Algebraic Structures and Their Applications Vol. X No. X (20XX) pp XX-XX.

Research Paper

MODULE STRUCTURES ON L-ALGEBRAS

MONA AALY KOLOGANI, MOHAMMAD MOHSENI TAKALLO AND RAJAB ALI BORZOOEI*

ABSTRACT. In this paper, we apply the modules theory to L-algebras and introduce the concept of an L-module. Then we construct L-modules by using power sets and De Morgan algebras. Moreover, we investigate some properties of modules such as sub-module and other related results. Finally, we introduce the concepts of multiplication L-modules and co-multiplication L-modules and we will represent each sub-module of an L-module by using them.

1. Introduction

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump [12]. Many examples shown that L-algebras are very useful. Yang and Rump [14], characterized pseudo MV-algebras and Bosbach's non-commutative bricks as L-algebras. Wu and Yang [20] proved that orthomodular lattices form a special class of L-algebras in

DOI: 10.22034/as.2025.23530.1816

MSC(2010): Primary: 06F15, 06F05, 03G25.

Keywords: L-algebra, L-module, (co-)multiplication L-module

Received: 16 August 2025, Accepted: 20 October 2025.

∗Corresponding author

different ways. It was shown that every lattice-ordered effect algebra has an underlying L-algebra structure in Wu et al. [19]. Also, other mathematicians studied the relationship between basic algebras and L-algebras. They proved that a basic algebra which satisfies $(z \oplus \neg x) \oplus \neg(y \oplus \neg x) = (z \oplus \neg y) \oplus \neg(x \oplus \neg y)$, can be converted into an L-algebra. Conversely, if an L-algebra with 0 and some conditions such that it is an involutive bounded lattice can be organized into a basic algebra, it must be a lattice-ordered effect algebra. For more study about L-algebras see [2, 3, 4, 8, 9, 10, 11, 21, 22].

Every module is an action of a ring on a certain group. This is, indeed, a source of motivation to study the action of certain algebraic structures on groups. It seems quite natural to introduce "modules" over L-algebras, analogously to modules over BCK-algebras and MV-algebras and try to build a theory that is parallel to the standard ring theory on L-algebras. For instance, in [6], authors defined the MV-modules over PMV-algebras and they proved that these are structures that naturally correspond to ℓu-modules over ℓu-rings. Fixing an ℓu-ring (R, ν), they showed that the equivalence between the category of ℓu-modules over (R, ν) and the category of MV-modules over $\Gamma(R, \nu)$. They also introduced the truncated modules, that are MV-algebras endowed with an external multiplication defined for any element in the positive cone of an ℓu-ring and showed that the natural equivalence between MV-modules and truncated modules. For more study about modules on logical algebras see [5, 6, 15, 17, 18].

Considering the importance of L-algebras in different fields of mathematics and physics, as well as the importance of modules in algebra, in the article [16], the authors introduce the L-module, in which an L-algebra acts on an arbitrary Abelian group, and therefore, the operations of the Abelian group in question are used in the definition of the L-module. Then, they explain the concept of the extended of the L-module and examine some results in this regard. In this article, we have used two L-algebras, which we have constructed with the help of one of the L-algebras and the definition of the operation $+$ an Abelian group, and then the other L-algebra acts on that and an L-module is constructed. Therefore, the definition presented is used with the help of the operation of implication in the L-algebra (with the help of which the Abelian group is defined). Then we decided to introduce the concept of L-modules and obtain some interesting examples and results. Moreover, we define the concepts of multiplication and co-multiplication modules in L-modules and investigate some results about them.

2. Preliminaries

This section lists the known default contents that will be used later.

An *L-algebra* [7] is an algebraic structure $(L; \to, 1)$ of type $(2, 0)$ satisfying

$(L1)$ $x \to x = x \to 1 = 1$ and $1 \to x = x$,

$(L2)$ $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,
$(L3)$ if $x \to y = y \to x = 1$, then $x = y$,
for any $x, y, z \in L$. Condition $(L1)$ states that 1 is a logical unit, while $(L2)$ is related to the quantum Yang-Baxter equation. Note that a logical unit is always unique. By $(L3)$, the relation

$$x \leq y \text{ if and only if } x \to y = 1,$$

defines a partial order for any L-algebra L. If L admits a smallest element 0, then it is called a *bounded L-algebra* (see [7]).

We say that a bounded L-algebra L has *negation* if the map $x \longmapsto x'$ is bijective, where $x' = x \to 0$.

Example 2.1. Let $(G, +, -, \wedge, \vee, 0)$ be an arbitrary ℓ-group. For an arbitrary element $a \in G$, where $0 \leq a$ define the binary operation $\to$ on $G[a] = [0, a]$ as follows,

$$x \to y = (y - x + a) \wedge a,$$

for any $x, y \in G[a]$. Then $(G[a], \to, a)$ is an L-algebra.

Definition 2.2. [13] An L-algebra L which satisfies

$$x \to (y \to x) = 1, \qquad (K)$$

for any $x, y \in L$ is called a *KL-algebra*.

A *CKL-algebra* is an L-algebra which satisfies

$$x \to (y \to z) = y \to (x \to z), , , , \qquad (C)$$

for any $x, y, z \in L$ (see [13]).

Clearly, every CKL-algebra is a KL-algebra, since for any $x, y \in L$, we have

$$x \to (y \to x) = y \to (x \to x) = y \to 1 = 1.$$

Proposition 2.3. [14] *Let L be an L-algebra. Then $x \leq y$ implies $z \to x \leq z \to y$, for any $x, y, z \in L$.*

Proposition 2.4. [14] *For an L-algebra L, the following are equivalent:*
(i) $x \leq y \to x$,
(ii) *if* $x \leq z$, *then* $z \to y \leq x \to y$,
(iii) $((x \to y) \to z) \to z \leq ((x \to y) \to z) \to ((y \to x) \to z)$,
for any $x, y, z \in L$.

Definition 2.5. [12] A subset I of an L-algebra L is called an *ideal of* L if it satisfies the following conditions for all $x, y \in I$,
(I_1) $1 \in I$,
(I_2) if $x \in I$ and $x \to y \in I$, then $y \in I$,
(I_3) if $x \in I$, then $(x \to y) \to y \in I$,
(I_4) if $x \in I$, then $y \to x \in I$ and $y \to (x \to y) \in I$.

