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Research Paper

SOME RESULTS ON THE GRAPH OF DERANGEMENTS

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Abstract. The graph of derangements, denoted by $\Gamma(D_n)$, is a simple graph whose vertex set is the set of all derangements on $[n]$ and two distinct vertices f and g are adjacent if and only if $f(i) \neq g(i)$, for every $i \in [n]$. In this paper, some properties of this graph are presented. The clique number and the vertex chromatic number of this graph are determined. Then we show that for every positive integer $n \geq 5$, $\Gamma(D_n)$ is neither a perfect graph nor a cograph. Moreover, this graph can not be a line graph unless $n \leq 4$. Maximum cliques and maximum independent sets are studied, too.

1. Introduction

We begin with some definitions and notations on graphs. Let $G = (V(G), E(G))$ be a simple graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The complete graph with n vertices and the complete bipartite graph with parts of size n_1, n_2 are denoted by K_n and K_{n_1,n_2}, respectively; in particular, if either $n_1 = 1$ or $n_2 = 1$, the complete bipartite graph is called a *star* graph.

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The star graph $K_{1,3}$ is called a *claw*, too. A *claw-free* graph is a graph with no claw as its induced subgraph. A graph is said to be a *refinement of a star* graph if it contains a vertex which is adjacent to any other vertices.

Here, a cycle (path) with n-vertices is denoted by C_n (P_n). For a graph G, let $\chi(G)$ denote the *vertex chromatic number* of the graph G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices receive different colors. A *clique* of a graph is its complete subgraph, and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. An independent set of G is a set A of its vertices which are pairwise nonadjacent; the maximum cardinality of an independent set is called the *independence number* and it is denoted by $\alpha(G)$. For any graph G and every subset $X \subseteq V(G)$ the *induced subgraph* of G on X, denoted by $G[X]$, is a graph with the vertex set X and to distinct vertices $x_1, x_2 \in X$ are adjacent if and only if $x_1x_2 \in E(G)$. A graph G is called *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$. Also, a *cograph* is a graph which does not contain P_4 as its induced subgraph. The *line graph* of G, denoted by $L(G)$, is a graph whose vertex set is the edge set of G and two distinct vertices of $L(G)$ are adjacent if they have a common vertex as edges of G. The graph G is called a line graph if it is isomorphic with the line graph of some graph. For more background on graph theory, see [2, 6].

A $m \times n$ *Latin rectangle* is an $m \times n$ matrix, with symbols from a set of cardinality n, such that each symbol occurs only once in each row and only once in each column. Also, if $m = n$ then the matrix is called a *Latin square* of order n (see [5] for more details). In this paper, by $[n]$ we mean the set $\{1, 2, \ldots, n\}$. The set of all functions from $[m]$ to $[n]$ is denoted by $F_{m,n}$ and the set of all permutations on $[n]$ is denoted by S_n. A fixed point of $f \in F_{n,n}$ is an element $i \in [n]$ such that $f(i) = i$. Some of the deepest theorems in mathematics involve fixed points. Here, we will focus on permutations that have no fixed points. A permutation with no fixed points is called a *derangement*. The set of derangements in S_n is denoted D_n and its cardinality is denoted by d_n. It is well-known that the nth derangement number, $d_n = |D_n|$, is the integer closest to $n!/e$. For example, $|D_4| = 9$ and $|D_5| = 44$. For simplicity, in this paper, we denote any derangement d on $[n]$, by its image $d = (d(1), d(2), \ldots, d(n))$. In fact, d is a combinatorial permutation of the numbers $1, 2, \ldots, n$ with no fixed point. For more details we refer the reader to [7].

The *graph of derangements*, denoted by $\Gamma(D_n)$, is a simple graph whose vertex set is D_n and two distinct vertices f, g are adjacent if they are derangements of each other, that is, $f(r) \neq g(r)$, for every $r \in [n]$. In Section 2, we study the coloring of this graph. It is shown that $\omega(\Gamma(D_n)) = \chi(\Gamma(D_n)) = n - 1$. Also, the independent sets are studied.

2. Main results

We start this section with the main definition of this paper.

