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Research Paper

# SOME RESULTS ON THE GRAPH OF DERANGEMENTS

HOSSEIN MOSHTAGH* AND FARZAD SHAVEISI

Abstract. The graph of derangements, denoted by $\Gamma(D_n)$, is a simple graph whose vertex set is the set of all derangements on $[n]$ and two distinct vertices $f$ and $g$ are adjacent if and only if $f(i) \neq g(i)$, for every $i \in [n]$. In this paper, some properties of this graph are presented. The clique number and the vertex chromatic number of this graph are determined. Then we show that for every positive integer $n \geq 5$, $\Gamma(D_n)$ is neither a perfect graph nor a cograph. Moreover, this graph can not be a line graph unless $n \leq 4$. Maximum cliques and maximum independent sets are studied, too.

## 1. Introduction

We begin with some definitions and notations on graphs. Let $G = (V(G), E(G))$ be a simple graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The complete graph with $n$ vertices and the complete bipartite graph with parts of size $n_1, n_2$ are denoted by $K_n$ and $K_{n_1,n_2}$, respectively; in particular, if either $n_1 = 1$ or $n_2 = 1$, the complete bipartite graph is called a *star* graph.

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*Corresponding author

The star graph $K_{1,3}$ is called a *claw*, too. A *claw-free* graph is a graph with no claw as its induced subgraph. A graph is said to be a *refinement of a star* graph if it contains a vertex which is adjacent to any other vertices.

Here, a cycle (path) with $n$-vertices is denoted by $C_n$ $(P_n)$. For a graph $G$, let $\chi(G)$ denote the *vertex chromatic number* of the graph $G$, i.e., the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices receive different colors. A *clique* of a graph is its complete subgraph, and the number of vertices in the largest clique of graph $G$, denoted by $\omega(G)$, is called the *clique number* of $G$. An independent set of $G$ is a set $A$ of its vertices which are pairwise nonadjacent; the maximum cardinality of an independent set is called the *independence number* and it is denoted by $\alpha(G)$. For any graph $G$ and every subset $X \subseteq V(G)$ the *induced subgraph* of $G$ on $X$, denoted by $G[X]$, is a graph with the vertex set $X$ and to distinct vertices $x_1, x_2 \in X$ are adjacent if and only if $x_1x_2 \in E(G)$. A graph $G$ is called *perfect* if for every induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$. Also, a *cograph* is a graph which does not contain $P_4$ as its induced subgraph. The *line graph* of $G$, denoted by $L(G)$, is a graph whose vertex set is the edge set of $G$ and two distinct vertices of $L(G)$ are adjacent if they have a common vertex as edges of $G$. The graph $G$ is called a line graph if it is isomorphic with the line graph of some graph. For more background on graph theory, see [2, 6].

A $m \times n$ *Latin rectangle* is an $m \times n$ matrix, with symbols from a set of cardinality $n$, such that each symbol occurs only once in each row and only once in each column. Also, if $m = n$ then the matrix is called a *Latin square* of order $n$ (see [5] for more details). In this paper, by $[n]$ we mean the set $\{1, 2, \ldots, n\}$. The set of all functions from $[m]$ to $[n]$ is denoted by $F_{m,n}$ and the set of all permutations on $[n]$ is denoted by $S_n$. A fixed point of $f \in F_{n,n}$ is an element $i \in [n]$ such that $f(i) = i$. Some of the deepest theorems in mathematics involve fixed points. Here, we will focus on permutations that have no fixed points. A permutation with no fixed points is called a *derangement*. The set of derangements in $S_n$ is denoted $D_n$ and its cardinality is denoted by $d_n$. It is well-known that the $n$th derangement number, $d_n = |D_n|$, is the integer closest to $n!/e$. For example, $|D_4| = 9$ and $|D_5| = 44$. For simplicity, in this paper, we denote any derangement $d$ on $[n]$, by its image $d = (d(1), d(2), \ldots, d(n))$. In fact, $d$ is a combinatorial permutation of the numbers $1, 2, \ldots, n$ with no fixed point. For more details we refer the reader to [7].

The *graph of derangements*, denoted by $\Gamma(D_n)$, is a simple graph whose vertex set is $D_n$ and two distinct vertices $f, g$ are adjacent if they are derangements of each other, that is, $f(r) \neq g(r)$, for every $r \in [n]$. In Section 2, we study the coloring of this graph. It is shown that $\omega(\Gamma(D_n)) = \chi(\Gamma(D_n)) = n - 1$. Also, the independent sets are studied.