If we consider the ideal of CKL-algebra, the conditions (I_3) and (I_4) can be dropped. In fact, for any $x \in I$, by (C) and (I_1) we have

$$x \to ((x \to y) \to y) = (x \to y) \to (x \to y) = 1 \in I,$$

for any $y \in L$. It follows by (I_2) that $(x \to y) \to y \in I$. Thus (I_3) holds. Furthermore, if $x \in I$, then for any $y \in L$, by (K) we have $x \to (y \to x) = 1 \in I$ and by (I_2), $y \to x \in I$.

For an L-algebra, a binary relation $\sim$ is a *congruence relation* [12] on L if it is an equivalence relation such that for any $x, y, z \in L$,

$$x \sim y \Leftrightarrow \ (z \to x) \sim (z \to y) \ \text{ and } \ (x \to z) \sim (y \to z).$$

Theorem 2.6. [12] *Let $(L, \to, 1)$ be an L-algebra. Then every ideal I of L defines a congruence relation on L, for any $x, y \in L$, where*

$$x \sim y \ \Leftrightarrow \ \ x \to y, \ y \to x \in I.$$

Conversely, every congruence relation $\sim$ defines an ideal $I = \{x \in L \mid x \sim 1\}$.

Definition 2.7. [12] Let L and H be two L-algebras. Then a map $f : L \to H$ is called an *L-homomorphism* if for any $x, y \in L$ we have $f(x \to_L y) = f(x) \to_H f(y)$.

If f is an injective, then f is called a *monomorphism* and if f is onto, then f is called an *epimorphism*. In addition, if f is a bijective function, then f is called an *isomorphism*.

3. Construction of an L-module by an Abelian group

In this section, we introduce the concept of L-modules by using Abelian groups and investigate some properties about them.

Definition 3.1. Let $\mathcal{L} = (L, \to, 1)$ be an L-algebra and $(G, \rightsquigarrow, ', 0, 1)$ be a bounded L-algebra with negation, where $(G, +, 0)$ be an Abelian group and $\cdot : L \times G \to G$ be a mapping such that for any $m \in G$ and $x, y \in L$:
(LM_1) $1 \cdot m = m$,
(LM_2) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,
(LM_3) $(x \to y) \cdot m = m - (x \cdot m \rightsquigarrow y \cdot m)'$.
Then G is called an *L-module*, where the subtraction "-" is in fact the group subtraction in G.

Notation. For any $m \in G$ and $x \in L$, we write xm instead of $x \cdot m$, for short.

Example 3.2. Assume $L = \{\alpha, \beta, 1\}$ be a set, where

$\rightarrow$	α	β	1
α	1	α	1
β	α	1	1
1	α	β	1

Then $(L, \rightarrow, 1)$ is an L-algebra. Let $(G = \{0, a, b, 1\}, \leq)$ be a poset, where $0 \leq a, b \leq 1$. Define the operation $\rightsquigarrow$ on L as follows:

$\rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then $(G, \rightsquigarrow, 1)$ is an L-algebra. Now, define the operation $+$ on G as follows:

$+$	0	a	b	1
0	0	a	b	1
a	a	0	1	b
b	b	1	0	a
1	1	b	a	0

According to the above table, $(G, +, 0)$ is an Abelian group. Define $\cdot : L \times G \rightarrow G$ by $x \cdot m = m$, for any $x \in L$ and $m \in G$. Then it is easy to see that G is an L-module.

Proposition 3.3. *Let X be a non-empty set and $L = P(X)$ be the power set of X. If operation "$\rightarrow$" on $P(X)$, for any $A, B \in P(X)$, is defined by:*

$$A \rightarrow B = A^c \cup B,$$

then $L = (P(X), \rightarrow, X)$ is an L-algebra. Also, if $G = (P(X), +, \emptyset)$, where

$$A + B = A \Delta B = (A \cup B) - (A \cap B),$$

then G is an Abelian group and G is an L-module.

Proof. Let $A, B, C \in P(X)$. Then

$$A \rightarrow A = A^c \cup A = X, \qquad X \rightarrow A = X^c \cup A = \emptyset \cup A = A,$$

$$A \to X = A^c \cup X = X.$$

So, $(L1)$ holds.

Let $A, B, C \in P(X)$. Then, we have

$$\begin{aligned}(A \to B) \to (A \to C) &= (A^c \cup B) \to (A^c \cup C) = (A^c \cup B)^c \cup (A^c \cup C)\\ &= (A \cap B^c) \cup (A^c \cup C) = ((A \cap B^c) \cup A^c) \cup C\\ &= ((A \cup A^c) \cap (B^c \cup A^c)) \cup C = (B^c \cup A^c) \cup C\\ &= (X \cap (B^c \cup A^c)) \cup C = ((B^c \cup B) \cap (B^c \cup A^c)) \cup C\\ &= ((A^c \cap B) \cup B^c) \cup C = (A^c \cap B) \cup (B^c \cup C)\\ &= (A \cup B^c)^c \cup (B^c \cup C) = (B \to A)^c \cup (B \to C)\\ &= (B \to A) \to (B \to C).\end{aligned}$$

Hence we have $(L2)$. Suppose $A \to B = X$ and $B \to A = X$. Then

$$A \to B = A^c \cup B = X \;\Rightarrow\; A \subseteq B,$$

$$B \to A = B^c \cup A = X \;\Rightarrow\; B \subseteq A.$$

Thus, $A = B$, and so $(L3)$ holds. Clearly, $\emptyset$ and X are the smallest and the greatest elements of $P(X)$. Therefore, $L = (P(X), \to, X)$ is an L-algebra. By the similar way, $G = (P(X), \to, \emptyset, X)$ is a bounded L-algebra.

Now, we prove that $G = (P(X), +, \emptyset)$ is an Abelian group. Let $A, B \in P(X)$. Then,

$$A + B = A\Delta B = (A \cup B) - (A \cap B) = (B \cup A) - (B \cap A) = B\Delta A = B + A.$$

Also,

$$A\Delta\emptyset = (A \cup \emptyset) - (A \cap \emptyset) = A.$$

Thus $\emptyset$ is an identity element of $P(X)$. In addition,

$$A + A = A\Delta A = A - A = \emptyset.$$

Hence, inverse of every element of $P(X)$ is itself. Moreover, clearly the operation Δ has associated property. Therefore, $G = (P(X), +, \emptyset)$ is an Abelian group.

