexus:
$N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3), (2,2,1), (2,2,2), (2,3,1), (2,3,2)\}$

with the following diagram.

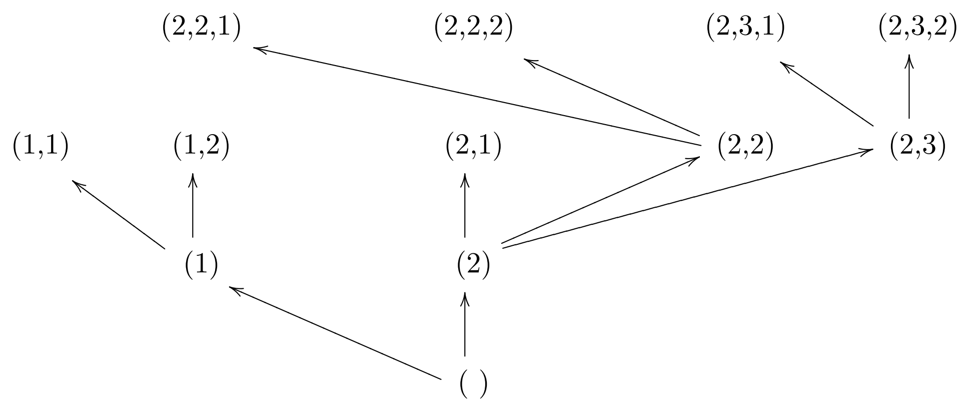

Fig. 1. Diagram of N.

Let $a = (2, 2)$ be an address of N and $A = \{(1, 1), (2, 2)\} \subseteq N$. Then

$q_a = \{(2, 2, 1), (2, 2, 2)\}$,

$Q_a = \{(2, 2), (2, 2, 1), (2, 2, 2), (2, 3), (2, 3, 1), (2, 3, 2)\}$,

$M_a = \{(2, 2), (2, 2, 1), (2, 2, 2)\}$,

$Q_A = \{(1, 1), (1, 2), (2, 2), (2, 2, 1), (2, 2, 2), (2, 3), (2, 3, 1), (2, 3, 2)\}$.

For $() \neq a, b \in N$, let $a = \{a_t\}_{t=1}^n$ and $b = \{b_t\}_{t=1}^m$. We define binary operations "+" and "·" on N as following:

(I) $() + a = a + () = a,$.

(II) $a + b = (a_1 \vee b_1)$,

(III) $a \cdot b = (a_1, \ldots, a_{i-1}, a_i \wedge b_i)$,

where $i\{a, b\} = \min\{t : a_t \neq b_t\}$ (briefly, $i\{a, b\} := i$). If there is no such that i, then $a = b$ and $i\{a, a\} = l(a)$.

Lemma 2.5. ([18]) *Let* $a = \{a_t\}_{t=1}^n$, $b = \{b_t\}_{t=1}^n$, $c = \{c_t\}_{t=1}^n \in N$. *Then*

(I) *if* $i\{a, b\} = i\{a, c\} = r$, *then* $i\{b, c\} \geqslant r$,

(II) *if* $i\{a, b\} = r$, $i\{a, c\} = s$ *and* $r \neq s$, *then* $i\{b, c\} = \min\{r, s\}$.

Theorem 2.6. ([18]) *The algebra* $(N, +, \cdot, ())$ *is a semi-ring.*

Example 2.7. ([18]) Let $N = \{(), (1), (2), (1, 1), (2, 1)\}$. By defined the binary operations "+" and "·" on N, we have: for every $a \in N$

$a + () = () + a = a$, $(1) + (1) = (1) + (1, 1) = (1, 1) + (1, 1) = (1)$, $(2) + a = (2, 1) + a = (2)$,

$a \cdot () = () \cdot a = ()$, $a \cdot (1) = (1) \cdot a = (1)$, $(2) \cdot (2) = (2) \cdot (2, 1) = (2)$, $(1, 1) \cdot (1, 1) = (1, 1)$, $(2) \cdot (1, 1) = (1, 1) \cdot (2, 1) = (1)$, $(2, 1) \cdot (2, 1) = (2, 1)$.

Then $(N, +, \cdot, ())$ is a semi-ring.

Remark 2.8. ([18]) Notice that the semi-ring $(N, +, \cdot, ())$ can not be a ring. Since, if for every $0 \neq a \in N$, there exists $() \neq b \in N$, such that $a + b = b + a = ()$, then $a = b = ()$, which is a contradiction.

In the sequel, for briefly, we denote the semi-ring $(N, +, \cdot, ())$ related to a nexus N only by N. Let $a, b \in N$, we say that $a|b$, if there exist $c \in R$ such that $a \cdot c = b$. An element $p \in N$ is prime if for every $a, b \in N$, $p|a \cdot b$ empies $p|a$ or $p|b$.

Let $() \in I \subseteq N$. We say that I is an ideal of N, if it satisfies the following conditions:

(I) for every $a, b \in I$, $a + b \in I$,

(II) for every $a \in I$ and every $b \in N$, $a \cdot b \in I$.

An ideal I of N is prime if for every $a, b \in I$, $a \cdot b$ implies $a \in I$ or $b \in I$. Let I be an ideal of N and $a \in N$. Consider the set $a + I = \{a + b : b \in I\}$ and define $\frac{N}{I} = \{a + I : a \in N\}$.

Lemma 2.9. ([18]) *Let I be an ideal of N and $a, a' \in N$. Then*

(I) *$a + I = I$ if and only if $a = ()$.*

(II) *$a + I = a' + I$ if and only if $stem\, a = stem\, a'$.*

Example 2.10. ([18]) Consider a nexus:
$N = \{(), (1), (2), (3), (4), (1, 1), (1, 2), (1, 1, 1), (1, 1, 2), (1, 1, 3), (2), (2, 1), (2, 1, 1), (2, 1, 2),$
$(3, 1), (3, 2), (3, 3), (4, 1)\}$
with the following diagram.

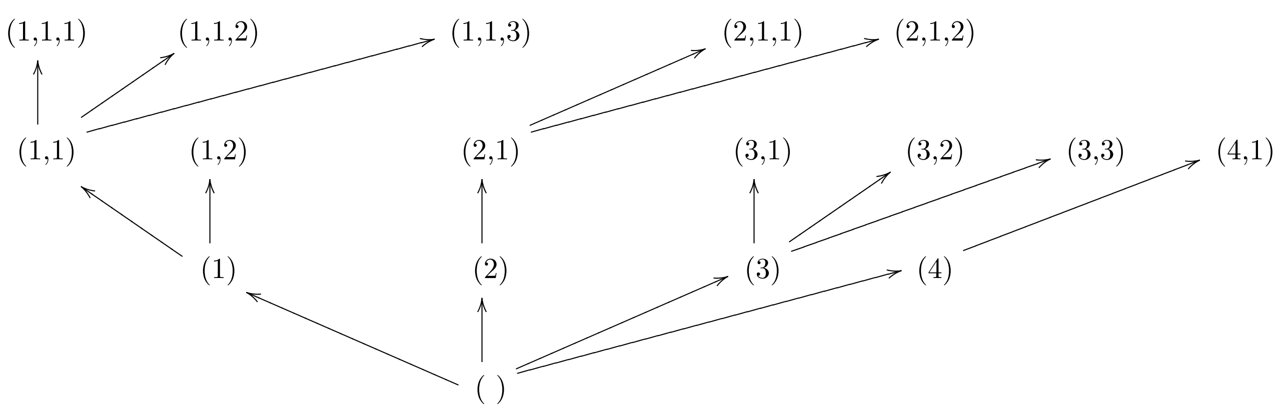

Fig. 2. Diagram of N.