## 2. Main results

We start this section with the main definition of this paper.

**Definition 2.1.** Let $D_n$ be the set of derangements on $[n]$. The graph of derangements, denoted by $\Gamma(D_n)$, is a simple graph whose vertex set is $D_n$ and two distinct vertices $f, g$ are adjacent if and only if they are derangements of each other, that is, $f(r) \neq g(r)$, for every $r \in [n]$.

This graph, in fact, is an induced subgraph of $Cay(S_n, D_n)$ on derangements [4].

**Example 2.2.** (i) It is clear that $\Gamma(D_1)$ has no vertex, $\Gamma(D_2) \cong K_1$ and $\Gamma(D_3) \cong K_2$ .

(ii) For $n = 4$, we have

$$\begin{aligned} D_4 &= \{(2,1,4,3),(4,3,1,2),(3,4,2,1),(3,4,1,2),(4,3,2,1),\\ &\quad (3,1,4,2),(2,4,1,3),(2,3,4,1),(4,1,2,3)\}, \end{aligned}$$

and

$$K_1 = \{(2,1,4,3),(3,4,2,1),(4,3,1,2)\}, K_2 = \{(2,1,4,3),(4,3,1,2),(3,4,1,2)\},$$
$$K_3 = \{(2,1,4,3),(3,1,4,2),(4,3,2,1)\}, K_4 = \{(2,3,4,1),(3,4,2,1),(4,1,2,3)\},$$

are four maximum cliques in $\Gamma(D_4)$. This graph has 9 vertices, 12 edges and diameter 3. In fact, this graph is pictured as in Figure 1.

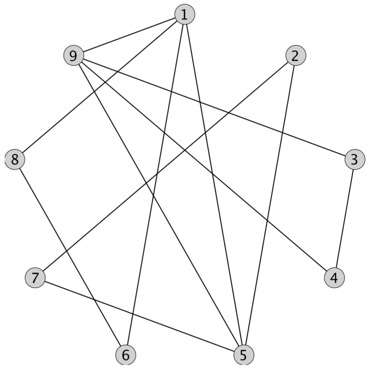

Figure 1. The graph $\Gamma(D_4)$.

(iii) The graph $\Gamma(D_5)$ has 44 vertices, 276 edges and diameter 3. This graph can be drawn as in Figure 2.

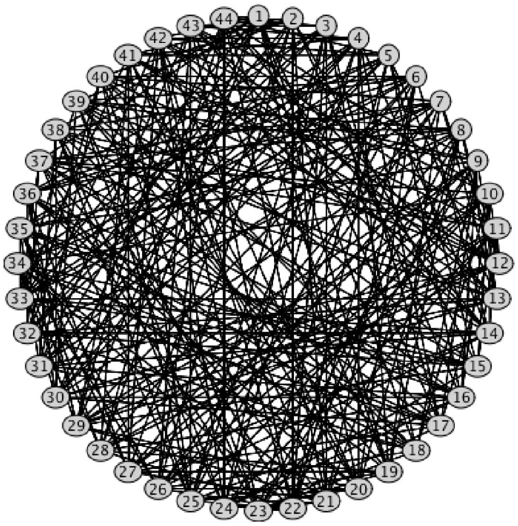

Figure 2. The graph $\Gamma(D_5)$.

From classical graph theory, we know that for any graph $G$, $\omega(G) \leq \chi(G)$. In the following theorem, it is shown that the clique number and the chromatic number of the graph of derangements are equal.

**Theorem 2.3.** *For any positive integer* $n$, $\omega(\Gamma(D_n)) = \chi(\Gamma(D_n)) = n-1$.

*Proof.* First we define a map $c : V(\Gamma(D_n)) = D_n \longrightarrow \{1, 2, \dots, n-1\}$ by $c(f) = f(1)$, for every derangements $f$. Since $f(1) \neq 1$, so $2 \leq c(f) \leq n$. Also, if $f, g$ are two adjacent vertices in $\Gamma(D_n)$, then $f(1) \neq g(1)$. Thus $c$ is a proper vertex coloring for the graph $\Gamma(D_n)$. This yields that $\chi(\Gamma(D_n)) \leq n-1$.