Now, we define the operation $\cdot : L \times G \to G$, for any $A \in L$ and $C \in G$, by

$$A \cdot C = A \cap C.$$

Then we prove that G is an L-module. Let $A, B \in L$ and $C \in G$. Then,

(LM_1): $1C = X \cap C = C$.

(LM_2):

$$\begin{aligned}
AB + AC = (A \cap B) + (A \cap C) &= ((A \cap B) \cap (A \cap C)^c) \cup ((A \cap C) \cap (A \cap B)^c) \\
&= ((A \cap B) \cap (A^c \cup C^c)) \cup ((A \cap C) \cap (A^c \cup B^c)) \\
&= (B \cap ((A \cap A^c) \cup (A \cap C^c))) \cup (C \cap ((A \cap A^c) \cup (A \cap B^c))) \\
&= (B \cap (A \cap C^c)) \cup (C \cap (A \cap B^c)) \\
&= (A \cap (B \cap C^c)) \cup (A \cap (C \cap B^c)) \\
&= A \cap ((B \cap C^c) \cup (C \cap B^c)) \\
&= A(B + C).
\end{aligned}$$

(LM3):

$$\begin{aligned}
C + ((A \cdot C) \to (B \cdot C))^c &= C + ((A \cap C) \to (B \cap C))^c \\
&= C + ((A \cap C)^c \cup (B \cap C))^c \\
&= C + ((A^c \cup C^c) \cup (B \cap C))^c \\
&= C + (((A^c \cup C^c) \cup B) \cap ((A^c \cup C^c) \cup C))^c \\
&= C + (((A^c \cup C^c) \cup B) \cap X)^c \\
&= C + ((A^c \cup C^c) \cup B)^c \\
&= C \Delta ((A^c \cup C^c) \cup B)^c \\
&= (C - ((A^c \cup C^c) \cup B)^c) \cup (((A^c \cup C^c) \cup B)^c - C) \\
&= (C \cap ((A^c \cup C^c) \cup B)) \cup (((A^c \cup C^c) \cup B)^c \cap C^c) \\
&= (C \cap ((A^c \cup C^c) \cup B)) \cup (((A \cap C) \cap B^c) \cap C^c) \\
&= C \cap ((A^c \cup C^c) \cup B) \\
&= (C \cap A^c) \cup (C \cap C^c) \cup (C \cap B) \\
&= (C \cap A^c) \cup (C \cap B) \\
&= (A^c \cup B) \cap C \\
&= (A \to B) \cdot C.
\end{aligned}$$

Therefore, G is an L-module. □

Proposition 3.4. *Let $(L, \wedge, \vee, ', 0, 1)$ be a De Morgan algebra, such that for any $x \in L$, $x \vee x' = 1$ and $1' = 0$. If operation "$\to$" on L, for any $x, y \in L$, is defined by*

$$x \to y = x' \vee y,$$

then $L = (L, \to, 1)$ is an L-algebra. Also, if $G = (L, +, 0)$, where, for any $x, y \in L$,

$$x + y = ((x \to y) \wedge (y \to x))',$$

then G is an Abelian group and G is an L-module.

Proof. At first we prove that $L = (L, \to, 1)$ is an L-algebra. Let $x, y, z \in L$. Then by assumption,

$$x \to x = x' \vee x = 1, \quad x \to 1 = x' \vee 1 = 1, \quad 1 \to x = 1' \vee x = 0 \vee x = x.$$

So, $(L1)$ holds. Also,

$$\begin{aligned}(x \to y) \to (x \to z) &= (x' \vee y) \to (x' \vee z) = (x' \vee y)' \vee (x' \vee z) = (x'' \wedge y') \vee (x' \vee z)\\ &= [(x'' \wedge y') \vee x'] \vee z = [(x'' \vee x') \wedge (y' \vee x')] \vee z\\ &= (y' \vee x') \vee z,\end{aligned}$$

On the other side, we have

$$\begin{aligned}(y \to x) \to (y \to z) &= (y' \vee x) \to (y' \vee z) = (y' \vee x)' \vee (y' \vee z) = (y'' \wedge x') \vee (y' \vee z)\\ &= [(y'' \wedge x') \vee y'] \vee z = [(y'' \vee y') \wedge (y' \vee x')] \vee z\\ &= (y' \vee x') \vee z,\end{aligned}$$

Hence, $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$ and so $(L2)$ holds. Finally, suppose $x \to y = y \to x = 1$. Then

$$x \to y = 1 \text{ imply } x' \vee y = 1 \text{ and } y \to x = 1 \text{ imply } y' \vee x = 1.$$

Since $y' \wedge y = y'' \wedge y' = (y' \vee y)' = 0$, we have

$$y = y \wedge 1 = y \wedge (y' \vee x) = (y \wedge y') \vee (y \wedge x) = y \wedge x.$$

Thus $y \leq x$. By the similar way, $x \leq y$ and so $x = y$. Hence, $(L3)$ holds and so $(L, \to, 1)$ is an L-algebra. By the similar way, $G = (L, +, 0, 1)$ is a bounded L-algebra.

Now, we prove that $G = (L, +, 0)$ is an Abelian group. For this, assume $x, y, z \in G$. Then clearly $(G, +)$ is closed and $x + y = y + x$. Also, $x + x = 0$ and $x + 0 = x$. So, 0 is an identity element and inverse of any element is itself. In the following we prove associativity.