If $I := \{(), (1), (2), (1, 1), (1, 2), (2), (2, 1)\}$, then I is an ideal of N. Using Lemma 2.9, we can see that

$() + I = (1, 1) + I = (1, 2) + I = (1, 1, 1) + I = (1, 1, 2) + I = (1, 1, 3) + I,$

$(2) + I = (2, 1) + I = (2, 1, 1) + I = (2, 1, 2) + I,$

$(3) + I = (3, 1) + I = (3, 2) + I = (3, 3) + I,$

$(4) + I = (4, 1) + I$.
Then $\dfrac{N}{I} = \{I, (1) + I, (2) + I, (3) + I, (4) + I\}$.

Definition 2.11. ([18]) For every $a + I$, $b + I \in \dfrac{N}{I}$, we define the binary operations "$*$" and "$\circ$" on $\dfrac{N}{I}$ with the following:

(I) $(a + I) * (b + I) = a + b + I$,

(II) $(a + I) \circ (b + I) = a \cdot b + I$.

Theorem 2.12. ([18]) *Let I be an ideal of N. Then $(\dfrac{N}{I}, *, \circ, I)$ is a semi-ring.*

3. Prime elements of a semi-ring $(N, +, \cdot, ())$

In this section, at first we verify when a semi-ring N is a unitary, then we characterize all prime elements of N.

Proposition 3.1. *Let N be a semi-ring. N is unitary if and only if N is a cyclic nexus. In this case $N = \langle 1 \rangle$.*

Proof. It is easy to see that for every $a, b \in N$, $a \cdot b = a$, implies $a \leqslant b$. At first, we assume that N be unitary. Hence $a \cdot 1 = 1 \cdot a = a$ and so $a \leqslant 1$. Since N is a nexus and $1 \in N$, for every $a \leqslant 1$, $a \in N$. Therefore, $N = \langle 1 \rangle$.

Conversely, let $N = \langle b \rangle$ be a cyclic nexus. Hence for every $a \in N$, $a \leqslant b$ and so $a \cdot b = b \cdot a = a$. Therefore, N is unitary and $b = 1$. □

Proposition 3.2. *Let $N = \langle 1 \rangle$ be a unitary semi-ring. Then*

(I) *N is a principal ideal domain (briefly, PID),*

(II) *every ideal of N is a prime ideal,*

(III) *every element of N is a prime element.*

Proof. (I) Let I be an ideal of N. Since N is a cyclic nexus, for every $a, b \in N$, we have $a \leqslant b$ or $b \leqslant a$. Hence there exist $c \in I$, such that for every $a \in I$, $a \leqslant c$ and so $I = \langle c \rangle$ is a principal ideal. Therefore, N is a PID.

(II) Let I be an ideal of N and $a, b \in N$ such that $a \cdot b \in I$. Since $a \leqslant b$ or $b \leqslant a$, we have $a \cdot b = a$ or $a \cdot b = b$. Hence $a \in I$ or $b \in I$. Therefore, I is a prime ideal of N.

(III) Let p be an element of N and $a, b \in N$ such that $p | a \cdot b$. Since $a \cdot b = a$ or $a \cdot b = b$, $p | a$ or $p | b$. Therefore, p is a prime element of N. □

Proposition 3.3. *Let N be a semi-ring and $a, b \in N$. Then $a | b$ if and only if $b \leqslant a$.*

Proof. Let $b \leqslant a$. Then $a \cdot b = b$ and so $a|b$.

Conversely, let $a|b$. Then there exist $c \in N$ such that $a \cdot c = b$. Let $i\{a, c\} = i$. Hence $b = (a_1, \ldots, a_{i-1}, a_i \wedge c_i)$ and so $b \leqslant a$. □

Note: Let N be a semi-ring and $p \in N$. Hereafter, by Proposition 3.3, p is prime if and only if for every $a, b \in N$, $a \cdot b \leqslant p$, implies $a \leqslant p$ or $b \leqslant p$.

Definition 3.4. Let N be a semi-ring and $p = (p_1, \ldots, p_n) \in N$. We put

$$A(p) = \{k : 1 \leqslant k \leqslant n \text{ such that there exist } a = (p_1, \ldots, p_{k-1}, a_k) \in N, \text{ with } a_k > p_k\}.$$

Also, we put

$$B(p) = \{a \in N : \text{ there exist } 1 \leqslant k \leqslant n, a = (p_1, \ldots, p_{k-1}, a_k) \in N, \text{ with } a_k < p_k \text{ and } q_a \neq \emptyset\}.$$

Lemma 3.5. *Let N be a semi-ring and $p = (p_1, \ldots, p_n) \in N$ and p be a prime element of N. Then*

(I) *if $A(p) \neq \emptyset$, then there exist a unique $k(1 \leqslant k \leqslant n)$ such that $A(p) = \{k\}$,*
(II) *if $B(p) \neq \emptyset$, then there exist a unique $a \in N$ such that $B(p) = \{a\}$.*

Proof. (I) Let $k, r \in A(p)$, $k \neq r$ and $k < r$. Then there exist $a = (p_1, \ldots, p_{k-1}, a_k) \in N$ such that $a_k > p_k$ and $b = (p_1, \ldots, p_k, \ldots, p_{r-1}, b_r) \in N$ such that $b_r > p_r$. Then $a \cdot b = (p_1, \ldots, p_k) \leqslant p$, but $a \nleqslant p$ and $b \nleqslant p$, which is a contradiction. Hence there exist a unique k such that $A(p) = \{k\}$.

(II) Suppose that $a, b \in B(p)$, $a \neq b$ and $a = (p_1, \ldots, p_{k-1}, a_k), k \leqslant n, a_k < p_k, q_a \neq \emptyset$ and $b = (p_1, \ldots, p_k, \ldots, p_{r-1}, b_r), k \leqslant r \leqslant n, b_r < p_r, q_b \neq \emptyset$. Let $\alpha = (p_1, \ldots, p_{k-1}, a_k, x) \in q_a$ and $\beta = (p_1, \ldots, p_k, \ldots, p_{r-1}, b_r, y) \in q_b$. If $k = r$, we have $\alpha \cdot \beta = (p_1, \ldots, p_{k-1}, a_k \wedge b_k) \leqslant p$ and if $k < r$, we have $\alpha \cdot \beta = (p_1, \ldots, p_{k-1}, a_k) \leqslant p$, but $\alpha \nleqslant p$ and $\beta \nleqslant p$, which is a contradiction. Hence there exist a unique $a \in N$ such that $B(p) = \{a\}$. □

Theorem 3.6. *Let N be a semi-ring and $p = (p_1, \ldots, p_n) \in N$. Then p is prime element of N if and only if one of the fallowing conditions is true:*

(I) *if $q_p \neq \emptyset$, then $N = \langle p \rangle \cup q_p$,*
(II) *if $q_p = \emptyset$ and $A(p) = \{k\}$, then $N = \langle p \rangle \bigcup (\bigcup_{a_k > p_k} M_{(p_1, \ldots, p_{k-1}, a_k)})$,*
(III) *if $q_p = \emptyset$, $A(p) = \emptyset$ and $B(p) = \{a\}$, then $N = \langle p \rangle \cup q_a$,*
(IV) *if $q_p = \emptyset$, $A(p) = \emptyset$ and $B(p) = \emptyset$, then $N = \langle p \rangle$.*

Proof. (I) Assume p is a prime ideal and $a = (p_1, \ldots, p_n, x) \in q_p$. Let $b = (b_1, \ldots, b_n) \in N$. If $b_1 > p_1$, then $a \cdot b = (p_1) \leqslant p$, but $a \nleqslant p$ and $b \nleqslant p$, which is a contradiction. Hence $b_1 \leqslant p_1$. If $b_1 < p_1$ and $b_2 \neq 0$, then $a \cdot b = (b_1) \leqslant p$, but $a \nleqslant p$ and $b \nleqslant p$, which is a contradiction. Hence for every $b_1 < p_1$, we have $q_{(b_1)} = \emptyset$. Let $b = (b_1, \ldots, b_n) \in q_{(p_1)}$. If $b_2 > p_2$, then $a \cdot b = (p_1, p_2) \leqslant p$, but $a \nleqslant p$ and $b \nleqslant p$, which is a contradiction. Hence $b_2 \leqslant p_2$. If we improve this method, we have:

For every $b = (b_1, \ldots, b_n) \in N$, $b_1 \leqslant p_1$ and for every $1 \leqslant b_1 < p_1$, we have $q_{(b_1)} = \emptyset$.