On the other hand, if $\eta \in D_n$ is a derangement such that $\eta(i) = i+1$, for every $i \in [n]$, i.e, $\eta = (2, 3, \dots, n, 1)$, then one can easily check that $W = \{\eta, \eta^2, \dots, \eta^{n-1}\}$ is a clique of order $n-1$ in $\Gamma(D_n)$. Thus $\chi(\Gamma(D_n)) \geq \omega(\Gamma(D_n)) \geq n-1$. Therefore, $\omega(\Gamma(D_n)) = \chi(\Gamma(D_n)) = n-1$.
□

Recall that, a Latin rectangle (square) is reduced if the values in the first row and column are in the natural order.

**Remark 2.4.** The proof of Theorem 2.3, shows that any maximal clique in $\Gamma(D_n)$ contains $n-1$ vertices. Also, the method of constructing cliques of order $n-1$ shows that the number of these cliques in $\Gamma(D_n)$ equals to the number of reduced Latin squares on $\{1, 2, \dots, n\}$.

Theorem 2.3 shows that the clique number of $\Gamma(D_n)$ equals its chromatic number. However, $\Gamma(D_n)$ for $n \geq 5$ may have some induced subgraph whose clique number is less than its chromatic number. This means that $\Gamma(D_n)$ is not a perfect graph for every $n \geq 5$. In the next theorem, it is shown that for every $n \geq 5$, $\Gamma(D_n)$ contains the cycle $C_5$ (the path $P_4$) as its induced subgraph. First, we prove the following two lemmas.

**Lemma 2.5.** *If* $n \geq 5$ *and* $C_5$ *is an induced subgraph of* $\Gamma(D_n)$*, then it is an induced subgraph of* $\Gamma(D_{n+4})$*, too.*

*Proof.* Let $\{f_1, f_2, f_3, f_4, f_5\} \subseteq D_n$ induce a cycle of length five in $\Gamma(D_n)$. Since there are nine derangements on $\{n+1, n+2, n+3, n+4\}$, we can choose three derangements $\alpha, \beta, \gamma$ on this set. Now, one can easily check that

$$\{f_1 \times \alpha, f_2 \times \beta, f_3 \times \alpha, f_4 \times \beta, f_5 \times \gamma\},$$

is a subset of derangements on $[n+4]$ and induce $C_5$ in $\Gamma(D_{n+4})$. □

**Lemma 2.6.** *If* $n \geq 5$ *and* $P_4$ *is an induced subgraph of* $\Gamma(D_n)$*, then it is an induced subgraph of* $\Gamma(D_{n+3})$*, too.*

*Proof.* Let $\{f_1, f_2, f_3, f_4\} \subseteq D_n$ induces $P_4$ in $\Gamma(D_n)$. There are two derangements on the set $\{n+1, n+2, n+3\}$, say $\alpha, \beta$ on this set. Now, one can easily check that

$$\{f_1 \times \alpha, f_2 \times \beta, f_3 \times \alpha, f_4 \times \beta\},$$

is a subset of derangements on $[n+3]$ and iduces $P_4$ in $\Gamma(D_{n+3})$. □

**Theorem 2.7.** *For every* $n \geq 5$, $\Gamma(D_n)$ *contains both* $P_4$ *and* $C_5$ *as its induced subgraphs.*

*Proof.* Let

$$\begin{aligned}
X &= \{(2,3,4,5,1),(3,4,5,1,2),(2,5,4,3,1),(3,1,2,5,4)\},\\
Y &= \{(2,3,4,5,6,1),(3,4,5,6,1,2),(5,6,4,1,2,3),(3,5,2,6,4,1)\},\\
Z &= \{(2,3,4,5,6,7,1),(3,4,5,6,7,1,2),(2,3,4,7,6,5,1),(3,4,5,6,1,7,2)\},\\
T &= \{(2,3,4,5,6,7,8,1),(3,4,5,6,7,8,1,2),(2,3,4,7,8,5,6,1),(3,4,5,6,1,7,8,2)\}.
\end{aligned}$$

Then $X$, $Y$, $Z$ and $T$ induce $P_4$ in graphs $\Gamma(D_5)$, $\Gamma(D_6)$, $\Gamma(D_7)$ and $\Gamma(D_8)$, respectively.