$$
\begin{aligned}
&(x+y)+z\\
&=[(x\to y)\wedge(y\to x)]'+z=\big[(x'\vee y)\wedge(y'\vee x)\big]'+z\\
&=\Big(\big[\big((x'\vee y)\wedge(y'\vee x)\big)\vee z\big]\wedge\Big[z'\vee\big((x'\vee y)\wedge(y'\vee x)\big)'\Big]\Big)'\\
&=\big(\big[(x'\vee y\vee z)\wedge(y'\vee x\vee z)\big]\wedge\big[z'\vee\big((x\wedge y')\vee(y\wedge x')\big)\big]\big)'\\
&=\big(\big[(x'\vee y\vee z)\wedge(y'\vee x\vee z)\big]\wedge\big[\big(z'\vee(x\wedge y')\big)\vee(y\wedge x')\big]\big)'\\
&=\big(\big[(x'\vee y\vee z)\wedge(y'\vee x\vee z)\big]\wedge\big[\big((z'\vee x)\wedge(z'\vee y')\big)\vee(y\wedge x')\big]\big)'\\
&=\big(\big[(x'\vee y\vee z)\wedge(y'\vee x\vee z)\big]\wedge\big[\big((z'\vee x)\vee(y\wedge x')\big)\wedge\big((z'\vee y')\vee(y\wedge x')\big)\big]\big)'\\
&=\big(\big[(x'\vee y\vee z)\wedge(y'\vee x\vee z)\big]\wedge\big[\big((z'\vee x\vee y)\wedge(z'\vee x\vee x')\big)\wedge\big((z'\vee y'\vee y)\wedge(z'\vee y'\vee x')\big)\big]\big)'\\
&=\big(\big[(x'\vee y\vee z)\wedge(y'\vee x\vee z)\big]\wedge\big[(z'\vee x\vee y)\wedge(z'\vee y'\vee x')\big]\big)'\\
&=\big(\big[(z'\vee x\vee y)\wedge(y'\vee x\vee z)\big]\wedge\big[(x'\vee y\vee z)\wedge(z'\vee y'\vee x')\big]\big)'\\
&=\big(\big[x\vee((z'\vee y)\wedge(y'\vee z))\big]\wedge\big[(x'\vee y\vee z)\wedge(x'\vee y\vee y')\wedge(z'\vee y'\vee x')\wedge(x'\vee z'\vee z)\big]\big)'\\
&=\big(\big[x\vee((z'\vee y)\wedge(y'\vee z))\big]\wedge\big[\big((x'\vee y)\vee(y'\wedge z)\big)\wedge\big((x'\vee z')\vee(y'\wedge z)\big)\big]\big)'\\
&=\big(\big[x\vee((z'\vee y)\wedge(y'\vee z))\big]\wedge\big[\big((x'\vee y)\wedge(x'\vee z')\big)\vee(y'\wedge z)\big]\big)'\\
&=\big(\big[x\vee((z'\vee y)\wedge(y'\vee z))\big]\wedge\big[\big(x'\vee(y\wedge z')\big)\vee(y'\wedge z)\big]\big)'\\
&=\big(\big[x\vee((z'\vee y)\wedge(y'\vee z))\big]\wedge\big[x'\vee\big((y\wedge z')\vee(y'\wedge z)\big)\big]\big)'\\
&=\Big(\Big[x\vee\big((z'\vee y)\wedge(y'\vee z)\big)''\Big]\wedge\Big[x'\vee\big((y'\vee z)\wedge(y\vee z')\big)'\Big]\Big)'\\
&=\Big(\Big[\big((z'\vee y)\wedge(y'\vee z)\big)'\to x\Big]\wedge\Big[x\to\big((y'\vee z)\wedge(y\vee z')\big)'\Big]\Big)'\\
&=\big(\big[((z\to y)\wedge(y\to z))'\to x\big]\wedge\big[x\to((y\to z)\wedge(z\to y))'\big]\big)'\\
&=([(y+z)\to x]\wedge[x\to(y+z)])'\\
&=x+(y+z).
\end{aligned}
$$

Thus, the operation $+$ is associative. Therefore, $(G,+)$ is an Abelian group.

Now, we prove that G is an L-module, where $xm = x\wedge m$. For this, let $x,y\in L$ and $m_1,m_2,m\in G$. Then

(LM_1) Since L is an L-algebra and $G=L$, we have $1m = 1\wedge m = m$.

(LM_2)

$$x(m_1+m_2) = x\wedge(m_1+m_2) = x\wedge((m_1\to m_2)\wedge(m_2\to m_1))'$$
$$= x\wedge\big((m_1'\vee m_2)\wedge(m_2'\vee m_1)\big)'$$
$$= x\wedge\big((m_1\wedge m_2')\vee(m_2\wedge m_1')\big)$$
$$= \big(x\wedge(m_1\wedge m_2')\big)\vee\big(x\wedge(m_2\wedge m_1')\big)\,. \tag{1}$$

and

$$xm_1+xm_2 = (x\wedge m_1)+(x\wedge m_2)$$
$$= [((x\wedge m_1)\to(x\wedge m_2))\wedge((x\wedge m_2)\to(x\wedge m_1))]'$$
$$= \big[\big((x\wedge m_1)'\vee(x\wedge m_2)\big)\wedge\big((x\wedge m_2)'\vee(x\wedge m_1)\big)\big]'$$
$$= \big[\big((x'\vee m_1')\vee(x\wedge m_2)\big)\wedge\big((x'\vee m_2')\vee(x\wedge m_1)\big)\big]'$$
$$= \big[\big((x'\vee m_1')\vee x\big)\wedge\big((x'\vee m_1')\vee m_2\big)\wedge\big((x'\vee m_2')\vee x)\wedge((x'\vee m_2')\vee m_1)\big)\big]'$$
$$= \big[\big((x'\vee m_1')\vee m_2\big)\wedge((x'\vee m_2')\vee m_1)\big]'$$
$$= \big((x\wedge m_1)\wedge m_2'\big)\vee((x\wedge m_2)\wedge m_1'). \tag{2}$$

Then by Equations (1) and (2), we have $x(m_1+m_2)=xm_1+xm_2$.
(LM_4)

$$m-(xm\to ym)' = m+((x\wedge m)\to(y\wedge m))'$$
$$= m+((x\wedge m)'\vee(y\wedge m))'$$
$$= m+((x\wedge m)\wedge(y\wedge m)')$$
$$= \big[\big(m\to((x\wedge m)\wedge(y\wedge m)')\big)\wedge\big(((x\wedge m)\wedge(y\wedge m)')\to m\big)\big]'$$
$$= \big[\big(m'\vee((x\wedge m)\wedge(y\wedge m)')\big)\wedge\big(((x\wedge m)\wedge(y\wedge m)')'\vee m\big)\big]'$$
$$= \big[\big(m'\vee((x\wedge m)\wedge(y\wedge m)')\big)\wedge\big((x\wedge m)'\vee(y\wedge m)\vee m\big)\big]'$$
$$= \big[\big(m'\vee(x\wedge m)\big)\wedge\big(m'\vee(y\wedge m)'\big)\wedge\big(((x\wedge m)'\vee m)\vee(y\wedge m)\big)\big]'$$
$$= \big[(m'\vee x)\wedge(m'\vee y')\big]'$$
$$= \big[m'\vee(x\wedge y')\big]'$$
$$= m\wedge(x'\vee y)$$
$$= (x'\vee y)\wedge m$$
$$= (x\to y)m.$$

Therefore, G is an L-module. □

Proposition 3.5. *Suppose $(L, \to, ', 0, 1)$ is an L-algebra with negation, $(G, \rightsquigarrow, \sim, 0, 1)$ is a bounded L-algebra, $G = (G, +, 0)$ is an Abelian group and G is an L-module. Then for any $x, y \in L$ and $m \in G$, the following statements hold:*
(i) $x0 = 0$,
(ii) $x(-m) = -(xm)$,
(iii) If $x \leq y$, then $xm \leq ym$,
(iv) If $0m = 0$, then $(x')m = m - xm$, where $x' = x \to 0$.

