Algebraic Structures and Their Applications
Yazd University

Algebraic Structures and Their Applications Vol. X No. X (20XX) pp XX-XX.

Research Paper

ON THE EXTENSION FUNCTORS OF CERTAIN MODULES OF SMALL DIMENSION

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Abstract. Let R be a commutative Noetherian ring with nonzero identity, I an ideal of R, and M an R-module such that $\operatorname{Ext}^i_R(R/I, M)$ is minimax for all $i \leq \dim M$. We prove that if $\operatorname{Supp}_R(M) \subseteq V(I)$, then for every finitely generated R-module N with $\dim N/IN \leq 1$, the R-module $\operatorname{Ext}^i_R(N, M)$ is I-cominimax for all $i \geq 0$. In particular, for a finitely generated R-module N with $\operatorname{Supp}_R(N/IN) \subseteq \operatorname{Max}(R)$, we show that $\operatorname{Ext}^i_R(N, M)$ is Artinian and I-cofinite for all $i \geq 0$.

1. Introduction

Let R be a commutative Noetherian ring with identity, I an ideal of R, and M and N two R-modules. To provide a counterexample to Grothendieck's conjecture [15], Hartshorne [16]

DOI: 10.22034/as.2025.21182.1703

MSC(2010): Primary: 05C50.

Keywords: Arithmetic rank, Cominimax modules, Local cohomology modules, Minimax modules, Weakly cofinite modules.

Received: 26 January 2024, Accepted: 20 September 2025.

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introduced the concept of cofiniteness with respect to an ideal. An R-module M is said to be I-*cofinite* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, M)$ is finitely generated for all j.

One of the interesting problems in commutative algebra is determining when the R-modules $\operatorname{Ext}^i_R(M, N)$ and $\operatorname{Tor}^R_i(M, N)$ are I-cofinite. Several papers are devoted to this problem; see [1, 4, 7, 10, 11, 19, 26].

In [29], Zöschinger introduced the class of minimax modules and provided equivalent conditions for a module to be minimax in [29] and [30]. An R-module M is called *minimax* if there exists a finitely generated submodule N of M such that M/N is Artinian. The concept of I-cominimax modules was introduced in [6] as a generalization of I-cofinite modules. An R-module M is said to be I-*cominimax* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, M)$ is minimax for all j. Since minimax modules naturally generalize finitely generated modules, many authors have studied the cominimaxness of extension and torsion modules; see [3, 5, 17, 18, 20].

In this paper, we continue studying minimaxness and cominimaxness of extension modules and extend certain results from [7].

According to [7, Theorem 2.2], if M is an I-cofinite R-module, then for every finitely generated R-module N with $\dim N/IN \leq 1$, the R-module $\operatorname{Ext}^i_R(N, M)$ is I-cofinite for all $i \geq 0$. One of the main goals of this paper is to generalize this result to the class of minimax modules. More precisely, we prove the following theorem in Theorem 3.3. Compare also with [17, Theorem 3.2].

Theorem 1.1. *Let I be an ideal of R and let M be an R-module of dimension n such that* $\operatorname{Supp}_R(M) \subseteq V(I)$ *and* $\operatorname{Ext}^i_R(R/I, M)$ *is minimax for all $i \leq n$ (note that M is not necessarily I-cominimax). Then for every finitely generated R-module N with $\dim N/IN \leq 1$, the R-module* $\operatorname{Ext}^i_R(N, M)$ *is I-cominimax for all $i \geq 0$.*

In this direction, we improve [7, Lemma 2.1], which states that if M is an I-cofinite R-module, then for every finitely generated R-module N with $\operatorname{Supp}_R(N/IN) \subseteq \operatorname{Max}(R)$, the R-module $\operatorname{Ext}^i_R(N, M)$ is Artinian and I-cofinite for all $i \geq 0$. In particular, in Theorem 3.2, we show the following:

Theorem 1.2. *Let I be an ideal of R and let M be an R-module of dimension n such that* $\operatorname{Supp}_R(M) \subseteq V(I)$ *and* $\operatorname{Ext}^i_R(R/I, M)$ *is minimax for all $i \leq n$ (not necessarily I-cofinite or I-cominimax). Then for every finitely generated R-module N with* $\operatorname{Supp}_R(N/IN) \subseteq \operatorname{Max}(R)$*, the R-module* $\operatorname{Ext}^i_R(N, M)$ *is Artinian and I-cofinite for all $i \geq 0$.*

According to [14] and [13], an R-module X is said to be *weakly Laskerian* if the set of associated primes of any quotient module of X is finite. An R-module X is said to be I-*weakly cofinite* if $\operatorname{Supp}_R(X) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, X)$ is weakly Laskerian for all $i \geq 0$.

