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Research Paper

RATIONAL NUMBERS WHOSE SUM AND PRODUCT ARE BOTH INTEGERS

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ABSTRACT. For $n > 2$, there are noninteger rational numbers r_i, $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} r_i$ and $\prod_{i=1}^{n} r_i$ are both integers. In this paper, while proving this proposition and with the help of the fundamental theorem of arithmetic and the properties of prime numbers, we will present an algorithm to generate numbers with this feature. Finally, we present a modified version of the original problem that requires further exploration.

1. Introduction

A satisfactory discussion of the main concepts of mathematics must be based on an accurately defined number concept. The number theory, sometimes called higher arithmetic, it is among the oldest and most natural of mathematical pursuits. Number theory is a branch of pure mathematics devoted primarily to the study of the integers and arithmetic functions.

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Number theorists study prime numbers as well as the properties of mathematical objects constructed from integers (for example, rational numbers), or defined as generalizations of the integers (for example, algebraic integers).

Integers can be considered either in themselves or as solutions to equations (Diophantine geometry). Questions in number theory can often be understood through the study of analytical objects, such as the Riemann zeta function, that encode properties of the integers, primes or other number-theoretic objects in some fashion (analytic number theory). One may also study real numbers in relation to rational numbers, as for instance how irrational numbers can be approximated by fractions (Diophantine approximation).

The main problem in this paper is the existence or non-existence of a solution for a particular Diophantine equation. The problem's origin is from the theory of Diophantine equations, and there are a remarkable number of related results in literature, more precisely, considering the non-existence side of the problem. A Diophantine equation is an equation, typically a polynomial equation in two or more unknowns with integer coefficients, for which only integer solutions are of interest. A linear Diophantine equation equates the sum of two or more unknowns, with coefficients, to a constant. An exponential Diophantine equation is one in which unknowns can appear in exponents.

Diophantine problems have fewer equations than unknowns and involve finding integers that solve all equations simultaneously. Because such systems of equations define algebraic curves, algebraic surfaces, or, more generally, algebraic sets, their study is a part of algebraic geometry that is called Diophantine geometry.

The word Diophantine refers to the Hellenistic mathematician of the 3rd century, Diophantus of Alexandria, who made a study of such equations and was one of the first mathematicians to introduce symbolism into algebra. The mathematical study of Diophantine problems that Diophantus initiated is now called Diophantine analysis.

While individual equations present a kind of puzzle and have been considered throughout history, the formulation of general theories of Diophantine equations, beyond the case of linear and quadratic equations, was an achievement of the twentieth century.

References [2], [3], [5] and [12] are good examples in this area. To better understand, review the details provided in [6], [7], [8], [9] and [11] for more details.

2. Main results

In this paper, we give a very elementary, self-contained answer of the following interesting question in number theory.

Question 1. *Are there noninteger rational numbers r_i, finite or infinite, such that $\sum_i r_i$ and $\prod_i r_i$ are both integers?*

We start with two numbers that we need to remember the following. Let us recall some definitions. In mathematics, Gauss's lemma,[1] is a lemma referring to polynomials over $\mathbb{Z}$. Gauss's lemma is the basis of all the theories of decomposition and divisibility of such polynomials.

A corollary of Gauss's lemma, sometimes also called Gauss's lemma, is that a primitive polynomial(a polynomial over $\mathbb{Z}$ that no prime number divides all the coefficients) is irreducible over the integers if and only if it is irreducible over the rational numbers. More generally, a primitive polynomial has the same complete factorization over $\mathbb{Z}$ and over $\mathbb{Q}$.

Now, by way of contradiction argument, suppose that there exist such numbers, i.e., there are $r,\ s \in \mathbb{Q} \setminus \mathbb{Z}$ such that $r+s$ and rs are integers. Then $p(x) = (x-r)(x-s)$ is primitive polynomial in $\mathbb{Z}[x]$ with roots in $\mathbb{Q} \setminus \mathbb{Z}$, which contradicts with Gauss's lemma.