Proof. Let $x, y \in L$ and $m \in G$. Then
(i) $x0 = x(0+0) = x0 + x0$. Hence, $x0 = 0$.
(ii) $0 = x0 = x(m-m) = x(m + (-m)) = xm + x(-m)$ and so $x(-m) = -(xm)$.
(iii) If $x \leq y$, then $x \to y = 1$ and so for any $m \in G$, we have:

$$m = 1m = (x \to y)m = m - (xm \rightsquigarrow ym)^{\sim},$$

Thus $(xm \rightsquigarrow ym)^{\sim} = 0$, and so $xm \rightsquigarrow ym = 1$. Hence, $xm \leq ym$.
(iv) Since G is with negation, we have $(xm)^{\sim\sim} = xm$, where $(xm)^{\sim} = (xm) \rightsquigarrow 0$ and so

$$(x)'m = (x \to 0)m = m - (xm \rightsquigarrow 0m)^{\sim} = m - (xm \rightsquigarrow 0)^{\sim} = m - (xm)^{\sim\sim} = m - xm.$$

□

Definition 3.6. Consider G to be an L-module and $\emptyset \neq H \subseteq G$. Then H is called a *sub-module of* G if for any $a, a_1, a_2 \in H$ and $x \in L$, we have
(S_1) $a_1 - a_2 \in H$,
(S_2) $xa \in H$.

Example 3.7. By Example 3.2, $H_1 = \{0, b\}$, $H_2 = \{0, a\}$, $H_3 = \{0, 1\}$ and $H_4 = \{0\}$ are sub-modules of L-module G.

In the following example we show that every ideal of an L-module G is not a sub-module of G, in general.

Example 3.8. According to Example 3.2, clearly $I_1 = \{a, 1\}$ and $I_2 = \{b, 1\}$ are two ideals of G, but they are not sub-modules of G, since $a + a = 0 \notin I_1$ and $b + b = 0 \notin I_2$.

Definition 3.9. Let G and H be two L-modules. Then the map $f : G \to H$ is called an *L-module homomorphism* if for all $m_1, m_2 \in G$ and $x \in L$ we have:
(H_1) $f(m_1 + m_2) = f(m_1) + f(m_2)$,
(H_2) $f(xm) = xf(m)$.

Example 3.10. According to Example 3.2, define $h(0) = 0$, $h(a) = b$, $h(b) = a$ and $h(1) = 1$. Then clearly, h is an *L-module homomorphism*.

Theorem 3.11. *Let $f : G \to X$ be an L-module homomorphism by assumption of Proposition 3.5. Then*
(i) $\ker f = \{m \in G \mid f(m) = 0\}$ *is a sub-module of G. If I is a sub-module of X, then $f^{-1}(I)$ is a sub-module of G.*
(ii) $Imf = \{f(m) \mid m \in G\}$ *is a sub-module of X. If H is a sub-module of G, then $f(H)$ is a sub-module of X.*

Proof. (*i*) Suppose $m_1, m_2 \in \ker f$. Then $f(m_1) = f(m_2) = 0$. Since f is an L-module homomorphism, by Proposition 3.5 we have

$$f(m_1 - m_2) = f(m_1) - f(m_2) = 0 - 0 = 0,$$

$$f(xm) = xf(m_1) = x0 = 0.$$

Then $\ker f$ is a sub-module of G. Now, suppose $x_1, x_2 \in f^{-1}(I)$. Then $f(x_1), f(x_2) \in I$. Since I is a sub-module of X, we have $f(x_1 - x_2) = f(x_1) - f(x_2) \in I$, and so $x_1 - x_2 \in f^{-1}(I)$. In addition, for any $m \in f^{-1}(I)$, we have $f(m) \in I$. Then for any $x \in L$, by assumption, $f(xm) = xf(m) \in I$ and so $xm \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is a sub-module of G.
(ii) Suppose $f(m_1), f(m_2) \in Imf$, where $m_1, m_2 \in G$. Since G is an Abelian group we have $m_1 - m_2 \in G$, and so $f(m_1) - f(m_2) = f(m_1 - m_2) \in Imf$. Also, if $f(m) \in Imf$ and $x \in L$, then since f is an L-module homomorphism, we get $xf(m) = f(xm)$. In addition, from $m \in G$ and G is an L-module we have $xm \in G$, and so $xf(m) \in Imf$. Therefore, Imf is a sub-module of X. The proof of other case is similar. □

Definition 3.12. Let G be an L-module. Then an ideal I of G is called an *L-ideal of G* if it satisfies in the following condition:

$$\text{If } m \in I \text{ and } x \in L, \text{ then } xm \in I.$$

Example 3.13. According to Example 3.2, set $I = \{a, 1\}$. Obviously, I is an L-ideal of G.

