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Research Paper

EXPLORING SOME MODULES THROUGH INCLUSION HYPERGRAPHS

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Abstract. Recent studies have shown that hypergraphs are useful in solving real-life problems. Hypergraphs have been successfully applied in various fields. Inspired by the importance, we introduce a new hypergraph assigned to a given module. In particular, vertices of this hypergraph (which we call inclusion hypergraph, denoted by $InH_R(M)$) are all nontrivial submodules of a module M and a subset E of the vertices is a hyperedge in case each two elements of E are comparable by inclusion and E is maximal with respect to this condition. We prove that the inclusion hypergraph of an R-module M is disconnected if and only if M can be written as a direct sum of its each two nontrivial submodules. The diameter of $InH_R(M)$ is shown to be at most 3.

1. Introduction

Based on [4, 5], a hypergraph $\mathcal{H}$ is composed of a set of vertices $V = \{v_1, \ldots, v_n\}$ and a collection of hyperedges $E = \{E_j \mid 1 \leq j \leq m\}$. Each hyperedge is a non-empty subset of

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vertices, and the union of all hyperedges equals the entire vertex set V. This means that in a hypergraph, each hyperedge links two or more vertices. There are several works related to various graphs and fuzzy graphs ([17], [24]).

Hypergraphs generalize traditional graphs by allowing edges called hyperedges to connect any number of vertices, rather than just pairs. In graphs, vertices typically represent individual elements of a set, and edges are limited to pairs of these elements. In contrast, hyperedges can represent subsets of any size or, more broadly, express complex relationships involving arbitrary subsets. Due to this flexibility, hypergraphs serve as powerful tools for modeling intricate structures and interactions among nodes. Over the last ten years, research has demonstrated the effectiveness of hypergraphs in addressing practical problems across diverse domains. They have found successful applications in areas such as network analysis, data structuring, engineering system modeling, cellular communications, image processing, machine learning, data mining, soft set theory, social network analysis, chemistry, and beyond. Notable studies in these applications include works such as [21], [11], [15], [7, 8, 9, 10], [12], [3], [16], [6, 13, 22], and [23].

From a theoretical standpoint, hypergraphs provide a natural extension of numerous graph theory results, often consolidating various graph theorems into a single statement within the hypergraph framework. For instance, Berge s weak perfect graph conjecture which characterizes a graph as perfect if and only if its complement is also perfect was established through the use of normal hypergraphs. On the practical side, hypergraphs are becoming more prominent than traditional graphs because they offer enhanced expressive capabilities. In graph theory, complete graphs play a fundamental role. Addressing this, the notions of co-intersection hypergraphs [19] and intersection hypergraphs [20] were introduced to describe particular classes of modules. These studies also quantified the number of complete subgraphs found in the co-intersection and intersection graphs constructed on modules. Additionally, in [14], the authors proposed a novel hypergraph defined on a module M, called the sum hypergraph. Its vertices consist of all nontrivial submodules of M, and a subset A of these vertices forms a hyperedge if the sum of any two elements in A equals M, with A being maximal under this condition. The authors characterized some semisimple modules via sum hypergraphs.

In the past twenty years, algebraic graph theory has attracted significantly increased interest from researchers (see [1, 2]). Recently, Mahdavi and Talebi defined the inclusion graph of submodules of a module M, denoted by $In(M)$ (see [18]). The graph's vertices consist of all nontrivial submodules of M, with two distinct vertices N and K connected if one is contained within the other. They investigated properties such as the connectivity, girth, and diameter of $In(M)$ ([18]).

In this work, we focus on the inclusion hypergraph associated with the submodules of a module M, which serves as a valuable tool for exploring the structure of certain module classes. Our goal is to gain deeper insights into modules by examining their corresponding inclusion hypergraphs. Alongside this, we also investigate fundamental characteristics of these hypergraphs, such as connectedness, diameter, and independence number. Notably, we establish that the inclusion hypergraph of an R-module M is null if and only if M can be expressed as a direct sum of every pair of its nontrivial submodules.

Note throughout the text, R denotes an associative ring with identity $1 \neq 0$ and all modules will be assumed as unitary right R-modules.

2. **Inclusion hypergraph of submodules of a module**

Inclusion graph of submodules of a module has been introduced and studied in [18]. Let M be a module. Then $In_R(M)$ is a simple undirected graph where the vertices are all nontrivial submodules of M and two distinct vertices N and K are adjacent in case $N \subset K$ or $K \subset N$. Inspired by [18], we shall introduce a new hypergraph on a module via inclusion. Note that the inclusion graph of a module M is itself the inclusion hypergraph of M.