Definition 2.1. Let D_n be the set of derangements on $[n]$. The graph of derangements, denoted by $\Gamma(D_n)$, is a simple graph whose vertex set is D_n and two distinct vertices f, g are adjacent if and only if they are derangements of each other, that is, $f(r) \neq g(r)$, for every $r \in [n]$.

This graph, in fact, is an induced subgraph of $Cay(S_n, D_n)$ on derangements [4].

Example 2.2. (i) It is clear that $\Gamma(D_1)$ has no vertex, $\Gamma(D_2) \cong K_1$ and $\Gamma(D_3) \cong K_2$.

(ii) For $n = 4$, we have

$$\begin{aligned} D_4 &= \{(2,1,4,3),(4,3,1,2),(3,4,2,1),(3,4,1,2),(4,3,2,1), \\ &\quad (3,1,4,2),(2,4,1,3),(2,3,4,1),(4,1,2,3)\}, \end{aligned}$$

and

$$K_1 = \{(2,1,4,3),(3,4,2,1),(4,3,1,2)\}, K_2 = \{(2,1,4,3),(4,3,1,2),(3,4,1,2)\},$$
$$K_3 = \{(2,1,4,3),(3,1,4,2),(4,3,2,1)\}, K_4 = \{(2,3,4,1),(3,4,2,1),(4,1,2,3)\},$$

are four maximum cliques in $\Gamma(D_4)$. This graph has 9 vertices, 12 edges and diameter 3. In fact, this graph is pictured as in Figure 1.

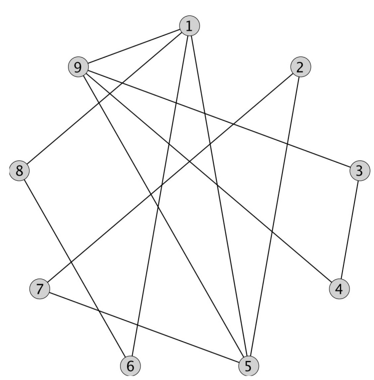

Figure 1. The graph $\Gamma(D_4)$.

(iii) The graph $\Gamma(D_5)$ has 44 vertices, 276 edges and diameter 3. This graph can be drawn as in Figure 2.

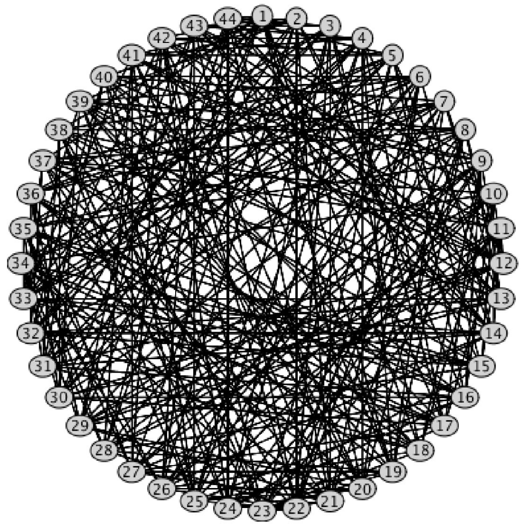

Figure 2. The graph $\Gamma(D_5)$.

From classical graph theory, we know that for any graph G, $\omega(G) \leq \chi(G)$. In the following theorem, it is shown that the clique number and the chromatic number of the graph of derangements are equal.

Theorem 2.3. *For any positive integer n, $\omega(\Gamma(D_n)) = \chi(\Gamma(D_n)) = n - 1$.*

Proof. First we define a map $c : V(\Gamma(D_n)) = D_n \longrightarrow \{1, 2, \ldots, n-1\}$ by $c(f) = f(1)$, for every derangements f. Since $f(1) \neq 1$, so $2 \leq c(f) \leq n$. Also, if f, g are two adjacent vertices in $\Gamma(D_n)$, then $f(1) \neq g(1)$. Thus c is a proper vertex coloring for the graph $\Gamma(D_n)$. This yields that $\chi(\Gamma(D_n)) \leq n - 1$.