Proposition 3.14. *Assume $h : G_1 \to G_2$ is an L-homomorphism such that $h(xm) = xh(m)$, for any $x \in L$ and $m \in G_1$. Then the set $K = \{m \in G_1 \mid h(m) = 1\}$ is an ideal of G_1.*

Proof. Since h is an L-homomorphism, clearly, $1 \in K \neq \emptyset$. Suppose $m_1, m_1 \to m_2 \in K$. Then $h(m_1) = h(m_1 \to m_2) = 1$, and so

$$h(m_2) = 1 \to h(m_2) = h(m_1) \to h(m_2) = h(m_1 \to m_2) = 1.$$

Thus $m_2 \in K$. Suppose $m_1 \in K$. Then for any $m_2 \in G_1$, we have

$$h((m_1 \to m_2) \to m_2) = (h(m_1) \to h(m_2)) \to h(m_2) = (1 \to h(m_2)) \to h(m_2) = h(m_2) \to h(m_2) = 1,$$

$$h(m_2 \to m_1) = h(m_2) \to h(m_1) = h(m_2) \to 1 = 1,$$

$$h(m_2 \to (m_1 \to m_2)) = h(m_2) \to (h(m_1) \to h(m_2)) = h(m_2) \to (1 \to h(m_2)) = h(m_2) \to h(m_2) = 1.$$

Hence $m_2 \to m_1, (m_1 \to m_2) \to m_2, m_2 \to (m_1 \to m_2) \in K$. Therefore, K is an ideal of G_1. □

Note. We have to notice that the set K defined in the above proposition is not an L-ideal, and it depends on how we define the act in L-module. For example, if according to Example 3.2 and Proposition 3.4, respectively, we define $x \cdot m = m$ or $x \cdot m = x \vee m$, then for $m \in K$, we have

$$h(xm) = xh(m) = x1 = 1 \Rightarrow \; xm \in K,$$

or

$$h(xm) = xh(m) = x \vee 1 = 1 \;\Rightarrow\; xm \in K.$$

But in general, $h(xm) = xh(m) = x \cdot 1$ and it is not equal to 1.

Proposition 3.15. *Let $L_1 = (L_1, \to_1, 1)$ and $L_2 = (L_2, \to_2, 1)$ be two L-algebras and $\psi : L_1 \to L_2$ be an L-homomorphism. If G is an L_2-module, by defining $x \cdot m = \psi(x)m$, where $x \in L_1$ and $m \in G$, then G becomes an L_1-module.*

Proof. Since $\psi(1_{L_1}) = \psi(x \to_{L_1} x) = \psi(x) \to_{L_2} \psi(x) = 1_{L_2}$, we have

$$1_{L_1} \cdot m = \psi(1_{L_1})m = 1_{L_2}m = m.$$

So, (LM_1) holds. Now, suppose $m_1, m_2 \in G$ and $x \in L_1$. Then

$$x \cdot (m_1 + m_2) = \psi(x)(m_1 + m_2) = \psi(x)m_1 + \psi(x)m_2 = x \cdot m_1 + x \cdot m_2.$$

Thus, (LM_2) holds. Finally, for $x, y \in L_1$ and $m \in G$, we have

$$\begin{aligned}(x \to_{L_1} y) \cdot m &= \psi(x \to_{L_1} y)m = (\psi(x) \to_{L_2} \psi(y))m \\ &= m - (\psi(x)m \rightsquigarrow_{L_2} \psi(y)m)^{\sim} = m - (x \cdot m \rightsquigarrow_{L_2} y \cdot m)^{\sim}.\end{aligned}$$

Hence, (LM_3) holds and so G is an L_1-module. □

Definition 3.16. Let L be an L-algebra, I be an ideal of L, G be an L-module, and H be a sub-module of G. Define

$$(H :_G I) = \{m \in G \mid xm \in H, \text{ for any } x \in I\}.$$

Note. We denote annihilator of H by $Ann_L(H)$, which is defined by

$$Ann_L(H) = (0 :_L H) = \{x \in L \mid xH = 0\},$$

where L is a bounded L-algebra.

Example 3.17. (i) According to Example 3.2, let $H = \{0, b\}$, $I = \{a, 1\}$. Then clearly $(H :_G I) = \{0, b\}$.
(ii) Clearly $Ann_L(H)$ is not an ideal of L, in general, since if $Ann_L(H)$ is an ideal of L, then $1 \in Ann_L(H)$, and so $1H = 0$. Hence, $H = \{0\}$.

Proposition 3.18. *Let G be an L-module, H be a sub-module of G and I be an ideal of L. Then $(H :_G I)$ is a sub-module of G, where $x(am) = a(xm)$, for any $x \in I$, $a \in L$ and $m \in G$.*

Proof. Suppose $m_1, m_2 \in (H :_G I)$. Then for any $x \in I$, $xm_1, xm_2 \in H$. By assumption, since H is a sub-module of G, we have $xm_1 - xm_2 \in H$. From G is an L-module we have $x(m_1 - m_2) = xm_1 - xm_2 \in H$. Hence $m_1 - m_2 \in (H :_G I)$. Now, suppose $m \in (H :_G I)$. Then for any $x \in I$, $xm \in H$. Since H is a sub-module of G, for any $a \in L$, we have $a(xm) \in H$. Thus by hypothesis, $x(am) \in H$, and so $am \in (H :_G I)$. Hence, $(H :_G I)$ is a sub-module of G. □

Theorem 3.19. *Let G be an L-module, where L be a chain, and I be an ideal of L such that $am = bm$, for any $a, b \in I$. Define $\cdot : \frac{L}{I} \times G \to G$ by $\cdot([x], m) = xm$. Then G is an $\frac{L}{I}$-module.*

Proof. Let $[x] = [y]$, for $x, y \in L$ and $m \in G$. Since L is a chain, we get $x \leq y$ or $y \leq x$. Without loss of generality, suppose $x \leq y$. By Proposition 3.5(iii), for any $m \in G$, $xm \leq ym$. From $[x] = [y]$, we have $x \to y = 1 \in I$ and $y \to x \in I$, and so by assumption, $(y \to x)m = (x \to y)m$. Hence $m - (ym \rightsquigarrow xm)^\sim = 1 \cdot m = m$, i.e. $(ym \rightsquigarrow xm)^\sim = 0$, and so $ym \rightsquigarrow xm = 1$. Therefore, $ym \leq xm$, and so $xm = ym$. Thus the operation $\cdot$ is well-defined. Now, we prove that G is an $\frac{L}{I}$-module. For this, suppose $x, y \in L$ and $m, m_1, m_2 \in G$. Then
(LM_1) $[1] \cdot m = 1m = m$.
(LM_2) $[x] \cdot (m_1 + m_2) = x(m_1 + m_2) = xm_1 + xm_2 = [x] \cdot m_1 + [x] \cdot m_2$.
(LM_3) $([x] \to [y]) \cdot m = [x \to y] \cdot m = (x \to y)m = m - (xm \rightsquigarrow ym)^\sim = m - ([x] \cdot m \rightsquigarrow [y] \cdot m)^\sim$.
Therefore, G is an $\frac{L}{I}$-module. □

The following example confirms the Theorem 3.19.