For every $b = (b_1, \ldots, b_n) \in q_{(p_1)}$, $b_2 \leqslant p_2$ and for every $1 \leqslant b_2 < p_2$, we have $q_{(p_1,b_2)} = \emptyset$.

For every $b = (b_1, \ldots, b_n) \in q_{(p_1,p_2)}$, $b_3 \leqslant p_3$ and for every $1 \leqslant b_3 < p_3$, we have $q_{(p_1,p_2,b_3)} = \emptyset$.

$\vdots$

For every $b = (b_1, \ldots, b_n) \in q_{(p_1,\ldots,p_{n-1})}$, $b_n \leqslant p_n$ and for every $1 \leqslant b_n < p_n$, we have $q_{(p_1,\ldots,p_{n-1},b_n)} = \emptyset$.

Therefore, $N = \{\} \bigcup (\bigcup_{b_1=1}^{p_1} \{(b_1)\}) \bigcup (\bigcup_{b_2=1}^{p_2} \{(p_1, b_2)\}) \bigcup \cdots \bigcup (\bigcup_{b_n=1}^{p_n} \{(p_1 \ldots, p_{n-1}, b_n)\}) \bigcup q_p$. Hence $N = \langle p \rangle \bigcup q_p$.

Conversely, let $N = \langle p \rangle \bigcup q_p$ and $a \cdot b \leqslant p$. If $a, b \in q_p$, then $a \cdot b \nleqslant p$, which is a contradiction. Hence $a \notin q_p$ or $b \notin q_p$ and so $a \in \langle p \rangle$ or $b \in \langle p \rangle$. Hence $a \leqslant p$ or $b \leqslant p$. Therefore, p is a prime element of N.

(II) Let p be prime and $a = (p_1, \ldots, p_{k-1}, a_k)$ such that $a_k > p_k$. Alike the proof of (I), we can abtain $N = \langle p \rangle \bigcup (\bigcup\limits_{a_k > p_k} M_{(p_1,\ldots,p_{k-1},a_k)})$.

Conversely, let $a \cdot b \leqslant p$. If $a, b \in \bigcup\limits_{a_k > p_k} M_{(p_1,\ldots,p_{k-1},a_k)}$, then $a \cdot b \nleqslant p$, which is a contradiction. Hence $a \in \langle p \rangle$ or $b \in \langle p \rangle$ and so $a \leqslant p$ or $b \leqslant p$. Therefore, p is a prime element of N.

(III) Let p be prime and $a = (p_1, \ldots, p_{k-1}, a_k)$ such that $a_k < p_k$ be the unique element of $B(p)$. Alike the proof of (I) and by the proof of Lemma 3.5 (II), we can abtain $N = \langle p \rangle \cup q_a$.

Conversely, let $b \cdot c \leqslant p$. If $b, c \in q_a$, then $b \cdot c \nleqslant p$, which is a contradiction. Hence $b \in \langle p \rangle$ or $c \in \langle p \rangle$ and so $b \leqslant p$ or $c \leqslant p$. Therefore, p is a prime element of N.

(IV) It is clear. □

Example 3.7. Consider a nexus:

$N = \{(), (1), (1,1), (1,2), (1,2,1), (1,2,2), (1,2,3), (1,2,1,1), (1,2,1,2)\}$,

with the following digram.

(I) If $p = (1,2,1)$. Since $q_{(1,2,1)} = \{(1,2,1,1), (1,2,1,2)\}$ and $N \neq \langle (1,2,1) \rangle \bigcup q_{(1,2,1)}$, by Theorem 3.6 (I), $(1,2,1)$ is not a prime element of N.

(II) If $p = (1,2,1,2)$. We have $q_{(1,2,1,2)} = \emptyset$ and $A((1,2,1,2)) = \{3\}$. Now since we have $N = \langle (1,2,1,2) \rangle \bigcup M_{(1,2,2)} \bigcup M_{(1,2,3)}$, by Theorem 3.6 (II), $(1,2,1,2)$ is a prime element of N.

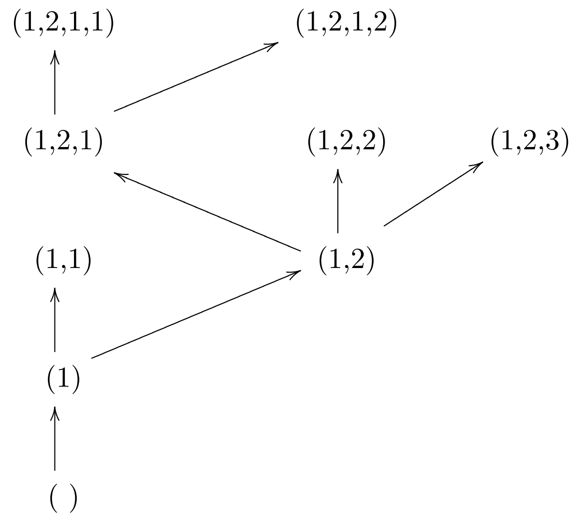

Fig. 3. Diagram of N.

(III) If $p = (1,2,3)$. We have $q_{(1,2,3)} = \emptyset$, $A((1,2,3)) = \emptyset$ and $B((1,2,3)) = \{(1,2,1)\}$. Now since $N = \langle(1,2,3)\rangle \bigcup q_{(1,2,1)}$, by Theorem 3.6 (III), $(1,2,3)$ is a prime element of N.

4. Prime ideals of a semi-ring $(N, +, \cdot, ())$

In this section we characterize all prime ideals of semi-ring $(N, +, \cdot, ())$ related to a nexus N.

Definition 4.1. An ideal I of N is prime if for every $a, b \in I$, $a \cdot b$ implies $a \in I$ or $b \in I$.

Proposition 4.2. *Let N be a semi-ring and $() \in I \subseteq N$. Then I is an ideal of N, if and only if it is a sub-nexus of N.*

Proof. Assume I is a sub-nexus of N, $a, b \in I$ and $c \in N$. Clearly, $a + b \in I$. If $a = c$, then $a \cdot c = a \in I$. Let $a \neq c$. Then $a \cdot c \leqslant a$. This shows that $a \cdot c \in I$.

Conversely, suppose I be an ideal of N, $a \in I$ and $b \leqslant a$. Then $a \cdot b = b \in I$. Therefore, I is a sub-nexus of N. □

Example 4.3. Consider a nexus:
$N = \{(), (1), (2), (3), (1,1), (1,2), (2,1), (2,2), (2,3), (2,3,1), (3,1)\}$,

with the following diagram.