We are thus led to the following consequence of Theorem 1.1 and Lemma 2.5.

Corollary 1.3. *Let* $(R, \mathfrak{m})$ *be a local ring,* M *an* I*-cominimax* R*-module, and* N *a finitely generated* R*-module. Then the* R*-module* $\operatorname{Ext}_R^i(N, M)$ *is* I*-weakly cofinite for all* $i \geq 0$ *if one of the following conditions holds:*

(i) $\dim N/IN \leq 2$;

(ii) N *is an* I*-torsion module with* $\dim N \leq 3$.

Throughout this paper, we assume that R is a commutative Noetherian ring with nonzero identity, I is an ideal of R, $V(I)$ is the set of all prime ideals of R containing I, and $\operatorname{Max}(R)$ is the set of all maximal ideals of R. For each R-module L, we denote by $\operatorname{Assh}_R(L)$ the set $\{\mathfrak{p} \in \operatorname{Ass}_R(L) \mid \dim R/\mathfrak{p} = \dim L\}$ and by $\operatorname{mAss}_R(L)$ the set of minimal elements of $\operatorname{Ass}_R(L)$ with respect to inclusion. Finally, for any ideal J of R, the radical of J, denoted by $\sqrt{J}$, is defined to be the set $\{x \in R \mid x^n \in J \text{ for some } n \in \mathbb{N}\}$. For any unexplained notation and terminology, we refer the reader to [12] and [23].

2. Preliminaries

Recall that a class $\mathcal{S}$ of R-modules is a *Serre subcategory* of the category of R-modules if it is closed under taking submodules, quotient modules, and extensions. For example, the classes of Noetherian modules, Artinian modules, minimax modules, and weakly Laskerian modules are Serre subcategories. We let $\mathcal{S}$ stand for a Serre subcategory of the category of R-modules. The following lemma, which is needed in the next section, follows immediately from the definition of Ext and Tor modules.

Lemma 2.1. *Let* M *be a finitely generated* R*-module and* $N \in \mathcal{S}$*. Then* $\operatorname{Ext}_R^i(M, N) \in \mathcal{S}$ *and* $\operatorname{Tor}_i^R(M, N) \in \mathcal{S}$ *for all* $i \geq 0$.

Lemma 2.2. *Let* M *be a finitely generated* R*-module and* N *an arbitrary* R*-module. Suppose that for some* $t \geq 0$, $\operatorname{Ext}_R^i(M, N) \in \mathcal{S}$ *for all* $i \leq t$*. Then for any finitely generated* R*-module* X *with* $\operatorname{Supp}_R(X) \subseteq \operatorname{Supp}_R(M)$, $\operatorname{Ext}_R^i(X, N) \in \mathcal{S}$ *for all* $i \leq t$.

Proof. See [2, Lemma 2.2]. □

Remark 2.3. The following statements hold true:

(i) The class of minimax modules contains all finitely generated and all Artinian modules.

(ii) Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Then M is minimax if and only if L and N are both minimax (see [8, Lemma 2.1]). Thus, any submodule and any quotient of a minimax module is minimax.

(iii) The set of associated primes of any minimax R-module is finite.

(iv) If M is a minimax R-module and $\mathfrak{p}$ is a non-maximal prime ideal of R, then $M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module.

(v) Let M be a minimax R-module that is annihilated by the product of finitely many (not necessarily distinct) maximal ideals of R. Then M has finite length. (See [17, Lemma 2.5].)

Lemma 2.4. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a ring homomorphism.*

(i) *If φ is a flat ring homomorphism and N is a minimax R-module, then $N \otimes_R S$ is a minimax S-module.*

(ii) *If φ is a faithfully flat ring homomorphism, then N is a minimax R-module if and only if $N \otimes_R S$ is a minimax S-module.*

Proof. See [28, Lemma 3.4]. □

Lemma 2.5. *Let M be an R-module such that $\operatorname{Ext}^i_R(R/I, M)$ is minimax for all $i \leq \dim M$. Then the following statements hold:*

(i) *The R-module $\operatorname{Ext}^i_R(N, M)$ is minimax for all $i \geq 0$ and for any finitely generated R-module N with $\operatorname{Supp}_R(N) \subseteq V(I)$ and $\dim N \leq 1$.*

(ii) *All Bass numbers $\mu^i(\mathfrak{p}, M)$ and all Betti numbers $\beta_i(\mathfrak{p}, M)$ of M are finite for all $\mathfrak{p} \in V(I)$.*