Definition 2.1. Suppose that M is an R-module. Then the inclusion hypergraph on M, which we denote by $InH_R(M)$, can be defined in the same way as the intersection hypergraph on M. In this way, the set of all nontrivial submodules of M is the set of vertices and a set A of some vertices of $InH_R(M)$ forms a hyperedge provided each pair of elements of A is strictly comparable by inclusion and A is maximal with respect to this property.

Note that by the definition, any hyperedge in $InH_R(M)$ forms a chain of submodules of M, and the converse also holds. In fact, any hyperedge of $InH_R(M)$, introduces a complete subgraph of $In_R(M)$.

Throughout this manuscript, we consider modules with at least two nontrivial submodules.

The first challenge is to determine when the inclusion hypergraph contains an isolated vertex. We address this problem.

Theorem 2.2. *Let M be a module. Then the following statements are equivalent:*

(1) *$InH_R(M)$ has an isolated vertex N;*

(2) *N is a simple maximal submodule of M;*

(3) *For each nontrivial submodule K of M distinct from N, we have $M = N \oplus K$;*

(4) *M can be written as a direct sum of every pair of its nontrivial submodules;*

(5) *The hypergraph $InH_R(M)$ is null.*

Proof. (1) $\Rightarrow$ (2) Suppose that N is an isolated vertex in $InH_R(M)$. If N is strictly contained in a proper submodule of M, namely K, then $\{N, K\}$ will be contained in a hyperedge of

$InH_R(M)$, while N is isolated. This shows that N is a maximal submodule of M. In other words, if N contains a nontrivial submodule T of M, then N can not be isolated. In fact, N is a simple maximal submodule.

(2) $\Rightarrow$ (3) Consider an arbitrary submodule K of M distinct from N. Then $N \cap K = \{0\}$, as N is simple. Being N a maximal submodule of M implies $N + K = M$, hence $M = N \oplus K$.

(3) $\Rightarrow$ (4) The assumption implies each submodule of M is simple. Hence for each pair of nontrivial submodules D and T of M, we have $M = D \oplus T$.

(4) $\Rightarrow$ (5) If $InH_R(M)$ contains a hyperedge E_i where $\mid E_i \mid \geq 2$, then for each pair of elements of E_i we must have a direct decomposition. This provide a contradiction.

(5) $\Rightarrow$ (1) Obvious. □

Recall that by $l_R(M)$, we mean the length $l_R(M)$ of the R-module M. In other words, we say M has length $n \in \mathbb{N}$, provided n is the length of the largest chain of submodules of M. If no such largest chain exists, then $l_R(M) = \infty$. From the definition of a hyperedge in $InH_R(M)$, we can say $l_R(M) = \mid E_m \mid +1$ where E_m is a hyperedge with maximum number of elements.

Following Theorem 2.2, $InH_R(M)$ is null if and only if M is a semisimple module with length 2.

Example 2.3. (1) The hypergraph $InH_{\mathbb{Z}}(\mathbb{Z}_n)$ is null if and only if $n = pq$ where p and q are two distinct prime numbers.

(2) The inclusion hypergraph of the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is null where p is a prime.

Recall that a module M is uniserial if its submodules are linearly ordered by inclusion. We next characterize modules M for which $InH_R(M)$ has just one hyperedge containing all vertices.

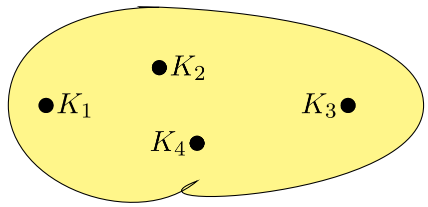
Proposition 2.4. *The hypergraph $InH_R(M)$ has exactly one hyperedge containing all vertices if and only if M is uniserial.*

Proof. Suppose that $InH_R(M)$ has exactly one hyperedge containing all vertices. It follows that any two submodules of M are comparable by inclusion. Hence, M is uniserial. The converse is obvious. □

From Proposition 2.4, the inclusion hypergraph of the $\mathbb{Z}$-modules $\mathbb{Z}_{p^n}$ $(n \geq 2)$ and $\mathbb{Z}_{p^\infty}$ have just one hyperedge.