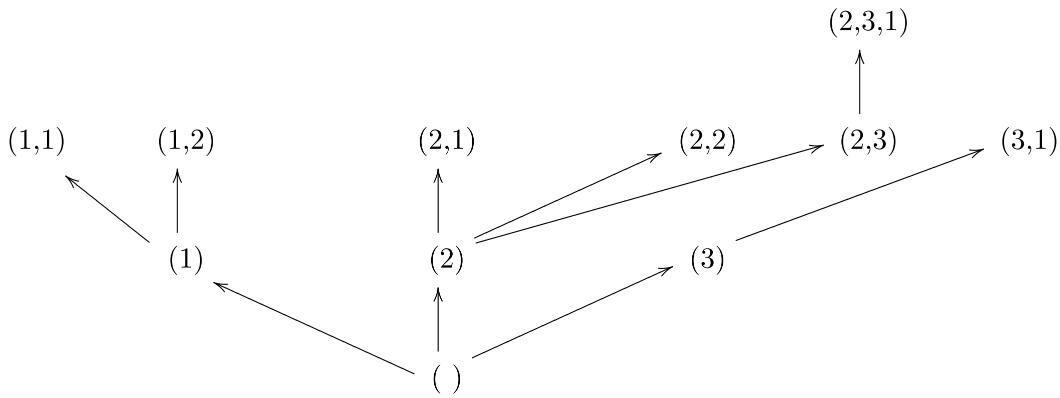

Fig. 4. Diagram of N.

If $I := \{(), (1), (2), (1,1), (1,2), (2,1), (2,2)\}$, then it is an ideal of semi-ring N. Also, if $J =: \{(), (1), (2), (3,1)\}$, then it is not an ideal of semi-ring N, since $(3) \cdot (3,1) = (3)$, but $(3) \notin J$.

Hereafter in this section, N is a finite semi-ring.

Remark 4.4. For every proper ideal I of N, we put $st(I) = \max\{l : (l) \in I\}$. If $st(I) = st(N) = t$, then there exist $i_1 \leqslant t$ and $a \in N \setminus I$ such that $stem\, a = i_1$.

Lemma 4.5. *If N is a semi-ring and P is a prime ideal of N such that $st(P) = st(N) = t$ and i_1 as in Remark 4.4, then i_1 is unique.*

Proof. If $s, s' \leqslant t$, $s \gneqq s'$ and $a \in M_{(s)} \setminus P$ and $a' \in M_{(s')} \setminus P$, then we have $a \cdot a' = (s) \in P$. Since P is a prime ideal, we get $a \in P$ or $a' \in P$, which is a contradiction. Therefore, i_1 is unique. □

Proposition 4.6. *Let P be a proper ideal of N, $() \in P$, $st(N) \neq st(P)$ and $st(P) = t$. Then P is a prime ideal of N if and only if $P = \bigcup_{s \leqslant t} M_{(s)} \bigcup \{()\}$.*

Proof. Assume P is a prime ideal of N, $s \leqslant t$ and $b \in M_{(s)}$. Since P is a proper ideal, there exist $t < l$ and $a \in N \setminus M_{(l)}$. Since $a \notin P$ and $b \cdot a = (s) \in P$, we get $b \in P$. Consequently, $M_{(s)} \subseteq P$. Clearly, for every $t < l$, $M_{(l)} \bigcap P = \emptyset$. Therefore, $P = \bigcup_{s \leqslant t} M_{(s)} \bigcup \{()\}$.

Conversely, let $P = \bigcup_{s \leqslant t} M_{(s)} \bigcup \{()\}$, $a, b \in N$ and $a \cdot b \in P$. If $stem\, a = a_1 > t$ and $stem\, b = b_1 > t$, then $stem\, a \cdot b > t$. Let $stem\, a \cdot b = r$. Thus, $(r) \in P$, which is a contradiction. If $a_1 \leqslant t$, then $a \in M_{(a_1)} \subseteq P$. Therefore, P is a prime ideal of N. □

Lemma 4.7. *Suppose P is a prime ideal of N, $st(P) = st(N) = t$, i_1 as in Remark 4.4 and $Q = \bigcup_{l \neq i_1} M_{(l)} \bigcup \{()\}$. Then $Q \subseteq P$.*

Proof. Let $l \neq i_1$ and $b \in M_{(l)}$. Since $P \neq N$, there exist $a \in N \setminus P$ with $stem\, a = i_1$. Hence $a \cdot b = (l \wedge i_1) \in P$, and so $b \in P$. Therefore, $Q \subseteq P$. $\square$

Proposition 4.8. *If P is a prime ideal of N and $st(P) = st(N) = t$ and i_1 as in Remark 4.4 and Q as in Lemma 4.7. Put $i_2 := \max\{r \in \mathbb{N}^* : (i_1, r) \in P \text{ and } (i_1, r+1) \notin P\}$. Then*

(I) if $i_2 = 0$, then $P = Q \bigcup \{(i_1)\}$,
(II) if $i_2 \neq 0$ and $(i_1, i_2 + 1) \in N$, then $P = Q \bigcup \{(i_1)\} \bigcup_{1 \leqslant r \leqslant i_2} M_{(i_1, r)}$.

Proof. (I) Since $i_2 = 0$, we have $q_{(i_1)} \bigcap P = \emptyset$. Hence by Lemma 4.7, we get $P = Q \bigcup \{(i_1)\}$.

(II) Let $1 \leqslant r \leqslant i_2$ and $b \in M_{(i_1,r)}$. Then $b(i_1, i_2 + 1) = (i_1, r) \in P$. This shows that $b \in P$. Therefore, $\bigcup_{1 \leqslant r \leqslant i_2} M_{(i_1,r)} \subseteq P$. Now, it is easy to see that $M_{(i_1, i_2+1)} \bigcap P = \emptyset$. Therefore, $P = Q \bigcup \{(i_1)\} \bigcup_{1 \leqslant r \leqslant i_2} M_{(i_1,r)}$. $\square$

Remark 4.9. By using the notation of Proposition 4.8, let $(i_1, i_2 + 1) \notin N$. Since $P \neq N$, there exist $1 \leqslant r \leqslant i_2$ such that $q_{(i_1,r)} \bigcap P \neq \emptyset$. We put

$$i_{r3} = \max\{s \in \mathbb{N}^* : (i_1, r, s) \in P \text{ and } (i_1, r, s + 1) \notin P\},$$

where $1 \leqslant r \leqslant i_2$. Let $A = \{1 \leqslant r \leqslant i_2 : i_{r3} = 0\}$ and $B = \{1 \leqslant r \leqslant i_2 : i_{r3} \neq 0\}$.

Proposition 4.10. *Let P be a prime ideal of N and $st(P) = st(N) = t$ and i_1, i_2, A, B as in Remark 4.9, and for every $r \in B$, $(i_1, r, i_{r3} + 1) \in N$. Then*
$P = Q \bigcup \{\bigcup_{r \in A} \langle (i_1, r) \rangle\} \bigcup \{\bigcup_{r \in B, 1 \leqslant s \leqslant i_{r3}} M_{(i_1,r,s)}\}$.

Proof. For every $r \in A$, since $i_{r3} = 0$, $q_{(i_1,r)} \bigcap P = \emptyset$. Now, let $r \in B$, $1 \leqslant s \leqslant i_{r3}$ and $b \in M_{(i_1,r,s)}$. Then $b(i_1, r, i_{r3} + 1) = (i_1, r, s) \in P$. So, $b \in P$. This complete the proof. $\square$

Remark 4.11. Suppose $r \in B$ and $(i_1, r, i_{r3} + 1) \notin N$. For every $1 \leqslant s \leqslant i_{r3}$, we put $i_{rs4} = \max\{m \in \mathbb{N}^* : (i_1, r, s, m) \in P, \text{ and } (i_1, r, s, m + 1) \notin P\}$. If we improve this method to end, since N is finite, we can find all prime ideals of N. See the following example:

Example 4.12. Consider a nexus:
$N = \{(), (1), (2), (3), (1, 1), (1, 1, 2), (1, 1, 3), (1, 2), (2, 1), (2, 2), (2, 3), (2, 3, 1), (3, 1)\}$,
with the following diagram.