(iii) *If $(R, \mathfrak{m})$ is a Noetherian local ring, then the R-module $\operatorname{Ext}^i_R(N, M)$ is minimax for all $i \geq 0$ and for any finitely generated R-module N with $\operatorname{Supp}_R(N) \subseteq V(I)$ and $\dim N \leq 2$.*

Proof. See [27, Theorem 2.7 and Corollary 2.8] for (i) and (ii) and [20, Theorem 2.5] for (iii). □

3. Main results

We begin by recalling the concept of the arithmetic rank of an ideal. For any proper ideal I of R, the arithmetic rank of I, denoted by $\mathfrak{a}ra(I)$, is the least number of elements of R required to generate an ideal with the same radical as I, i.e.,

$$\mathfrak{a}ra(I) := \min\left\{n \in \mathbb{N}_0 \mid \exists a_1, \ldots, a_n \in R \text{ with } \sqrt{(a_1, \ldots, a_n)} = \sqrt{I}\right\}.$$

Also, for an R-module M, the arithmetic rank of I with respect to M, denoted by $\mathfrak{a}ra_M(I)$, is defined as the arithmetic rank of the ideal $I + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)$ in the ring $R/\operatorname{Ann}_R(M)$. Moreover, the I-torsion submodule of M, denoted by $\Gamma_I(M)$, is the set of elements of M

annihilated by some power of I, i.e., $\Gamma_I(M) := \bigcup_{n\in\mathbb{N}_0}(0 :_M I^n)$.

The following lemma is needed in the proof of Theorem 3.2.

Lemma 3.1. *Let I be an ideal of R and M an R-module such that $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is minimax for all $i \leq \dim M$. Then for every finite length R-module N, the R-module $\operatorname{Ext}^i_R(N, M)$ is of finite length for all $i \geq 0$.*

Proof. We may assume that $N \neq 0$. Then, by assumption, the set $\operatorname{Supp}_R(N)$ is a finite nonempty subset of $\operatorname{Max}(R)$. Let $\operatorname{Supp}_R(N) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ and set $J := \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r$. We show that the R-module $\operatorname{Ext}^i_R(R/J, M)$ is of finite length for all $i \geq 0$. Since $\operatorname{Ext}^i_R(R/J, M) \cong \bigoplus_{j=1}^r \operatorname{Ext}^i_R(R/\mathfrak{m}_j, M)$, we may assume, without loss of generality, that $r = 1$ and $J = \mathfrak{m}_1$. Clearly, $\mathfrak{m}_1 \in \operatorname{Supp}_R(M) \subseteq V(I)$. Therefore, by Lemma 2.5, the R-module $\operatorname{Ext}^i_R(R/\mathfrak{m}_1, M)$ is minimax for all $i \geq 0$. Thus, by Remark 2.3(v), the module $\operatorname{Ext}^i_R(R/\mathfrak{m}_1, M)$ is of finite length for all $i \geq 0$, and hence so does $\operatorname{Ext}^i_R(R/J, M)$. This completes the proof of the claim. Consequently, since $\operatorname{Supp}_R(N) = V(J)$, Lemma 2.2 implies that $\operatorname{Ext}^i_R(N, M)$ also is of finite length for all $i \geq 0$. □

Theorem 3.2. *Let I be an ideal of R and M an R-module such that $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is minimax for all $i \leq \dim M$ (not necessarily I-cofinite or I-cominimax). Then for every finitely generated R-module N with $\operatorname{Supp}_R(N/IN) \subseteq \operatorname{Max}(R)$, the R-module $\operatorname{Ext}^i_R(N, M)$ is Artinian and I-cofinite for all $i \geq 0$.*

Proof. If $IN = N$, then for every $i \geq 0$,

$$\begin{aligned}\operatorname{Supp}_R(\operatorname{Ext}^i_R(N, M)) &\subseteq \operatorname{Supp}_R(N) \cap \operatorname{Supp}_R(M)\\ &\subseteq \operatorname{Supp}_R(N) \cap V(I)\\ &= \operatorname{Supp}_R(N/IN) = \emptyset.\end{aligned}$$

Thus, $\operatorname{Ext}^i_R(N, M) = 0$ for all $i \geq 0$, and there is nothing to prove. Hence, we assume that $N/IN \neq 0$. Therefore, $I + \operatorname{Ann}_R(N) \neq R$. We prove the assertion by induction on $t = \mathfrak{a}ra_N(I) = \mathfrak{a}ra(I + \operatorname{Ann}_R(N)/\operatorname{Ann}_R(N))$. If $t = 0$, then by definition, $I \subseteq \operatorname{Ann}_R(N)$. Thus, $\operatorname{Supp}_R(N) = V(\operatorname{Ann}_R(N)) \subseteq V(I)$, and hence $\operatorname{Supp}_R(N) = \operatorname{Supp}_R(N) \cap V(I) = \operatorname{Supp}_R(N/IN)$. Therefore, by assumption, N is a finitely generated R-module with support in $\operatorname{Max}(R)$, which means that N is of finite length. Hence, the result follows from Lemma 3.1.