On the other hand, if $\eta \in D_n$ is a derangement such that $\eta(i) = i + 1$, for every $i \in [n]$, i.e, $\eta = (2, 3, \ldots, n, 1)$, then one can easily check that $W = \{\eta, \eta^2, \ldots, \eta^{n-1}\}$ is a clique of order $n-1$ in $\Gamma(D_n)$. Thus $\chi(\Gamma(D_n)) \geq \omega(\Gamma(D_n)) \geq n-1$. Therefore, $\omega(\Gamma(D_n)) = \chi(\Gamma(D_n)) = n-1$. □

Recall that, a Latin rectangle (square) is reduced if the values in the first row and column are in the natural order.

Remark 2.4. The proof of Theorem 2.3, shows that any maximal clique in $\Gamma(D_n)$ contains $n - 1$ vertices. Also, the method of constructing cliques of order $n - 1$ shows that the number of these cliques in $\Gamma(D_n)$ equals to the number of reduced Latin squares on $\{1, 2, \ldots, n\}$.

Theorem 2.3 shows that the clique number of $\Gamma(D_n)$ equals its chromatic number. However, $\Gamma(D_n)$ for $n \geq 5$ may have some induced subgraph whose clique number is less than its chromatic number. This means that $\Gamma(D_n)$ is not a perfect graph for every $n \geq 5$. In the next theorem, it is shown that for every $n \geq 5$, $\Gamma(D_n)$ contains the cycle C_5 (the path P_4) as its induced subgraph. First, we prove the following two lemmas.

Lemma 2.5. *If $n \geq 5$ and C_5 is an induced subgraph of $\Gamma(D_n)$, then it is an induced subgraph of $\Gamma(D_{n+4})$, too.*

Proof. Let $\{f_1, f_2, f_3, f_4, f_5\} \subseteq D_n$ induce a cycle of length five in $\Gamma(D_n)$. Since there are nine derangements on $\{n+1, n+2, n+3, n+4\}$, we can choose three derangements α, β, γ on this set. Now, one can easily check that

$$\{f_1 \times \alpha, f_2 \times \beta, f_3 \times \alpha, f_4 \times \beta, f_5 \times \gamma\},$$

is a subset of derangements on $[n+4]$ and induce C_5 in $\Gamma(D_{n+4})$. □

Lemma 2.6. *If $n \geq 5$ and P_4 is an induced subgraph of $\Gamma(D_n)$, then it is an induced subgraph of $\Gamma(D_{n+3})$, too.*

Proof. Let $\{f_1, f_2, f_3, f_4\} \subseteq D_n$ induces P_4 in $\Gamma(D_n)$. There are two derangements on the set $\{n+1, n+2, n+3\}$, say α, β on this set. Now, one can easily check that

$$\{f_1 \times \alpha, f_2 \times \beta, f_3 \times \alpha, f_4 \times \beta\},$$

is a subset of derangements on $[n+3]$ and iduces P_4 in $\Gamma(D_{n+3})$. □

Theorem 2.7. *For every $n \geq 5$, $\Gamma(D_n)$ contains both P_4 and C_5 as its induced subgraphs.*

Proof. Let

$$\begin{aligned}
X &= \{(2,3,4,5,1), (3,4,5,1,2), (2,5,4,3,1), (3,1,2,5,4)\},\\
Y &= \{(2,3,4,5,6,1), (3,4,5,6,1,2), (5,6,4,1,2,3), (3,5,2,6,4,1)\},\\
Z &= \{(2,3,4,5,6,7,1), (3,4,5,6,7,1,2), (2,3,4,7,6,5,1), (3,4,5,6,1,7,2)\},\\
T &= \{(2,3,4,5,6,7,8,1), (3,4,5,6,7,8,1,2), (2,3,4,7,8,5,6,1), (3,4,5,6,1,7,8,2)\}.
\end{aligned}$$

Then X, Y, Z and T induce P_4 in graphs $\Gamma(D_5)$, $\Gamma(D_6)$, $\Gamma(D_7)$ and $\Gamma(D_8)$, respectively.

Also, the subsets

$$X' = X \cup \{(4,5,1,3,2)\},\ Y' = Y \cup \{(6,4,1,3,2,5)\},$$
$$Z' = Z \cup \{(4,1,6,2,7,5,3)\},\ T' = T \cup \{(4,1,7,2,3,8,6,5)\},$$

induce C_5 in graphs $\Gamma(D_5)$, $\Gamma(D_6)$, $\Gamma(D_7)$ and $\Gamma(D_8)$, respectively. So, the assertion follows from induction and Lemmas 2.5 and 2.6. □

Since $\chi(C_5) \neq \omega(C_5)$, from Theorem 2.7, we have the following immediate corollary.