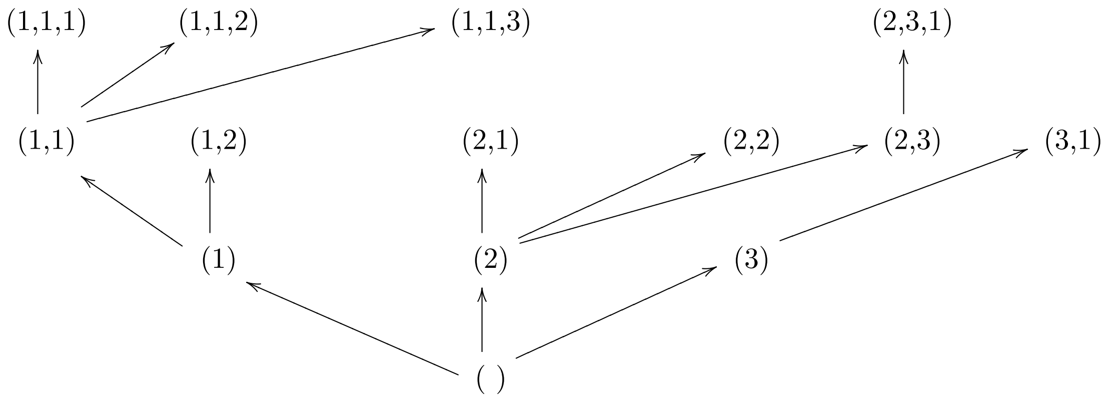

Fig. 5. Diagram of N.

In the following, we list all prime ideals of N.

If $st(P) = 0$, then $P_0 = \{0\}$.

If $st(P) = 1$, then $P_1 = \{0, (1), (1, 1), (1, 1, 2), (1, 1, 3), (1, 2)\}$.

If $st(P) = 2$, then $P_2 = \{0, (1), (2), (1, 1), (1, 1, 2), (1, 1, 3), (1, 2), (2, 1), (2, 2), (2, 3), (2, 3, 1)\}$.

If $st(P) = 3$, $i_1 = 1$ and $i_2 = 0$, then $P_3 = \{0, (1), (2), (3), (2, 1), (2, 2), (2, 3), (2, 3, 1), (3, 1)\}$.

If $st(P) = 3$, $i_1 = 1$ and $i_2 = 1$, then $P_4 = P_3 \bigcup \{(1, 1), (1, 1, 1), (1, 1, 2), (1, 1, 3)\}$.

If $st(P) = 3$, $i_1 = 1$, $i_2 = 2$ and $i_{13} = 0$, then $P_5 = P_3 \bigcup \{(1, 1), (1, 2)\}$.

If $st(P) = 3$, $i_1 = 1$, $i_2 = 2$ and $i_{13} = 1$, then $P_6 = P_3 \bigcup \{(1, 1), (1, 1, 1), (1, 2)\}$.

If $st(P) = 3$, $i_1 = 1$, $i_2 = 2$ and $i_{13} = 2$, then $P_7 = P_3 \bigcup \{(1, 1), (1, 1, 1), (1, 1, 2), (1, 2)\}$.

If $st(P) = 3$, $i_1 = 2$, $i_2 = 0$, then $P_8 = \{0, (1), (2), (3), (1, 1), (1, 1, 1), (1, 1, 3), (3, 1)\}$.

If $st(P) = 3$, $i_1 = 2$, $i_2 = 1$, then $P_9 = P_8 \bigcup \{(2, 1)\}$.

If $st(P) = 3$, $i_1 = 2$, $i_2 = 2$, then $P_{10} = P_8 \bigcup \{(2, 1), (2, 2)\}$.

If $st(P) = 3$, $i_1 = 2$, $i_2 = 3$ and $i_{33} = 0$, then $P_{11} = P_8 \bigcup \{(2, 1), (2, 2), (2, 3)\}$.

If $st(P) = 3$, $i_1 = 3$, $i_2 = 0$, then

$P_{12} = \{0, (1), (2), (3), (1, 1), (1, 1, 2), (1, 1, 3), (1, 2), (2, 1), (2, 2), (2, 3), (2, 3, 1)\}$.

5. Localization of a semi-ring $(N, +, \cdot, ())$

As usual we can define a semi-ring homomorphism on two given semi-rings N and M, the function $f : (N, +_1, \cdot_1, ()) \longrightarrow (M, +_2, \cdot_2, ())$ is called a semi-ring homomorphism ([18]) if it satisfies the following conditions: for all $a = \{a_t\}_{t=1}^n$, $b = \{b_t\}_{t=1}^n \in N$

(I) $f(()) = ()$,

(II) $f(a +_1 b) = f(a) +_2 f(b)$,

(III) $f(a \cdot_1 b) = f(a) \cdot_2 f(b)$.

Proposition 5.1. *Let* $(N, +_1, \cdot_1, ())$ *and* $(M, +_2, \cdot_2, ())$ *be two semi-rings related two nexuses* N *and* M*, and* $f : (N, +_1, \cdot_1, ()) \longrightarrow (M, +_2, \cdot_2, ())$ *be a semi-ring homomorphism,* $a, b \in N$ *and* $c \in M$*. Then*

(I) f *is a nexus homomorphism,*

(II) $Imf \setminus \{0\} \subseteq Q_{f(1)}$,

(III) $f(stem\, a) = stem\, f(a)$,

(IV) *if* $stem\, a \geqslant stem\, b$, *then* $stem\, f(a) \geqslant stem\, f(b)$,

(V) $f(M_a) \subseteq M_{f((stem\, a))}$,

(VI) $f(Q_a) \subseteq Q_{f(a)}$,

(VII) $f^{-1}(Q_c) = \bigcup\limits_{c \leqslant c'} Q_{f^{-1}(c')}$.

Proof. (I) Assume $a, b \in N$ and $a \leqslant b$. Hence $a \cdot_1 b = a$. So, we have $f(a) = f(a \cdot_1 b) = f(a) \cdot_2 f(b)$. This shows that $f(a) \leqslant f(b)$. Therefore, f is a nexus homomorphism.

(II) Suppose $() \neq y \in Imf$. Then there exist $() \neq x \in N$ such that $f(x) = y$. Since $(1) \leqslant x$, using (I) we get $f((1)) \leqslant f(x) = y$. Thus $y \in Q_{f((1))}$.

(III) Let $f(a) = c$, $stem\, a = a_1$ and $stem\, c = c_1$. Then

$$f((a_1)) = f(a +_1 a) = f(a) +_2 f(a) = c +_2 c = (c_1).$$

(IV) Assume $f(a) = c$, $f(b) = d$, $stem\, a = a_1$, $stem\, b = b_1$, $stem\, c = c_1$ and $stem\, d = d_1$. By (III), $f((a_1)) = (c_1)$ and $f((b_1)) = (d_1)$. Then

$$(c_1 \vee d_1) = c +_2 d = f(a) +_2 f(b) = f(a +_1 b) = f((a_1 \vee b_1)) = f((a_1)) = (c_1).$$

Thus $c_1 \geqslant d_1$.

(V) Let $f(a) = c$, $stem\, a = a_1$ and $stem\, c = c_1$. Using (III) we get $f((a_1)) = c_1$. Assume $b \in M_a$. Then $stem\, b = b_1 = a_1$, snd so $stem\, f(b) = f(stem\, b) = f(b_1) = c_1$. This shows that $f(b) \in M_{(c_1)} = M_{f(a_1)}$.