Example 3.20. Let $(L = \{0, a, 1\}, \leq)$ be a chain. Define the operation $\to$ on L as follows:

$\to$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

Then $(L, \to, 1)$ is an L-algebra. Set $G = L$ and define the operation $+$ on L as follows:

$+$	0	a	1
0	0	a	1
a	a	0	a
1	1	a	0

According to the above table, inverse of any element is itself, since $x + x = 0$. Hence, $(G, +, 0)$ is an Abelian group. Define $\cdot : L \times G \to G$ where $x \cdot m = m$, for any $x \in L$ and $m \in G$. Then G is an L-module. In addition, $I = \{a, 1\}$ is an ideal of L, and for any $x, y \in I$ and $m \in G$, we have $xm = ym$. Hence, by Theorem 3.19, G is an $\dfrac{L}{I}$-module.

Definition 3.21. Let G be an L-module such that $0m = 0$, for all $m \in G$ and H be a sub-module of G. Then H is called a *prime sub-module of* G, if $H \neq G$ and by $xm \in H$, we have $m \in H$ or $x \in (H :_L G)$.

Example 3.22. Let G be an L-module as in Proposition 3.3 and $\emptyset \neq I \subseteq X$. Then $(P(I), \Delta)$ is a sub-module of G, since for any $A, B \in P(I)$, we have

$$A \Delta B = (A - B) \cup (B - A) \subseteq I,$$

and so $A \Delta B \in P(I)$. Also, for any $C \in P(X)$ and $A \in P(I)$, $C \cdot A = C \cap A \subseteq A \in P(I)$. Hence, $(P(I), \Delta)$ is a sub-module of G. Now, we show that $P(I)$ is a prime sub-module of G. For this, clearly, $P(I) \neq P(X)$. Let $C \cdot B \in P(I)$, for $C \in P(X)$, where $B \notin P(I)$. Since $(P(I), \Delta)$ is a sub-module of G, for any $A \in P(X)$, we get $C \cap A \cap B \in P(I)$. We prove $C \in (P(I) :_L G)$. Assume $C \notin (P(I) :_L G)$. Then $C \cdot G \notin P(I)$, and so for any $A \in G$, $C \cdot A \notin P(I)$. It means $C \cap A \notin P(I)$, for any $A \in G$. Thus, $C \cap A \cap B \subseteq C \cap A \notin P(I)$, which is a contradiction. Hence, $C \in (P(I) :_L G)$. Therefore, $P(I)$ is a prime submodule of G.

Proposition 3.23. *Let G be an L-module such that $0m = 0$, for all $m \in G$ and H be a proper sub-module of G. If H is prime, then for every ideal I of L and sub-module M of G, $IM \subseteq H$ implies that $M \subseteq H$ or $I \subseteq (H :_L G)$.*

Proof. Let H be a prime sub-module of G, $IM \subseteq H$, $M \nsubseteq H$ and $I \nsubseteq (H :_L G)$. Then there exists $m \in M$ such that $m \notin H$ and $x \in I$ such that $x \notin (H :_L G)$. Since $xm \in IM \subseteq H$ and H is prime, then $m \in H$ or $x \in (H :_L G)$ which is a contradiction. □

Proposition 3.24. *Let G_1 and G_2 be two L-modules such that $0m = 0$, for all $m \in G_1$ and $\psi : G_1 \to G_2$ be an L-module epimorphism. If H is a prime sub-module of G_2, then $\psi^{-1}(H)$ is a prime sub-module of G_1.*

Proof. Obviously, by Proposition 3.11(i), $\psi^{-1}(H)$ is a proper sub-module of G_1. Let $xm \in \psi^{-1}(H)$. Then $\psi(xm) = x\psi(m) \in H$, and so $\psi(m) \in H$ or $x \in (H :_L G_2)$, that is $m \in \psi^{-1}(H)$ or $xG_2 \subseteq H$. If $xG_2 \subseteq H$, then $xG_1 \subseteq \psi^{-1}(H)$, since ψ is an L-module epimorphism. Therefore, $\psi^{-1}(H)$ is a prime sub-module of G_1. □

4. Multiplication and co-multiplication L-modules

In this section, we introduce the concept of multiplication and co-multiplication of L-modules and we will represent each sub-module of an L-module by using them.

Definition 4.1. Let G be an L-module. Then G is called a *multiplication L-module*, if for any sub-module H, there exists an ideal I of L such that $H = IG$.

Example 4.2. Let $(G, +)$ be an Abelian group as in Example 3.2 and $L = G$ as an L-algebra. Define the operation $\cdot : L \times G \to G$ by $x \cdot m = x \wedge m$, for any $x \in L$ and $m \in G$. Clearly, G is an L-module. Assume $I = \{b, 1\}$ and $H = \{0, b\}$. Obviously, I is an ideal of L and H is a sub-module of G. By routine calculation, we can prove that $H = IG$. Hence G is a multiplication L-module.

Proposition 4.3. *Let G be an L-module. If G is a multiplication L-module, then $H = (H :_L G)G$, for every sub-module H of G.*

Proof. Assume G be a multiplication module. Then there exists an ideal I of L such that $H = IG$, for any sub-module H of G. Thus $I \subseteq (H :_L G)$, and so $H = IG \subseteq (H :_L G)G$. Conversely, $(H :_L G)G \subseteq H$, because if $a \in (H :_L G)G$, then there exist $x \in (H :_L G)$ and $m \in G$ such that $a = xm$. Since $x \in (H :_L G)$, for any $m \in G$, $xm \in H$, and so $a \in H$. Thus, $H = (H :_L G)G$. □

Proposition 4.4. *Homomorphic image of every multiplication module is a multiplication module.*

Proof. Consider $\psi : G_1 \to \psi(G_1)$ is an L-module epimorphism, G_1 is a multiplication L-module and H is a sub-module of $\psi(G_1)$. Then obviously, $\psi^{-1}(H)$ is a sub-module of G_1. Thus by hypothesis, there exists an ideal I of L such that $\psi^{-1}(H) = IG_1$, and so $H = I\psi(G_1)$. Therefore, $\psi(G_1)$ is a multiplication module. □

Definition 4.5. Assume G is an L-module. Then G is called a *co-multiplication L-module*, if for every sub-module H of G, there exists an ideal I of L such that $H = (0 :_G I)$.