Now, suppose that $t > 0$ and that the result holds for all smaller values of t. It is easy to see that $\mathfrak{a}ra_{\Gamma_I(N)}(I) = 0$ and

$$\operatorname{Supp}_R(\Gamma_I(N)/I\Gamma_I(N)) \subseteq \operatorname{Supp}_R(N) \cap V(I) = \operatorname{Supp}_R(N/IN) \subseteq \operatorname{Max}(R).$$

Thus, by the inductive hypothesis, the R-modules $\operatorname{Ext}^i_R(\Gamma_I(N), M)$ are Artinian and I-cofinite for all integers $i \geq 0$. On the other hand, set $\overline{N} := N/\Gamma_I(N)$. Then $\operatorname{Ann}_R(N) \subseteq \operatorname{Ann}_R(\overline{N})$. Hence, $\operatorname{Supp}_R(\overline{N}/I\overline{N}) \subseteq \operatorname{Supp}_R(N/IN)$ and $\mathfrak{a}ra_{\overline{N}}(I) \leq \mathfrak{a}ra_N(I)$. Now, consider [25, Corollary 4.4] and the long exact sequence

$$(1) \qquad \begin{aligned} 0 &\to \operatorname{Hom}_R(\overline{N}, M) \to \operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(\Gamma_I(N), M) \\ &\to \operatorname{Ext}^1_R(\overline{N}, M) \to \operatorname{Ext}^1_R(N, M) \to \operatorname{Ext}^1_R(\Gamma_I(N), M) \to \cdots, \end{aligned}$$

induced by the exact sequence $0 \to \Gamma_I(N) \to N \to \overline{N} \to 0$. Without loss of generality, we may assume that N is a finitely generated R-module with $\Gamma_I(N) = 0$, $\operatorname{Supp}_R(N/IN) \subseteq \operatorname{Max}(R)$, and $\mathfrak{a}ra_N(I) = t$. Then, by [12, Lemma 2.1.1], we have $I \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(N)} \mathfrak{p}$. Moreover, by definition, there exist elements $a_1, \ldots, a_t \in I$ such that

$$\sqrt{\frac{I + \operatorname{Ann}_R(N)}{\operatorname{Ann}_R(N)}} = \sqrt{\frac{(a_1, \ldots, a_t) + \operatorname{Ann}_R(N)}{\operatorname{Ann}_R(N)}}.$$

Therefore,

$$(a_1, \ldots, a_t) + \operatorname{Ann}_R(N) \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(N)} \mathfrak{p}.$$

But since

$$\operatorname{Ann}_R(N) \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(N)} \mathfrak{p},$$

it follows that

$$(a_1, \ldots, a_t) \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(N)} \mathfrak{p}.$$

Thus, by [23, Ex. 16.8], there exists $a \in (a_2, \ldots, a_t)$ such that

$$a_1 + a \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(N)} \mathfrak{p}.$$

Set $x := a_1 + a$. Therefore,

$$\sqrt{\frac{I + \operatorname{Ann}_R(N)}{\operatorname{Ann}_R(N)}} = \sqrt{\frac{(x, a_2, \ldots, a_t) + \operatorname{Ann}_R(N)}{\operatorname{Ann}_R(N)}}.$$

This implies that

$$\sqrt{\frac{I + \operatorname{Ann}_R(N/xN)}{\operatorname{Ann}_R(N/xN)}} = \sqrt{\frac{(a_2, \ldots, a_t) + \operatorname{Ann}_R(N/xN)}{\operatorname{Ann}_R(N/xN)}},$$

and hence $\mathfrak{a}ra_{N/xN}(I) \leq t - 1$. Furthermore, since $x \in I$, we have

$$\operatorname{Supp}_R((N/xN)/I(N/xN)) = \operatorname{Supp}_R(N/IN).$$

Therefore, N/xN is a finitely generated R-module of dimension zero, and by the inductive hypothesis, the R-module $\mathrm{Ext}^i_R(N/xN, M)$ is Artinian and I-cofinite for every integer $i \geq 0$. Now, the exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$ induces the long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(N, M) \xrightarrow{x} \mathrm{Ext}^i_R(N, M) \to \mathrm{Ext}^{i+1}_R(N/xN, M) \\ \to \mathrm{Ext}^{i+1}_R(N, M) \xrightarrow{x} \mathrm{Ext}^{i+1}_R(N, M) \to \cdots ,$$