(VI) Let $b \in Q_a$. Then $a \leqslant b$, and so $a \cdot_1 b = a$. If $f(a) = c$ and $f(b) = d$, we have $c = f(a) = f(a \cdot_1 b) = f(a) \cdot_2 f(b) = c \cdot_2 d$. Thus $f(a) = c \leqslant d = f(b)$. Therefore, $f(b) \in Q_{f(a)}$.

(VII) Let $x \in \bigcup\limits_{c \leqslant c'} Q_{f^{-1}(c')}$. Then there is $c' \in M$ such that $c \leqslant c'$ and $x \in Q_{f^{-1}(c')}$. Hence there exists $y \in f^{-1}(c')$ such that $y \leqslant x$. Now, by (I), $c \leqslant c' = f(y) \leqslant f(x)$, and so $f(x) \in Q_c$. Therefore, $x \in f^{-1}(Q_c)$.

Conversely, let $x \in f^{-1}(Q_c)$. Hence $f(x) \in Q_c$. Put $f(x) = c'$. Then we have $c \leqslant c' = f(x)$. Therefore, $x \in f^{-1}(c') \subseteq Q_{f^{-1}(c')}$ and the proof is complete. □

Proposition 5.2. *Let* $(N, +_1, \cdot_1, ())$ *and* $(M, +_2, \cdot_2, ())$ *be two semi-rings related two nexuses* N *and* M*, and* $f : (N, +_1, \cdot_1, ()) \longrightarrow (M, +_2, \cdot_2, ())$ *be a map,* $a \in N$ *and* $c \in M$ *such that*

$f(()) = ()$ and for every $a \in N$, $f(a) = c$. Then f is a semi-ring homomorphism if and only if $l(c) = 1$.

Proof. Let f be a semi-ring homomorphism and $a \in N$ with $l(a) = 1$. Since $f(a) = c$, by Proposition 5.1 (III), $l(c) = 1$.

Conversely, let $l(c) = 1$, then $c +_2 c = c$ and $c \cdot_2 c = c$. Hence for every $a, b \in N$, we have $f(a +_1 b) = c = c +_2 c = f(a) +_2 f(b)$ and $f(a \cdot_1 b) = c = c \cdot_2 c = f(a) \cdot_2 f(b)$. It follows that f is a semi-ring homomorphism. □

Remark 5.3. Let N and M be two semi-rings and $f : (N, +_1, \cdot_1, ()) \longrightarrow (M, +_2, \cdot_2, ())$ be a semi-ring homomorphism. We say that f satisfy in property (*) if for every $a, a' \in N$, $f(a) = f(a')$ if and only if, $stem\, a = stem\, a'$.

Example 5.4. Consider Example 2.10 and put $M := \{(), (1), (2), (3), (4)\}$. Define $f : N \longrightarrow M$ be by $f(a) = (a_1)$, for $a = (a_1, \ldots, a_n) \in N$. Let $a = (a_1, \ldots, a_n)$, $a' = (a'_1, \ldots, a'_n) \in N$. Since $stem\,(a + a') = a_1 \vee a'_1$ and $stem\,(a \cdot a') = a_1 \wedge a'_1$, we have $f(a + a') = (a_1 \vee a'_1) = (a_1) + (a'_1) = f(a) + f(a')$ and $f(a \cdot a') = (a_1 \wedge a'_1) = (a_1) \cdot (a'_1) = f(a) \cdot f(a')$. Hence f is a semi-ring homomorphism. Now, we have $f(a) = f(a')$ if and only if $(a_1) = (a'_1)$ if and only if $stem\, a = stem\, a'$. Therefore, f satisfy in property (*).

Definition 5.5. Let I be an ideal of N. We define $stem\,(I) = \{()\} \bigcup \{(i) : (i) \in I\}$.

Example 5.6. Consider Example 2.10 and let
$I = \{(), (1), (2), (3), (1, 1), (1, 2), (1, 1, 1), (1, 1, 2), (2), (2, 1), (2, 1, 1), (2, 1, 2), (3, 1)\}$.
Then we can see that I is an ideal of N, and $stem\,(I) = \{(), (1), (2), (3)\}$.

Proposition 5.7. *Let $(N, +_1, \cdot_1, ())$ and $(M, +_2, \cdot_2, ())$ be two semi-rings related to nexuses N and M and $0 \neq I, J$ be two ideals of N. Then*

(I) *if $f : N \longrightarrow M$ is a semi-ring epimorphism, then $\overline{f} : \dfrac{N}{I} \longrightarrow M$ such that for every $a \in N$, $\overline{f}(a +_1 I) = f(a)$ is a semi-ring isomorphism if and only if f satisfy in property $(*)$,*

(II) *the map $g : \dfrac{N}{I} \longrightarrow \dfrac{N}{J}$ such that for every $a \in N$, $g(a +_1 I) = a +_1 J$ is a semi-ring isomorphism,*

(III) *the map $h : \dfrac{N}{I} \longrightarrow stem\,(I)$ such that for every $a \in N$, $h(a +_1 I) = (stem\, a)$, is a semi-ring isomorphism.*

Proof. (I) Since f is onto, we get $\overline{f}$ is onto. Also, since f satisfy in property (*), by Lemma 2.9, $f(a) = f(a')$ if and only if $stem\, a = stem\, a'$ if and only if $a +_1 I = a' +_1 I$. Hence $\overline{f}$ is one to one. Clearly, $\overline{f}$ is a semi-ring homomorphism. Therefore, $\overline{f}$ is a semi-ring isomorphism.

(II) Clearly, g is a semi-ring homomorphism and onto. Now, by Lemma 2.9, for every $a, a' \in N$, $a +_1 I = a' +_1 I$ if and only if $\mathit{stem}\, a = \mathit{stem}\, a'$ if and only if $a +_1 J = a' +_1 J$. Hence g is one to one. Thus, it is a semi-ring isomorphism.

(III) The proof is similar to the proof (II).□

Now, we verify the concept localization of semi-ring N.

The subset S of N is a multiplicatively closed subset of N or (m.c.s), if every $a, b \in S$ implies $a \cdot b \in S$.

Let $X \subseteq N$. We put $\langle X \rangle_s := \{\prod_{i=1}^{n} a_1 \cdot a_2 \cdot \cdots \cdot a_n : \{a_1, a_2, \ldots, a_n\} \subseteq X, i \in \mathbb{N}\}$. Since for every $a \in N$, $a \cdot a = a$, it is easy to see that $\langle X \rangle_s$ is a (m.c.s) of N. Conversely, let S be a (m.c.s) of N. Clearly, $S = \langle S \rangle_s$. Hence for every (m.c.s) S of N, there exist a subset X of N such that $S = \langle X \rangle_s$ and X is called generated set of S. X is called the minimal generated set of S, if for every generated set Y of S, $X \subseteq Y$. Hereafter $S = \langle X \rangle_s$ it means that S is a (m.c.s) and X is a minimal generated set of S.

For every $a \in N$, $\{a\} = \langle a \rangle_s$ is a (m.c.s) of N.

For every $a, b \in N$, $S = \{a, b, a \cdot b\}$ is a (m.c.s) of N and $X = \{a, b\}$.

Now, we define a relation on $N \times S$. For every $a, a' \in N$ and $t, t' \in S$, $(a, t) \sim (a', t')$, if and only if there exist $s \in S$ such that $s \cdot t' \cdot a = s \cdot t \cdot a'$. It is easy to see that $\sim$ is an equivalence relation on $N \times S$.