For triples and quadruplets, etc., we can find an answer to the Question 1 with the help of the basic theorem of arithmetic and properties of prime numbers. For example, for triples, with prime numbers 2 and 3, it is as follows:

$$\frac{2^2}{3}, \frac{3^2}{2}, \frac{1}{2\times 3}.$$

For triples we refer to a recent paper, but for triples, the problem has different dimensions and has a long history and is not so easy:
In 1997, Garaev [4] proved that if k is of the form $4n, 8n-1$ *or* $2^{2m+1}(2n-1)+3$, where $n, m \in \mathbb{N}$, then there are no positive rational numbers p, q, r satisfying

$$pqr = 1, p+q+r = k.$$

Also, Tho [12] proved that: if $k \in \mathbb{N}$ is an odd number and if either $m \equiv 0 \pmod 4$ or $m \equiv 7 \pmod 8$.
Then there are no positive rational numbers p, q, and r so that

$$pqr = k, p+q+r = km.$$

A representative example of our result is the following statement: assume that $k, m \in \mathbb{N}$ are such that at least one of the following conditions hold:

- $m \equiv 0 \pmod 4$,
- $m \equiv 7 \pmod 8$,
- $m \equiv 0 \pmod 4$,
- $m \equiv 0 \pmod 2$ and $n \equiv 3 \pmod 4$,
- $k^2 m^3 = 2^{2n+1}(2l-1) + 27$ *for some* $l, n \in \mathbb{N}$.

Then the system of equations

$$pqr = k, p + q + r = km.$$

has no solutions in positive rational numbers p, q, r [5].

However, it is not enough to study this for two and three rational numbers. Since we doubted that the answer to Question 1 would be positive for $n > 3$, we found counterexamples, e.g. These four:

$$\frac{(4k+1)^2}{2}, \frac{1}{2}, \frac{4k}{4k+1}, \frac{1}{4k+1}.$$

The sum is clearly an integer $8k^2 + 4k + 2$, the product is clearly 1. Also, these quintuplets:

$$\frac{(5k-1)^3}{5}, \frac{1}{5}, \frac{5k}{5k-1}, \frac{5}{5k-1}, \frac{-6}{5k-1}.$$

The sum is clearly an integer $25k^3 - 15k^2 + 3k + 1$, the product is clearly $-6k$. Also, these sixtuples:

$$\frac{(6k-1)^4}{3}, \frac{-1}{6k-1}, \frac{1}{6k-1}, \frac{-1}{3}, \frac{3k}{6k-1}, \frac{3k-3}{6k-1}.$$

The sum and product are integers.
For infinite, suppose $r_i = \frac{1}{i(i+1)}$, so

$$\sum_{i=1}^{\infty} r_i = 1, \qquad 0 \leq \prod_{i=1}^{\infty} r_i \leq \prod_{i=1}^{\infty} \frac{1}{2} = 0.$$

Also, if $r_i = \frac{1}{2^i}$, so

$$\sum_{i=1}^{\infty} r_i = 1, \qquad 0 \leq \prod_{i=1}^{\infty} r_i \leq \prod_{i=1}^{\infty} \frac{1}{2} = 0.$$

To complete the discussion in an infinite state, let us recall some definitions. In the sense of Blaschke product, the product of previous examples is divergence. A sequence $\{a_n\}$ of complex numbers whose $|a_n| < 1$ satisfies the Blaschke condition if

$$\sum_n (1 - |a_n|) < \infty. \tag{1}$$

The Blaschke product for the sequence $\{a_n\}$ that satisfy to the condition (1) is defined as:

$$B(z) = \prod_n B(a_n, z),$$

where $B(a_n, z) = \dfrac{|a_n|}{a_n} \dfrac{a_n - z}{1 - \overline{a_n} z}$ provided $a_n \neq 0$. Here $\overline{a_n}$ is the complex conjugate of a_n. Take $B(0, z) = z$.

The Blaschke product is a holomorphic function on the unit disk, which becomes zero only at the a_n and is bounded in a neighborhood of the unit disk, i.e. $B(z) \in H^{\infty}$. For more details see [10].

A sequence of complex numbers that satisfies relation (1) is called a Blashke sequence. With the condition relation (1), it is not possible for both

$$\sum_{i=1}^{\infty} r_i, \quad \prod_{i=1}^{\infty} r_i,$$

to converge at the same time.