for all $i \geq 0$. Thus, we obtain the short exact sequence

$$0 \to \mathrm{Ext}^i_R(N, M)/x\,\mathrm{Ext}^i_R(N, M) \to \mathrm{Ext}^{i+1}_R(N/xN, M) \to (0 :_{\mathrm{Ext}^{i+1}_R(N,M)} x) \to 0,$$

for all $i \geq 0$. Therefore, by [25, Corollary 4.4], the R-modules

$$\mathrm{Ext}^i_R(N, M)/x\,\mathrm{Ext}^i_R(N, M) \quad \text{and} \quad (0 :_{\mathrm{Ext}^{i+1}_R(N,M)} x),$$

are I-cofinite for all $i \geq 0$. Also, from the exact sequence

$$0 \to \mathrm{Hom}_R(N/xN, M) \to \mathrm{Hom}_R(N, M) \xrightarrow{x} \mathrm{Hom}_R(N, M),$$

and the inductive hypothesis, it follows that the R-module $(0 :_{\mathrm{Hom}_R(N,M)} x)$ is I-cofinite. Therefore, for every $i \geq 0$, the modules $(0 :_{\mathrm{Ext}^i_R(N,M)} x)$ and $\mathrm{Ext}^i_R(N, M)/x\,\mathrm{Ext}^i_R(N, M)$ are I-cofinite. Hence, by [25, Corollary 3.4], the R-module $\mathrm{Ext}^i_R(N, M)$ is I-cofinite for all $i \geq 0$. Furthermore, it is clear that

$$\mathrm{Supp}_R(\mathrm{Ext}^i_R(N, M)) \subseteq \mathrm{Supp}_R(N) \cap V(I) = \mathrm{Supp}_R(N/IN) \subseteq \mathrm{Max}(R).$$

Thus, the R-module $\mathrm{Hom}_R(R/I, \mathrm{Ext}^i_R(N, M))$ is of finite length. Finally, by [25, Proposition 4.1], the R-module $\mathrm{Ext}^i_R(N, M)$ is Artinian and I-cofinite, as desired. □

Now, we are ready to prove the main result of the paper, which extends [7, Theorem 2.2] to the class of minimax modules.

Theorem 3.3. *Let I be an ideal of R and M an R-module such that $\mathrm{Supp}_R(M) \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I, M)$ is minimax for all $i \leq \dim M$. Then for every finitely generated R-module N with $\dim N/IN \leq 1$, the R-module $\mathrm{Ext}^i_R(N, M)$ is I-cominimax for all $i \geq 0$.*

Proof. In view of Theorem 3.2, it suffices to prove the assertion when $\dim N/IN = 1$. We proceed by induction on $t = \mathfrak{ara}_N(I)$. If $t = 0$, then by definition, $\mathrm{Supp}_R(N) \subseteq V(I)$, and the assertion follows from Lemma 2.5. Thus, assume that $t > 0$ and that the result holds for all values less than t. Set $\overline{N} := N/\Gamma_I(N)$. Then $\mathfrak{ara}_{\overline{N}}(I) \leq \mathfrak{ara}_N(I)$ and $\dim \overline{N}/I\overline{N} \leq \dim N/IN$. On the other hand, since $\mathrm{Supp}_R(\Gamma_I(N)) \subseteq \mathrm{Supp}_R(N/IN)$, the assumption and Lemma 2.5(i) imply that the R-module $\mathrm{Ext}^i_R(\Gamma_I(N), M)$ is minimax. Thus, in view of the long exact sequence (1) and Theorem 3.2, we may assume, without loss of generality, that N is a finitely generated

R-module with $\Gamma_I(N)=0$, $\dim N/IN=1$, and $\mathfrak{a}ra_N(I)=t$. Hence, by [12, Lemma 2.1.1], we have $I \not\subseteq \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(N)} \mathfrak{p}$.

Fix an integer $k \geq 0$. Let

$$\mathcal{S}_k := \bigcup_{i=0}^{k} \mathrm{Supp}_R\left(\mathrm{Ext}^i_R(N,M)\right) \quad \text{and} \quad \mathcal{T} := \{\mathfrak{p}\in \mathcal{S}_k \mid \dim R/\mathfrak{p}=1\}.$$

It is easy to see that $\mathcal{T} \subseteq \mathrm{Assh}_R(N/IN)$. Therefore, $\mathcal{T}$ is finite by assumption. Moreover, by [23, Ex. 7.7] and Lemma 2.4, for each $\mathfrak{p}\in\mathcal{T}$, the $R_\mathfrak{p}$-module $\mathrm{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p})$ is minimax for all $i \leq \dim M_\mathfrak{p} \leq n$, and $N_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module with

$$\mathrm{Supp}_{R_\mathfrak{p}}(N_\mathfrak{p}/IN_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\} = \mathrm{Max}(R_\mathfrak{p}).$$