For every $(a, t) \in N \times S$, put $\frac{a}{t} = \{(a', t') : (a, t) \sim (a', t')\}$.

Remark 5.8. Let S be a (m.c.s) of N. We put $\sigma = \prod_{s \in S} s$. For every $a \in N$ and $s \in S$, $\frac{a}{\sigma} = \frac{a}{s}$. Also, for every $a \in N$, $\frac{a}{\sigma} = \frac{()}{\sigma}$ if and only if $a = ()$.

Lemma 5.9. *Let $S \subseteq N$ be a (m.c.s) of N. Then for every $a, a' \in N$, $\frac{a}{\sigma} = \frac{a'}{\sigma}$ if and only if $\sigma \cdot a = \sigma \cdot a'$.*

Proof. Since for every $s \in S$, we get $\sigma \cdot s = \sigma$, the proof is clear. □

Theorem 5.10. *Let $S \subseteq N$ be a (m.c.s) of N, $a, a' \in N$, σ as in Remark 5.8 and $\mathit{stem}\, \sigma = k$. Then*

(I) *if* $\min\{\mathit{stem}\, a, \mathit{stem}\, a'\} > k$, *then* $\frac{a}{\sigma} = \frac{a'}{\sigma}$,

(II) $\max\{\mathit{stem}\, a, \mathit{stem}\, a'\} < k$, $\frac{a}{\sigma} = \frac{a'}{\sigma}$ *if and only if* $\mathit{stem}\, a = \mathit{stem}\, a'$,

(III) *if* $\sigma = (k)$ *and* $\min\{\mathit{stem}\, a, \mathit{stem}\, a'\} \geqslant k$, *then* $\frac{a}{\sigma} = \frac{a'}{\sigma}$,

(IV) *if* $\sigma \neq (k)$, *then* $\frac{(k)}{\sigma} = \frac{a}{\sigma}$ *if and only if* $(k) = a$ *or* $\mathit{stem}\, a > k$,

(V) *if* $\sigma \neq (k)$, *then* $\dfrac{\sigma}{\sigma} = \dfrac{a}{\sigma}$ *if and only if* $\sigma \leqslant a$,

(VI) *if* $\sigma \neq (k)$ *and* $a, a' < \sigma$, *then* $\dfrac{a}{\sigma} = \dfrac{a'}{\sigma}$, *if and only if* $a = a'$.

Proof. Let $a, a' \in N$, $stem\, a = a_1$ and $stem\, a' = a'_1$. If $a_1, a'_1 > k$, then $i\{a, \sigma\} = i\{a', \sigma\} = 1$ and $a \cdot \sigma = a' \cdot \sigma = (k)$. Now, by Lemma 5.9, we get $\dfrac{a}{\sigma} = \dfrac{a'}{\sigma}$.

(II) Let $a, a' \in N$, $stem\, a = a_1$ and $stem\, a' = a'_1$. If $a_1, a'_1 < k$, then $i\{a, \sigma\} = i\{a', \sigma\} = 1$. Using Lemma 5.9, we get $\dfrac{a}{\sigma} = \dfrac{a'}{\sigma}$ if and only if $a \cdot \sigma = a' \cdot \sigma$ if and only if $(a_1) = (a'_1)$ if and only if $a_1 = a'_1$.

(III) Let $a, a' \in N$, $stem\, a = a_1$ and $stem\, a' = a'_1$. If $a_1, a'_1 \geqslant k$, then $i\{a, \sigma\} = i\{a', \sigma\} = 1$ and $a \cdot \sigma = a' \cdot \sigma = (k)$. Now, by Lemma 5.9, we get $\dfrac{a}{\sigma} = \dfrac{a'}{\sigma}$.

(IV) Let $a \in N$, by Lemma 5.9, $\dfrac{(k)}{\sigma} = \dfrac{a}{\sigma}$ if and only if $\sigma \cdot (k) = \sigma \cdot a$. Since $\sigma \neq (k)$ and $stem\, \sigma = k$, $\sigma \cdot (k) = (k)$, we get $\sigma \cdot a = (k)$. Now, we prove $\sigma \cdot a = (k)$ if and only if $stem\, a > k$ or $a = (k)$. If $a_1 = stema > k$, then $i\{a, \sigma\} = 1$, and so $\sigma \cdot a = (a_1 \wedge k) = (k)$. If $a = (k)$, then $i\{a, \sigma\} = 1$, and so $\sigma \cdot a = \sigma \cdot (k) = (k)$.

Conversely, let $\sigma \cdot a = (k)$ and $a_1 = stem\, a \leqslant k$. We show that $a = (k)$. Since $\sigma \cdot a = (k)$ and $a_1 = stem\, a \leqslant k$, $k = stem(\sigma \cdot a) = a_1 \wedge k = a_1$. Let $l(a) \geqslant 2$. Since $\sigma \neq (k)$, $l(\sigma) \geqslant 2$, and hence $l(a \cdot \sigma) \geqslant 2$. Since $\sigma \cdot a = (k)$, $l((k)) \geqslant 2$, which is a contradiction. Therefore, $stem\, a > k$ or $a = (k)$. Hence the proof is complete.

(V) Let $a \in N$, by Lemma 5.9, $\dfrac{\sigma}{\sigma} = \dfrac{a}{\sigma}$ if and only if $\sigma \cdot \sigma = a \cdot \sigma$ if and only if $a \cdot \sigma = \sigma$ if and only if $\sigma \leqslant a$.

(VI) Let $a, a' \in N$ and $a, a' < \sigma$, by Lemma 5.9, $\dfrac{a}{\sigma} = \dfrac{a'}{\sigma}$ if and only if $\sigma \cdot a = a' \cdot \sigma$ if and only if $a = a'$. □

Example 5.11. Consider a nexus:

$N = \{(), (1), (1,1), (1,1,2), (1,2), (1,2,1), (1,2,2), (2), (2,1), (2,2), (3), (3,1), (3,2), (4)\}$

with the following diagram.

(I) Let $S := \langle (1,1), (1,1,2) \rangle_s$. Then $\sigma = (1,1)$. Using Theorem 5.10, we get:

$\dfrac{(1,1)}{(1,1)} = \dfrac{(1,1,2)}{(1,1)} = \dfrac{(1,2)}{(1,1)} = \dfrac{(1,2,1)}{(1,1)} = \dfrac{(1,2,2)}{(1,1)}$ and

$\dfrac{(1)}{(1,1)} = \dfrac{(2)}{(1,1)} = \dfrac{(2,1)}{(1,1)} = \dfrac{(2,2)}{(1,1)} = \dfrac{(3)}{(1,1)} = \dfrac{(3,1)}{(1,1)} = \dfrac{(3,2)}{(1,1)} = \dfrac{(4)}{(1,1)}$.

Hence $N_s = \{\dfrac{0}{(1,1)}, \dfrac{(1)}{(1,1)}, \dfrac{(1,1)}{(1,1)}\}$.

(II) Let $S := \langle (1,2,1), (1,2,2) \rangle_s$. Then $\sigma = (1,2)$.