According to the above explanations, it is better to modify question 1 as follows to make the topic interesting. The modified question asks: Are there positive noninteger rational numbers r_i such that $\sum_i r_i$ and $\prod_i r_i$ are both integers?

From all the issues raised, it can be conjecture that: for $n > 2$, there are noninteger rational numbers $r_i,\ i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} r_i$ and $\prod_{i=1}^{n} r_i$ are both integers.

If n is even and $n > 5$, then the following numbers

$$\frac{(4k+1)^2}{2}, \frac{2^{n-2}k}{4k+1}, \frac{1}{4k+1}, \frac{1}{2}, \ldots, \frac{1}{2}.$$

confirm the conjecture, when the numbers of $\frac{1}{2}$ is $n-3$.

Also, for example (there are many),

- $\frac{k}{3}, \frac{8k}{3}, \frac{k}{2}, \frac{9k}{2}$, for k with $\gcd(k, 6) = 1$,
- $\frac{k}{3}, \frac{4k}{3}, \frac{7k}{3}, \frac{k}{2}, \frac{27k}{2}$, for k with $\gcd(k, 6) = 1$,
- $\frac{k}{3}, \frac{2k}{3}, \frac{4k}{3}, \frac{5k}{3}, \frac{k}{2}, \frac{81k}{2}$, for k with $\gcd(k, 6) = 1$,
- $\frac{k}{3}, \frac{2k}{3}, \frac{4k}{3}, \frac{16k}{3}, \frac{7k}{3}, \frac{k}{2}, \frac{243k}{2}$ for k with $\gcd(k, 6) = 1$.

Takes care of the cases $4 \leq n \leq 7$. Then by combining these (eg, taking two collections of the form for $n = 4$ above to get $n = 8$), one can find rational numbers (even distinct rational numbers) for all $n > 3$.

If n is odd and $n > 5$, then the following numbers

$$\frac{k}{2}, \frac{k}{3}, \frac{7k}{3}, \frac{27k}{2}, \frac{2^{n-3}k}{3}, \frac{1}{2}, \ldots, \frac{1}{2}.$$

for $\gcd(k, 6) = 1$, confirm the conjecture, when the numbers of $\frac{1}{2}$ is $n-5$.

Using the properties of prime numbers, it is possible to prove the existence of the answer to Question 1. In other words, if $p_1, p_2, p_3, \ldots, p_n$ are the first n prime numbers for $n \geq 2$, and

$$q_i = \prod_{1 \leq k \leq n, k \neq i} p_k, \quad i = 1, 2, 3, \ldots, n,$$

then, the $n+1$ pure rational numbers

$$\frac{p_1 p_2 \ldots p_n - r}{p_1 p_2 \ldots p_n}, \frac{p_1^n}{q_1}, \frac{p_2^n}{q_2}, \ldots, \frac{p_n^n}{q_n},$$

where, r is the remainder of divided by $p_1^{n+1} + p_2^{n+1} + \ldots + p_n^{n+1}$ by $p_1 p_2 \ldots p_n$ is an answer to Question 1 and it follows that the following main proposition:

Proposition 2.1. *For $n > 2$, there are non-integer rational numbers $r_i,\ i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} r_i$ and $\prod_{i=1}^{n} r_i$ are both integers.*

Conclusion. Just as the number 2 is an exception in many topics, including, the only even prime number, the first basis for counting and classifying objects, the basis of formal valuation or formal logic, and the basis of computer programming, $n = 2$ is also an exception in the Question 1.

Considering the problem from a different perspective, or ”another angle,” can involve reframing the subject to gain a broader understanding and challenge existing assumptions. This approach lead to innovative thinking. The problem of finding non-integer rational numbers $r_i,\ i = 1, 2, \ldots, n$, $n > 2$, such that $\sum_{i=1}^{k} r_i$ and $\prod_{i=1}^{k} r_i$ are both integers for all k, $3 \leq k \leq n$, is invalid because the sum of a non-integer rational number and an integer is not an integer. The modified question asks:

Question 2. *Is there the permutations ϕ of the set $\{1, 2, 3, \ldots, n\}$ such that $\sum_{i=1}^{n} r_i$ and*

$$\prod_{i\in\{\phi(1),\ldots,\phi(k)\}} r_i,$$

are integers for all k, $3 \leq k \leq n$?

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