Therefore, by [23, Ex. 7.7] and Theorem 3.2, the $R_\mathfrak{p}$-module $\left(\mathrm{Ext}^i_R(N,M)\right)_\mathfrak{p}$ is Artinian and $IR_\mathfrak{p}$-cofinite for all $0\leq i\leq k$. Write

$$\mathcal{T} = \{\mathfrak{p}_1,\ldots,\mathfrak{p}_n\}.$$

By [9, Lemma 2.5], we have

$$V(IR_{\mathfrak{p}_j}) \cap \mathfrak{Att}_{R_{\mathfrak{p}_j}}\left(\mathrm{Ext}^i_R(N,M)\right)_{\mathfrak{p}_j} \subseteq V(\mathfrak{p}_jR_{\mathfrak{p}_j}),$$

for all integers $0\leq i\leq k$ and $1\leq j\leq n$. Next, let

$$\sum := \bigcup_{i=0}^{k}\bigcup_{j=1}^{n}\left\{\mathfrak{q}\in\mathrm{Spec}(R) \mid \mathfrak{q}R_{\mathfrak{p}_j}\in \mathfrak{Att}_{R_{\mathfrak{p}_j}}\left(\mathrm{Ext}^i_R(N,M)\right)_{\mathfrak{p}_j}\right\}.$$

Then it is easy to see that $V(I)\cap\sum\subseteq\mathcal{T}$. Also, for each $\mathfrak{q}\in\sum$, we have

$$\mathfrak{q}R_{\mathfrak{p}_j}\in \mathfrak{Att}_{R_{\mathfrak{p}_j}}\left(\mathrm{Ext}^i_R(N,M)\right)_{\mathfrak{p}_j},$$

for some integers $0\leq i\leq k$ and $1\leq j\leq n$. This implies that

$$\mathrm{Ann}_R(N)R_{\mathfrak{p}_j} \subseteq \mathrm{Ann}_{R_{\mathfrak{p}_j}}\left(\left(\mathrm{Ext}^i_R(N,M)\right)_{\mathfrak{p}_j}\right) \subseteq \mathfrak{q}R_{\mathfrak{p}_j}.$$

Hence, $\mathrm{Ann}_R(N)\subseteq\mathfrak{q}$, and so $\sum\subseteq\mathrm{Supp}_R(N)$. On the other hand, by definition, there exist elements $y_1,\ldots,y_t\in I$ such that

$$\sqrt{\frac{I+\mathrm{Ann}_R(N)}{\mathrm{Ann}_R(N)}} = \sqrt{\frac{(y_1,\ldots,y_t)+\mathrm{Ann}_R(N)}{\mathrm{Ann}_R(N)}}.$$

Since $I \not\subseteq \left(\bigcup_{\mathfrak{q}\in\sum\setminus V(I)}\mathfrak{q}\right)\cup\left(\bigcup_{\mathfrak{p}\in\mathrm{Ass}_R(N)}\mathfrak{p}\right)$, we have

$$(y_1,\ldots,y_t)+\mathrm{Ann}_R(N) \not\subseteq \left(\bigcup_{\mathfrak{q}\in\sum\setminus V(I)}\mathfrak{q}\right)\cup\left(\bigcup_{\mathfrak{p}\in\mathrm{Ass}_R(N)}\mathfrak{p}\right).$$

But since

$$\mathrm{Ann}_R(N) \subseteq \left(\bigcup_{\mathfrak{q} \in \sum \setminus V(I)} \mathfrak{q} \right) \cup \left(\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(N)} \mathfrak{p} \right),$$

we conclude that

$$(y_1, \ldots, y_t) \nsubseteq \left(\bigcup_{\mathfrak{q} \in \sum \setminus V(I)} \mathfrak{q} \right) \cup \left(\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(N)} \mathfrak{p} \right).$$

Thus, by [23, Ex. 16.8], there exists $a \in (y_2, \ldots, y_t)$ such that

$$y_1 + a \notin \left(\bigcup_{\mathfrak{q} \in \sum \setminus V(I)} \mathfrak{q} \right) \cup \left(\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(N)} \mathfrak{p} \right).$$

Set $x := y_1 + a$. Therefore,

$$\sqrt{\frac{I + \mathrm{Ann}_R(N)}{\mathrm{Ann}_R(N)}} = \sqrt{\frac{(x, y_2, \ldots, y_t) + \mathrm{Ann}_R(N)}{\mathrm{Ann}_R(N)}}.$$

This implies that

$$\sqrt{\frac{I + \mathrm{Ann}_R(N/xN)}{\mathrm{Ann}_R(N/xN)}} = \sqrt{\frac{(y_2, \ldots, y_t) + \mathrm{Ann}_R(N/xN)}{\mathrm{Ann}_R(N/xN)}},$$

and hence $\mathfrak{a}ra_{N/xN}(I) \leq t - 1$. Therefore, by the inductive hypothesis, the R-module $N_i := \mathrm{Ext}^i_R(N/xN, M)$ is I-cominimax for all $i \geq 0$.