Example 2.5. Let p be a prime. Now, consider $\mathbb{Z}$-module $M = \mathbb{Z}_{p^5}$. Then $K_1 =< p >$, $K_2 =< p^2 >$, $K_3 =< p^3 >$ and $K_4 =< p^4 >$ are all nontrivial submodules of M. As M

is uniserial, we have $V = \{K_1, K_2, K_3, K_4\}$, $E = \{\{K_1, K_2, K_3, K_4\}\}$ and the hypergraph $InH_{\mathbb{Z}}(M)$ has the following form:



We shall investigate, when $InH_R(M)$ is connected.

Theorem 2.6. *For an R-module M, the hypergraph $InH_R(M)$ is disconnected if and only if $InH_R(M)$ is null.*

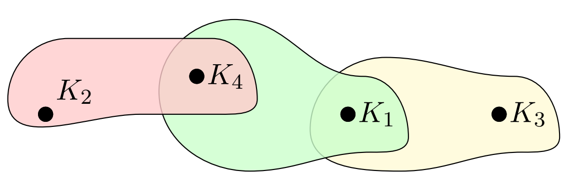
Proof. Let $InH_R(M)$ be disconnected. To the contrary, suppose that there is no isolated vertices ($InH_R(M)$ has at least one hyperedge). Then we can assume that there are two vertices A and C in $InH_R(M)$ such that there are no paths between them (note that if $InH_R(M)$ contains an isolated vertex, then it must be null by Theorem 2.2). By the definition, there are two hyperedges E_1 and E_2 with at least two elements. Suppose $A, B \in E_1$ and $C, D \in E_2$. Now, consider the submodule $A \cap C$. Since, there is no path between A and C so $A \cap C \neq A$ and $A \cap C \neq C$. As $A \cap C$ is contained in both A and C, then by assumption $A \cap C = \{0\}$. By same argument, $A + C = M$. In fact, $A \oplus C = M$. Hence we will have $M = A \oplus C = A \oplus D = B \oplus C = B \oplus D$. Note also that $A \subset B$ or $B \subset A$. If each of these two cases holds, then $A = B$ a contradiction. Therefore, $InH_R(M)$ is null. Other side is obvious. □

The following is immediate from Theorems 2.2 and 2.6.

Corollary 2.7. *The hypergraph $InH_R(M)$ is disconnected if and only if M can be written as a direct sum of its each two nontrivial submodules.*

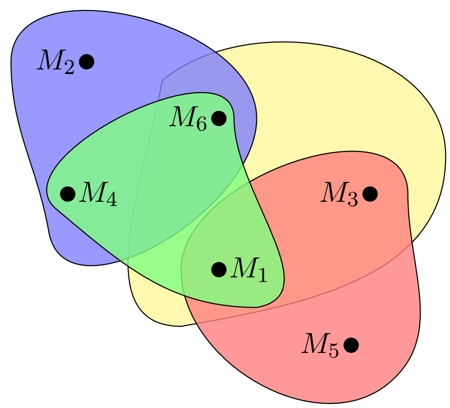
A hypergraph $\mathcal{H}$ is said to be k-uniform in case the cardinal of all hyperedges of $\mathcal{H}$ is k. The following includes a module M where $InH_R(M)$ is 2-uniform.

Example 2.8. Suppose that $M = \mathbb{Z}_{p^2q}$ as an $\mathbb{Z}$-module where p, q are distinct prime numbers and $p < q$. The list of all nontrivial submodules of M is $K_1 =< pq >$, $K_2 =< p^2 >$, $K_3 =< q >$ and $K_4 =< p >$. Then $InH_{\mathbb{Z}}(M)$ has three hyperedges $E_1 = \{K_1, K_3\}$, $E_2 = \{K_1, K_4\}$ and $E_3 = \{K_2, K_4\}$.



The following example introduces a module M such that $InH_R(M)$ is a 3-uniform hypergraph.

Example 2.9. Let $M = \mathbb{Z}_{p^3q}$ as an $\mathbb{Z}$-module. All nontrivial submodules are $M_1 =< p^2q >$, $M_2 =< p^3 >$, $M_3 =< pq >$, $M_4 =< p^2 >$, $M_5 =< q >$ and $M_6 =< p >$. Then $InH_{\mathbb{Z}}(M)$ has four hyperedges $E_1 = \{M_2, M_4, M_6\}$, $E_2 = \{M_1, M_3, M_5\}$, $E_3 = \{M_1, M_3, M_6\}$, $E_4 = \{M_1, M_4, M_6\}$. The corresponding inclusion hypergraph has the following figure:



Note that a Berge's cycle in a hypergraph is a sequence $x - E_1 - y_1 - \ldots - y_n - E_n - x$ where $x, y_1, \ldots, y_n$ are distinct vertices and $E_1, \ldots, E_n$ are distinct hyperedges. The girth of a hypergraph is the length of a shortest Berge's cycle if such a cycle exists. Unlike graphs, the girth of a hypergraph can be 2. This happens when there exist two distinct hyperedges with at least two common elements. If there is no Berge's cycle, then the girth is defined to be infinite. We try to compute the girth of $InH_R(M)$.