Also, the subsets

$$X' = X \cup \{(4,5,1,3,2)\},\ Y' = Y \cup \{(6,4,1,3,2,5)\},$$
$$Z' = Z \cup \{(4,1,6,2,7,5,3)\},\ T' = T \cup \{(4,1,7,2,3,8,6,5)\},$$

induce $C_5$ in graphs $\Gamma(D_5)$, $\Gamma(D_6)$, $\Gamma(D_7)$ and $\Gamma(D_8)$, respectively. So, the assertion follows from induction and Lemmas 2.5 and 2.6. □

Since $\chi(C_5) \neq \omega(C_5)$, from Theorem 2.7, we have the following immediate corollary.

**Corollary 2.8.** *For every $n \geq 5$, $\Gamma(D_n)$ is neither a perfect graph nor a cograph.*

**Remark 2.9.** From Theorem 2.3, we know that for every $n \geq 4$, $\Gamma(D_n)$ contains a triangle and so this graph is triangle-free (bipartite) if and only if $n \leq 3$.

A graph is said to be square-free if it contains no $C_4$ as its induced subgraph. In the following proposition, we study when $\Gamma(D_n)$ is a square-free graph.

**Proposition 2.10.** *$\Gamma(D_n)$ is a square-free graph if and only if $n \leq 4$.*

*Proof.* If $n \leq 4$, then as we saw in Example 2.2, $\Gamma(D_n)$ contains no square (as its induced subgraph). Also, it is easy to see that the sets

$$X = \{(2,3,4,5,1),(3,4,5,1,2),(5,1,4,2,3),(3,5,2,1,4)\},$$

$$Y = \{(2,3,4,5,6,1),(3,4,5,6,1,2),(5,6,4,1,2,3),(3,5,2,6,1,4)\}, \text{ and}$$

$$Z = \{(2,3,4,5,6,7,1),(3,4,5,6,7,1,2),(5,6,4,7,1,2,3),(3,5,2,6,7,1,4)\},$$

induce $C_4$ in the graphs $\Gamma(D_5)$, $\Gamma(D_6)$ and $\Gamma(D_7)$, respectively. Now, we claim that if $\Gamma(D_n)$ contains $C_4$ as its induced subgraph, then $\Gamma(D_{n+3})$ has induced $C_4$, too. To see this, let $\{f_1, f_2, f_3, f_4\}$ induces $C_4$ in $\Gamma(D_n)$, then $\alpha = (n+2, n+3, n+1)$ and $\beta = (n+3, n+1, n+2)$ are two distinct derangements on the set $\{n+1, n+2, n+3\}$. It is easy to check that $\{f_1 \times \alpha, f_2 \times \beta, f_3 \times \alpha, f_4 \times \beta\}$ induces $C_4$ in $\Gamma(D_{n+3})$. So, the claim is proved and the assertion follows from the claim. □

Recall that a graph is called claw-free if it does not contain $K_{1,3}$ as its induced subgraph.

**Theorem 2.11.** *$\Gamma(D_n)$ is a claw-free graph if and only if $n \leq 4$.*

*Proof.* If $n \leq 4$, then as we saw in Example 2.2, $\Gamma(D_n)$ is claw-free. Also, it is easy to see that the sets

$$X = \{(2,3,4,5,1),(3,4,5,1,2),(4,3,1,2,5),(2,3,1,5,4)\},$$

$$Y = \{(2,3,4,5,6,1),(3,4,5,6,1,2),(4,3,6,1,2,5),(2,3,6,1,4,5)\}, \text{ and}$$

$$Z = \{(2,3,4,5,6,7,1),(3,4,5,6,7,1,2),(4,3,6,7,1,2,5),(2,3,6,7,1,4,5)\},$$

induce $K_{1,3}$ in the graphs $\Gamma(D_5)$, $\Gamma(D_6)$ and $\Gamma(D_7)$, respectively. To complete the proof, we show that if $\Gamma(D_n)$ contains a claw as its induced subgraph, then $\Gamma(D_{n+3})$ has a claw, too. To see this, let $\{f, g_1, g_2, g_3\}$ induces $K_{1,3}$ with $f$ as its center in $\Gamma(D_n)$, then $\alpha = (n+2, n+3, n+1)$ and $\beta = (n+3, n+1, n+2)$ are two distinct derangements on the set $\{n+1, n+2, n+3\}$. It is easy to check that $\{f \times \alpha, g_1 \times \beta, g_2 \times \beta, g_3 \times \beta\}$ induces $K_{1,3}$ in $\Gamma(D_{n+3})$. So, we are done. □

**Theorem 2.12.** [3] *A graph $G$ is the line graph of some graph if and only if none of the nine graphs in Figure 3 is an induced subgraph of $G$.*