Now, the exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$ induces the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Ext}^i_R(N, M) \xrightarrow{x} \mathrm{Ext}^i_R(N, M) \to N_{i+1} \\ \to \mathrm{Ext}^{i+1}_R(N, M) \xrightarrow{x} \mathrm{Ext}^{i+1}_R(N, M) \to \cdots, \end{aligned}$$

for all $i \geq 0$. Thus, we obtain the short exact sequence

$$0 \to \mathrm{Ext}^i_R(N, M)/x \,\mathrm{Ext}^i_R(N, M) \to N_{i+1} \to (0 :_{\mathrm{Ext}^{i+1}_R(N,M)} x) \to 0,$$

for all $0 \leq i \leq k$. For each $0 \leq i \leq k$, let $L_i := \mathrm{Ext}^i_R(N, M)/x \,\mathrm{Ext}^i_R(N, M)$. By [9, Lemma 2.4], it is easy to see that $(L_i)_{\mathfrak{p}_j}$ is of finite length for all $0 \leq i \leq k$ and $1 \leq j \leq n$. Therefore, for each i and j, there exists a finitely generated submodule L_{ij} of L_i such that $(L_i)_{\mathfrak{p}_j} = (L_{ij})_{\mathfrak{p}_j}$. Set $L'_i := L_{i1} + \cdots + L_{in}$. Then L'_i is a finitely generated submodule of L_i such that

$$\mathrm{Supp}_R(L_i/L'_i) \subseteq \mathcal{S}_k \setminus \mathcal{T} \subseteq \mathrm{Max}(R).$$

Therefore, there exists a finitely generated submodule N'_{i+1} of N_{i+1} such that the sequence

$$0 \to L_i/L'_i \to N_{i+1}/N'_{i+1} \to (0 :_{\mathrm{Ext}^{i+1}_R(N,M)} x) \to 0,$$

is exact. We now show that L_i is a minimax R-module. To this end, note that for all $0 \leq i \leq k$, the module N_{i+1}/N'_{i+1} is I-cominimax. Hence, $\mathrm{Hom}_R(R/I, L_i/L'_i)$ is a minimax R-module. Since $\mathrm{Supp}_R(\mathrm{Hom}_R(R/I, L_i/L'_i)) \subseteq \mathrm{Supp}_R(L_i/L'_i) \subseteq \mathrm{Max}(R)$, it follows that

$\operatorname{Hom}_R(R/I, L_i/L_i')$ is Artinian. Moreover, since L_i/L_i' is I-torsion, [24, Theorem 1.3] implies that it is an Artinian R-module. This shows that L_i is a minimax R-module. Hence, from the exact sequence

$$0 \to L_i \to N_{i+1} \to (0 :_{\operatorname{Ext}^{i+1}_R(N,M)} x) \to 0,$$

we deduce that $(0 :_{\operatorname{Ext}^{i+1}_R(N,M)} x)$ is I-cominimax for all $0 \le i \le k$. In particular, from the exact sequence

$$0 \to \operatorname{Hom}_R(N/xN, M) \to \operatorname{Hom}_R(N, M) \xrightarrow{x} \operatorname{Hom}_R(N, M),$$

and the inductive hypothesis, it follows that the R-module $(0 :_{\operatorname{Hom}_R(N,M)} x)$ is I-cominimax. Therefore, for every $0 \le i \le k$, the modules $(0 :_{\operatorname{Ext}^i_R(N,M)} x)$ and $\operatorname{Ext}^i_R(N, M)/x \operatorname{Ext}^i_R(N, M)$ are I-cominimax. Hence, by [25, Corollary 3.3], the module $\operatorname{Ext}^i_R(N, M)$ is I-cominimax for all $0 \le i \le k$. Since k is arbitrary, it follows that the R-module $\operatorname{Ext}^i_R(N, M)$ is I-cominimax for all $i \ge 0$. This completes the proof. □