Using Lemma 5.9 and Theorem 5.10, we get:

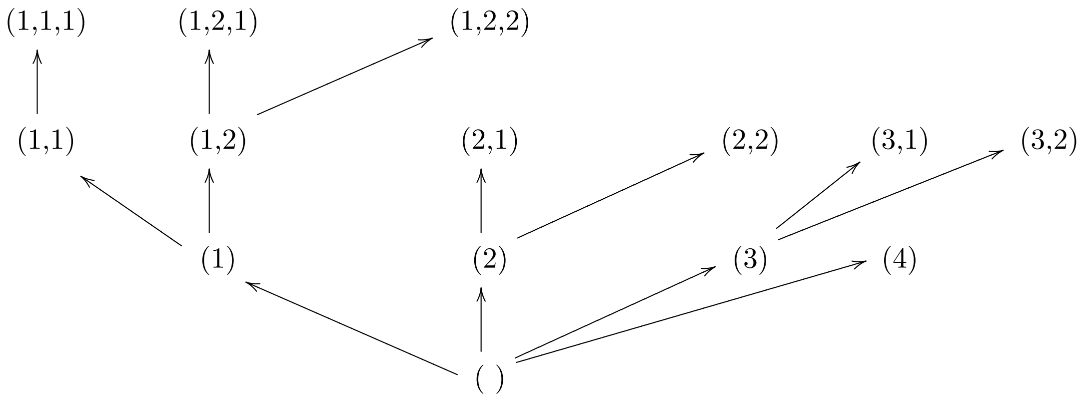

Fig. 6. Diagram of N.

$\frac{(1,1)}{(1,2)} = \frac{(1,1,2)}{(1,2)}, \frac{(1,2)}{(1,2)} = \frac{(1,2,1)}{(1,2)} = \frac{(1,2,2)}{(1,2)}$ and
$\frac{(1)}{(1,2)} = \frac{(2)}{(1,2)} = \frac{(2,1)}{(1,2)} = \frac{(2,2)}{(1,2)} = \frac{(3)}{(1,2)} = \frac{(3,1)}{(1,2)} = \frac{(3,2)}{(1,2)} = \frac{(4)}{(1,2)}$.
Hence $N_s = \{\frac{0}{(1,2)}, \frac{(1)}{(1,2)}, \frac{(1,1)}{(1,2)}, \frac{(1,2)}{(1,2)}\}$.

Definition 5.12. Let $S \subseteq N$ be a (m.c.s) of N and $\sigma = \prod_{s\in S} s$. We put

$$N_s := \{\frac{a}{\sigma} : (a,\sigma) \in N \times S\}.$$

For every $a, a' \in N$, we define the binary operations "$\oplus$" and "$\odot$" on N_S with the following:

$$\frac{a}{\sigma} \oplus \frac{a'}{\sigma} = \frac{a+a'}{\sigma} \text{ and } \frac{a}{\sigma} \odot \frac{a'}{\sigma} = \frac{a\cdot a'}{\sigma}.$$

Example 5.13. Consider Example 5.11 (II). We have
$\frac{(1,1)}{(1,2)} \oplus \frac{(1,2)}{(1,2)} = \frac{(1,1)\cdot(1,2)+(1,2)\cdot(1,2)}{(1,2)} = \frac{(1,1)+(1,2)}{(1,2)} = \frac{(1)}{(1,2)}$,
$\frac{(1,1)}{(1,2)} \odot \frac{(1,2)}{(1,2)} = \frac{(1,1)\cdot(1,2)}{(1,2)} = \frac{(1,1)}{(1,2)}$.

Proposition 5.14. *The algebra* $(N_s, \oplus, \odot, \frac{()}{\sigma})$ *is a semi-ring.*

Proof. Let $a, a', b, b' \in N$ and $\frac{a}{\sigma} = \frac{b}{\sigma}$ and $\frac{a'}{\sigma} = \frac{b'}{\sigma}$. Then by Lemma 5.9, $\sigma\cdot a = \sigma\cdot b$ and $\sigma\cdot a' = \sigma\cdot b'$. It follows that $\sigma\cdot a + \sigma\cdot a' = \sigma\cdot b + \sigma\cdot b'$ and $\sigma\cdot a\cdot a' = \sigma\cdot b\cdot b'$. Therefore, $\frac{\sigma\cdot a+\sigma\cdot a'}{\sigma} = \frac{\sigma\cdot b+\sigma\cdot b'}{\sigma}$ and $\frac{a\cdot a'}{\sigma} = \frac{b\cdot b'}{\sigma}$. Also, we have $\frac{a}{\sigma} + \frac{()}{\sigma} = \frac{a}{\sigma}$. Since $(N, +, \cdot, ())$ is a semi-ring, for $a, b, c \in N$, we have
$\frac{a}{\sigma} \oplus (\frac{b}{\sigma} \oplus \frac{c}{\sigma}) = \frac{a}{\sigma} \oplus \frac{b+c}{\sigma} = \frac{a+(b+c)}{\sigma} = \frac{(a+b)+c}{\sigma} = \frac{a+b}{\sigma} \oplus \frac{c}{\sigma} = (\frac{a}{\sigma} \oplus \frac{b}{\sigma}) \oplus \frac{c}{\sigma}$.
$\frac{a}{\sigma} \odot (\frac{b}{\sigma} \odot \frac{c}{\sigma}) = \frac{a}{\sigma} \odot \frac{b\cdot c}{\sigma} = \frac{a\cdot(b\cdot c)}{\sigma} = \frac{(a\cdot b)\cdot c}{\sigma} = \frac{a\cdot b}{\sigma} \odot \frac{c}{\sigma} = (\frac{a}{\sigma} \odot \frac{b}{\sigma}) \odot \frac{c}{\sigma}$.

$$\frac{a}{\sigma}\odot(\frac{b}{\sigma}\oplus\frac{c}{\sigma})=\frac{a}{\sigma}\odot\frac{b+c}{\sigma}=\frac{a\cdot(b+c)}{\sigma}=\frac{(a\cdot b)+(a\cdot c)}{\sigma}=\frac{a\cdot b}{\sigma}\oplus\frac{a\cdot c}{\sigma}=(\frac{a}{\sigma}\odot\frac{b}{\sigma})\oplus(\frac{a}{\sigma}\odot\frac{c}{\sigma}).$$

Therefore, $(N_s,\oplus,\odot,\frac{()}{\sigma})$ is a semi-ring. □

Example 5.15. Consider Example 5.11 (II). Hence $N_s=\{\frac{0}{(1,2)},\frac{(1)}{(1,2)},\frac{(1,1)}{(1,2)},\frac{(1,2)}{(1,2)}\}$. Thus, by Proposition 5.14, $(N_s,\oplus,\odot,\frac{()}{\sigma})$ is a semi-ring.

6. Conclusions and future works

In this work, we have explored the construction and properties of semi-rings over a nexus, contributing to the broader understanding of algebraic structures in this context. We began by presenting preliminary results on semi-rings constructed over a nexus, establishing conditions under which such a semi-ring is unitary or a principal ideal domain (PID). For a semi-ring N, we introduced the concept of $st(N)$ and characterized prime ideals and prime elements using panels and quasi-panels. For a proper ideal P of N, we defined $st(P)=\max\{l:(l)\in P\}$ and proved that if $st(P)\neq st(N)$ and $st(P)=t$, then P is a prime ideal of N if and only if $P=\bigcup_{s\leqslant t}M_{(s)}\cup\{0\}$. Furthermore, when $st(P)=st(N)$, we utilized Remarks 4.9 and 4.11 to characterize all prime ideals of N. Additionally, for an ideal I of N and a multiplicatively closed subset S of N, we defined the quotient semi-ring $\frac{N}{I}$ and the local semi-ring N_S.

Furthermore, we defined homomorphisms for these semi-rings and investigated their fundamental properties. Finally, we demonstrated the construction of quotient semi-rings induced by ideals of a semi-ring over a nexus and established the concept of localization. These results extend the theoretical framework of nexus algebras and provide new tools for analyzing algebraic structures in this setting.

As a direction of this study, we will be investigating the relationship between semi-rings over a nexus and other algebraic structures, such as modules or lattices. Further, extending the concept of localization to more general settings or exploring its applications in solving equations over semi-rings could provide practical tools for both theoretical and applied research.

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