Theorem 2.10. *For an R-module M, assume that $InH_R(M)$ contains a Berge's cycle. Then $gr(InH_R(M))$ is either 2 or 5.*

Proof. Assume that $InH_R(M)$ has at least two distinct hyperedges E_i and E_j such that $| E_i \cap E_j | \geq 2$. Set $A, B \in E_i \cap E_j$. Then $A - E_i - B - E_j - A$ is a cycle. Otherwise, suppose the intersection of each two distinct hyperedges of $InH_R(M)$ has at most one element. Now assume $A - E_i - B - E_j - C - E_t - A$ is a Berge's cycle with length 3. Now, eight cases can occur:

(1) $A \subset B, B \subset C, C \subset A$;
(2) $A \subset B, B \subset C, A \subset C$;
(3) $A \subset B, C \subset B, A \subset C$;
(4) $A \subset B, C \subset B, C \subset A$;
(5) $B \subset A, B \subset C, C \subset A$;

(6) $B \subset A, B \subset C, A \subset C$;

(7) $B \subset A, C \subset B, A \subset C$;

(8) $B \subset A, C \subset B, C \subset A$.

Some of the above cases are impossible and some of them introduce a chain.

Hence, $InH_R(M)$ can not include a cycle with length 3. Applying same arguments, $InH_R(M)$ can not include cycles of length 4 or $6, 7, \ldots$.

Therefore, the girth of $InH_R(M)$ is either 2 or 5. □

A subset I of $V(\mathcal{H})$ where $\mathcal{H}$ is a hypergraph, is said to be independent if for any two elements A, B in I, there does not exist a hyperedge including them. The independent number of a hypergraph $\mathcal{H}$ (denoted by $\alpha(\mathcal{H})$) is the cardinal number of a maximal independent set of $\mathcal{H}$.

Proposition 2.11. *Let M be an R-module. Then:*

(1) *Both the sets of minimal and maximal submodules of M are independent in $InH_R(M)$.*

(2) $\alpha(InH_R(M)) \geq max\{\mid Max(M) \mid, \mid Min(M) \mid\}$.

Proof. (1) This follows from the fact that any two maximal (minimal) submodules of M cannot be comparable by inclusion.

(2) Can be derived from (1). □

Recall that in a hypergraph $\mathcal{H}$, the distance between two distinct vertices N and K is the length of the shortest path between them. In this way, diameter of $\mathcal{H}$ is defined as $diam(\mathcal{H}) = max\{d(N, K) \mid N, K \in V(\mathcal{H})\}$. It can be of interest to compute the diameter of $InH_R(M)$.

Theorem 2.12. *Let M be an R-module such that $InH_R(M)$ is connected. Then $diam(InH_R(M)) \leq 3$.*

Proof. Let N and K be two nontrivial submodules of M including two distinct hyperedges E_i and E_j. Consider $N \cap K$. If $N \cap K$ is a nonzero submodule of M, then $N - E_i - (N \cap K) - E_j - K$ is a path with length 2. Otherwise, assume $N \cap K = \{0\}$. Now, we can consider $N + K$ as a submodule of M. Here, we shall check two possibilities. One case is $N + K \neq M$. Hence, $N - E_i - (N + K) - E_j - K$ will be a path. Suppose that $N + K = M$. In this way, $M = N \oplus K$. If both N and K are maximal submodules of M, then both are simple submodules of M implying that $InH_R(M)$ has isolated vertex and consequently $InH_R(M)$ is null. Therefore, one of N and K is not maximal; assume N is not maximal. Then N is contained in another proper submodule L of M. Now, L and K have two possibilities: either they are comparable by inclusion, or they are not. If first case happens, there is the path $N - E_i - L - E_j - K$. Otherwise, we have two following cases:

Case 1: $K \cap L = \{0\}$. Then $L = L \cap (N \oplus K) = N + (K \cap L) = N$, which is a contradiction.

Case 2: $K \cap L \neq \{0\}$. Then there exists another hyperedge including $K \cap L$ and L, say E_s. Therefore, $K - E_j - (K \cap L) - E_s - L - E_i - N$ is a path with length 3. □

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