Corollary 3.4. *Let I be an ideal of R and M an R-module such that* $\operatorname{Supp}_R(M) \subseteq V(I)$ *and* $\operatorname{Ext}^i_R(R/I, M)$ *is minimax for all $i \le \dim M$. Let N be a finitely generated R-module and set*

$$\Omega := \{\mathfrak{p} \in \operatorname{mAss}_R(N) \mid 0 \le \dim R/(I + \mathfrak{p}) \le 1\},$$

and $J := \bigcap_{\mathfrak{p} \in \Omega} \mathfrak{p}$. Then, the R-module $\operatorname{Ext}^i_R(\Gamma_J(N), M)$ *is I-cominimax for all $i \ge 0$.*

Proof. Since $\operatorname{Ass}_R(\Gamma_J(N)) = \operatorname{Ass}_R(N) \cap V(J)$, it is easy to see that

$$\operatorname{mAss}_R(\Gamma_J(N)) = \{\mathfrak{p} \in \operatorname{mAss}_R(N) \mid 0 \le \dim R/(I + \mathfrak{p}) \le 1\} = \Omega.$$

This implies that $0 \le \dim R/(I + \mathfrak{p}) \le 1$ for each $\mathfrak{p} \in \operatorname{mAss}_R(\Gamma_J(N))$. Therefore, $0 \le \dim \Gamma_J(N)/I\Gamma_J(N) \le 1$. So, the assertion follows from Theorem 3.3. □

Finally, as a consequence of Theorem 3.3 and Lemma 2.5, we prove the following result.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a local ring, M an R-module such that* $\operatorname{Ext}^i_R(R/I, M)$ *is minimax for all $i \le \dim M$, and N an R-module. Then the R-module* $\operatorname{Ext}^i_R(N, M)$ *is I-weakly cofinite for all $i \ge 0$ if one of the following statements holds:*

(i) $\operatorname{Supp}_R(M) \subseteq V(I)$ *and N is finitely generated with $\dim N/IN \le 2$;*

(ii) *N is minimax with* $\operatorname{Supp}_R(N) \subseteq V(I)$ *and $\dim N \le 3$.*

Proof. Let $\Lambda = \left\{\operatorname{Ext}^j_R\left(R/I, \operatorname{Ext}^i_R(N, M)\right) \mid i \ge 0, j \ge 0\right\}$. Suppose that $K \in \Lambda$ and K' is a submodule of K. By definition, it suffices to show that $\operatorname{Ass}_R(K/K')$ is finite. To do this, based on [23, Ex. 7.7] and [21, Lemma 2.1], we may assume, without loss of generality, that R is complete.

On the contrary, suppose that the set $\operatorname{Ass}_R(K/K')$ is infinite. So, there exists a countably infinite subset $\{\mathfrak{p}_r\}_{r=1}^{\infty}$ of non-maximal elements of $\operatorname{Ass}_R(K/K')$. Thus, $\mathfrak{m} \nsubseteq \bigcup_{r=1}^{\infty} \mathfrak{p}_r$ by [22, Lemma 3.2]. Let $S := R \setminus \bigcup_{r=1}^{\infty} \mathfrak{p}_r$. Now, we consider the following two cases:

Case 1: If $\dim N/IN = 2$, then $S^{-1}N/S^{-1}IS^{-1}N$ has dimension at most one. Hence, it follows from Lemma 2.4 and Theorem 3.3 that $\operatorname{Ext}^i_{S^{-1}R}(S^{-1}N, S^{-1}M)$ is $S^{-1}I$-cominimax.

Case 2: If $\dim N \leq 3$, then $S^{-1}N$ is a finitely generated $S^{-1}R$-module by Remark 2.3(iv) and has dimension at most two. Hence, it follows from Lemma 2.4 and Lemma 2.5(iii) that $\operatorname{Ext}^i_{S^{-1}R}(S^{-1}N, S^{-1}M)$ is minimax.

In both cases, it follows that $S^{-1}K/S^{-1}K'$ is a minimax $S^{-1}R$-module, and so $\operatorname{Ass}_{S^{-1}R}\left(S^{-1}K/S^{-1}K'\right)$ is finite by Remark 2.3. But

$$S^{-1}\mathfrak{p}_r \in \operatorname{Ass}_{S^{-1}R}\left(S^{-1}K/S^{-1}K'\right),$$

for all $r = 1, 2, \ldots$, a contradiction. □

Corollary 3.6. *Let $(R, \mathfrak{m})$ be a local ring, M an I-cominimax R-module, and N a finitely generated R-module. Then the R-module $\operatorname{Ext}^i_R(N, M)$ is I-weakly cofinite for all $i \geq 0$ if one of the following statements holds:*

(i) *$\dim N/IN \leq 2$;*

(ii) *N is an I-torsion module with $\dim N \leq 3$.*

Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

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