Proposition 4.6. *Consider G be an L-module. If G is a co-multiplication L-module, then $H = (0 :_G Ann_L(H))$, for every sub-module H of G.*

Proof. Suppose G is a co-multiplication L-module and H be a sub-module of G. Then there exists an ideal I of L such that $H = (0 :_G I)$, and so $IH = 0$. Thus, if $m \in H$, then $Ann_L(H)m = 0$, and so $m \in (0 :_G Ann_L(H))$. Hence,

$$H \subseteq (0 :_G Ann_L(H)) \subseteq (0 :_G I) = H.$$

Therefore, $H = (0 :_G Ann_L(H))$. □

Proposition 4.7. *Consider L to be an L-algebra, G be a co-multiplication L-module and suppose that L satisfies the ascending chain condition. Then G satisfies the descending chain condition.*

Proof. Assume $H_1, H_2, \cdots$ are sub-modules of G such that

$$H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots \supseteq H_n \supseteq \cdots,$$

is a descending chain of sub-modules G. Then

$$(0 :_L H_1) \subseteq (0 :_L H_2) \subseteq (0 :_L H_3) \subseteq \cdots \subseteq (0 :_L H_n) \subseteq \cdots,$$

Thus

$$Ann_L(H_1) \subseteq Ann_L(H_2) \subseteq Ann_L(H_3) \subseteq \cdots \subseteq Ann_L(H_n) \subseteq \cdots,$$

which is an ascending chain of L. Hence by assumption, there exists $n \in \mathbb{N}$ such that

$$Ann_L(H_k) = Ann_L(H_n), \text{ for all } k \geq n.$$

By Proposition 4.6, suppose $m \in H_n = (0 :_G Ann_L(H_n))$. Then

$$(Ann_L(H_n))m = 0 \text{ or } (Ann_L(H_k))m = 0, \quad \text{i.e. } \; m \in (0 :_G Ann_L(H_k)) = H_k.$$

Hence $H_k = H_n$, for all $k \geq n$. Therefore, G satisfies the descending chain condition. □

5. Conclusions and future works

In this paper, we continued the study of L-algebras, begun by Rump [12]. The main goal of this work was to introduce L-modules and we did this by introducing action of L-algebras on groups.

In classical ring theory, modules, multiplication and comultiplication modules are studied. We defined these concepts on L-modules and investigated some results. Also, we defined L-algebras of fractions and L-modules of fractions and studied some of their properties.

6. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

References

[1] A. Abdollahi, S. Janbaz and M. Oubodi, *Graphs cospectral with a friendship graph or its complement*, Trans. Combin., **2** No. 4 (2013) 37-52.

[2] M. Aaly Kologani, *On (semi)topology L-algebras*, CGASA, **18** No. 1 (2023) 81-103.

[3] M. Aaly Kologani, *Relation between L-algebras and other logical algebras*, JAHLA, **4** No. 1 (2023) 27-46.

[4] M. Aaly Kologani, *Results on L-algebras*, Soft Comput., **27** (2023) 13765-13777.

[5] R. A. Borzooei and S. Saidi Goraghani, *Free MV-modules*, J. Intell. Fuzzy Syst., **31** No. 1 (2016) 151-161.

[6] A. Di Nola, P. Flondor and I. Leustean, *MV-modules*, J. Algebra., **267** (2003) 21-40.

[7] V. G. Drinfeld, *On Some Unsolved Problems in Quantum Group Theory*, In P.P. Kulish (Ed.), Quantum Groups (Leningrad, 1990), Lecture Notes in Mathematics, Vol. 1510, Springer, Berlin, 1992.

[8] C. Flaut and D. Piciu, *Connections between commutative rings and some algebras of logic*, Iran. J. Fuzzy Syst., **19** No. 6 (2022) 93-110.

[9] M. Kondo, *Note on quotient structures of CKL-algebras*, JAHLA, **6** No. 1 (2025) 159-170.

[10] M. Mohseni Takallo, *Block code on L-algebras*, JAHLA, **4** No. 1 (2023) 47-60.

[11] W. Rump, *A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation*, Adv. Math., **193** (2005) 40-55.

[12] W. Rump, *L-algebras, self-similarity, and ℓ-groups*, J. Algebra., **320** (2008) 2328-2348.

[13] W. Rump, *A general Glivenko theorem*, **61** (2009) 455-473.

[14] W. Rump, Y. Yang, *Interval in ℓ-groups as L-algebras*, Algebra Universalis., **67** No. 2 (2012) 121-130.

[15] S. Saidi Goraghani and R. A. Borzooei, *On injective MV-modules*, Bull. Sect. Log., **47** No. 4 (2018) 283-298.

[16] S. Saidi Goraghani and R. A. Borzooei, *L-Modules*, Bull. Sect. Log., **53** No. 1 (2024) 125-144.

[17] S. Saidi Goraghani, R. A. Borzooei and S. S. Ann, *New definition of MV-semimoduls*, J. Intell. Fuzzy Syst., **41** (2021) 3715-3726.

[18] S. Saidi Goraghani, R. A. Borzooei, S. S. Ahn and Y. B. Jun, *New Kind of MV-Modules*, Int. J. Comput. Intell., **13** No. 1 (2020) 794-801.

[19] Y. L. Wu, J. Wang and Y. C. Yang, *Lattice-ordered effect algebras and L-algebras*, Fuzzy Sets Syst., **369** (2019) 103-113.

[20] Y. L. Wu and Y. C. Yang, *Orthomodular lattices as L-algebras*, Soft Comput., **24** (2020) 14391-14400.

[21] X. L. Xin, X. F. Yang and Y. C. Ma, *Pseudo L-algebras*, Iran J. Fuzzy Syst., **19** No. 6 (2022) 61-73.

[22] B. Zeighami, M. Mohseni Takallo, M. Aaly Kologani, S. S. Ahn and R. A. Borzooei, *Ideals of roughness in L-algebras*, JAHLA, **5** No. 1 (2024) 63-86.

Mona Aaly Kologani

Hatef Higher Education Institute,

Zahedan, Iran.

mona4011@gmail.com

Mohammad Mohseni Takallo

Department of Mathematics,

Faculty of Mathematical Sciences,

Shahid Beheshti University,

Tehran, Iran.

mohammad.mohseni1122@gmail.com

Rajab Ali Borzooei

Department of Mathematics,

Soft Computing and Artificial Intelligence Center,

Faculty of Mathematical Sciences,

Shahid Beheshti University,

Tehran, Iran.

borzooei@sbu.ac.ir