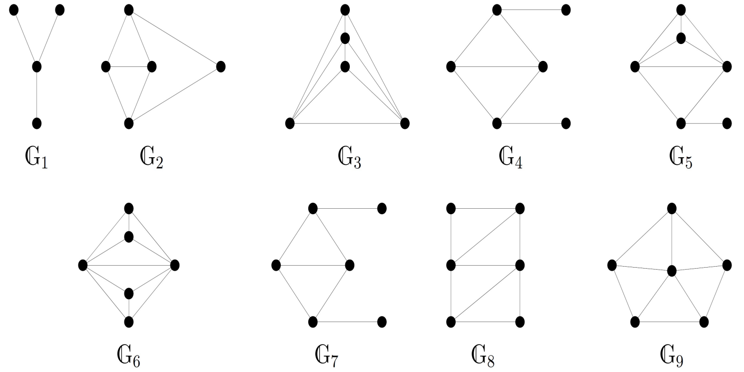

FIGURE 3. Forbidden induced subgraphs in a line graph.

**Corollary 2.13.** *The graph $\Gamma(D_n)$ is a line graph if and only if $n \leq 4$.*

*Proof.* This follows from Example 2.2 and Theorems 2.11 and 2.12. □

**Proposition 2.14.** *For $i, j \in [n]$, let $A_{ij} = \{f \in D_n | f(i) = j\}$. Then $|A_{ij}| = d_{n-2} + d_{n-1}$.*

*Proof.* Derangements $f \in A_{ij}$ can be divided into two types, those for which $f(j) = i$ and those for which $f(j) \neq i$. Counting derangements of the first type is easy: $f \in A_{ij}$ and $f(j) = i$ if and only if the restriction of $f$ to $[n] - \{i, j\}$ is a derangement of order $n - 2$. So, we have $|D_{n-2}|$ derangements of this type. Also, there is a one-to-one correspondence between the set of derangements of the second type and $D_{n-1}$. In fact, for any derangement $f$ of the second type, we can define the derangement $\overline{f} : [n] - \{j\} \to [n] - \{j\}$ by

$$\overline{f}(k) = \begin{cases} f(j), & k = i, \\ f(k), & k \neq i. \end{cases}$$

Therefore, $|A_{ij}| = d_{n-1} + d_{n-2}$. □

**Proposition 2.15.** *For $i, j \in [n]$, let $A_{ij} = \{f \in D_n | f(i) = j\}$. Then $A_{ij}$ is a maximal independent set of $\Gamma(D_n)$.*

*Proof.* It is clear that $A_{ij}$s are independent sets. We prove that $A_{ij}$ is a maximal independent set. To see this, suppose to the contrary $\alpha \in D_n - A_{ij}$ and $A_{ij} \cup \{\alpha\}$ is an independent set in $\Gamma(D_n)$. Now, consider the Latin rectangle

$$\begin{bmatrix} 1 & 2 & \dots & n \\ \alpha(1) & \alpha(2) & \dots & \alpha(n) \end{bmatrix}.$$

By Ryser's theorem (see [1, Theorem 7.5]), this Latin rectangle can be extended to an $n \times n$ Latin square, say $L$. Since $j$ appears in the $i$th column exactly one time, we can deduce that the ith component of one of the rows of $L$ is $j$. Represent this row as $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$. Since $\gamma_i = j$, so $\gamma \in A_{ij}$. Also, $\gamma$ and $\alpha$ are two rows of the above Latin square and $\gamma \in N(\alpha)$, which is a contradiction. □

**Conjecture 2.16.** *For any positive integer n, any $A_{ij}$ is a maximum independent set in $\Gamma(D_n)$ and so $\alpha(\Gamma(D_n)) = \frac{d_n}{n-1}$.*

**Remark 2.17.** If Conjecture 2.16 holds, then by Theorem 2.3, we have

$$\alpha(\Gamma(D_n))\chi(\Gamma(D_n)) = \alpha(\Gamma(D_n))\omega(\Gamma(D_n)) = d_n.$$

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**Hossein Moshtagh**

Department of Computer Science,

University of Garmsar, Garmsar, Semnan, Iran.

`h.moshtagh@fmgarmsar.ac.ir`

**Farzad Shaveisi**

Department of Mathematics, Faculty of Science,

Razi University, Kermanshah, Iran.

`f.shaveisi@razi.ac.ir`