Corollary 2.8. *For every $n \geq 5$, $\Gamma(D_n)$ is neither a perfect graph nor a cograph.*

Remark 2.9. From Theorem 2.3, we know that for every $n \geq 4$, $\Gamma(D_n)$ contains a triangle and so this graph is triangle-free (bipartite) if and only if $n \leq 3$.

A graph is said to be square-free if it contains no C_4 as its induced subgraph. In the following proposition, we study when $\Gamma(D_n)$ is a square-free graph.

Proposition 2.10. *$\Gamma(D_n)$ is a square-free graph if and only if $n \leq 4$.*

Proof. If $n \leq 4$, then as we saw in Example 2.2, $\Gamma(D_n)$ contains no square (as its induced subgraph). Also, it is easy to see that the sets

$$X = \{(2,3,4,5,1),(3,4,5,1,2),(5,1,4,2,3),(3,5,2,1,4)\},$$

$$Y = \{(2,3,4,5,6,1),(3,4,5,6,1,2),(5,6,4,1,2,3),(3,5,2,6,1,4)\}, \text{ and}$$

$$Z = \{(2,3,4,5,6,7,1),(3,4,5,6,7,1,2),(5,6,4,7,1,2,3),(3,5,2,6,7,1,4)\},$$

induce C_4 in the graphs $\Gamma(D_5)$, $\Gamma(D_6)$ and $\Gamma(D_7)$, respectively. Now, we claim that if $\Gamma(D_n)$ contains C_4 as its induced subgraph, then $\Gamma(D_{n+3})$ has induced C_4, too. To see this, let $\{f_1, f_2, f_3, f_4\}$ induces C_4 in $\Gamma(D_n)$, then $\alpha = (n+2, n+3, n+1)$ and $\beta = (n+3, n+1, n+2)$ are two distinct derangements on the set $\{n+1, n+2, n+3\}$. It is easy to check that $\{f_1 \times \alpha, f_2 \times \beta, f_3 \times \alpha, f_4 \times \beta\}$ induces C_4 in $\Gamma(D_{n+3})$. So, the claim is proved and the assertion follows from the claim. □

Recall that a graph is called claw-free if it does not contain $K_{1,3}$ as its induced subgraph.

Theorem 2.11. *$\Gamma(D_n)$ is a claw-free graph if and only if $n \leq 4$.*

Proof. If $n \leq 4$, then as we saw in Example 2.2, $\Gamma(D_n)$ is claw-free. Also, it is easy to see that the sets

$$X = \{(2,3,4,5,1),(3,4,5,1,2),(4,3,1,2,5),(2,3,1,5,4)\},$$

$$Y = \{(2,3,4,5,6,1),(3,4,5,6,1,2),(4,3,6,1,2,5),(2,3,6,1,4,5)\}, \text{ and}$$

$$Z = \{(2,3,4,5,6,7,1),(3,4,5,6,7,1,2),(4,3,6,7,1,2,5),(2,3,6,7,1,4,5)\},$$

induce $K_{1,3}$ in the graphs $\Gamma(D_5)$, $\Gamma(D_6)$ and $\Gamma(D_7)$, respectively. To complete the proof, we show that if $\Gamma(D_n)$ contains a claw as its induced subgraph, then $\Gamma(D_{n+3})$ has a claw, too. To see this, let $\{f, g_1, g_2, g_3\}$ induces $K_{1,3}$ with f as its center in $\Gamma(D_n)$, then $\alpha = (n+2, n+3, n+1)$ and $\beta = (n+3, n+1, n+2)$ are two distinct derangements on the set $\{n+1, n+2, n+3\}$. It is easy to check that $\{f \times \alpha, g_1 \times \beta, g_2 \times \beta, g_3 \times \beta\}$ induces $K_{1,3}$ in $\Gamma(D_{n+3})$. So, we are done.

□

Theorem 2.12. [3] *A graph G is the line graph of some graph if and only if none of the nine graphs in Figure 3 is an induced subgraph of G.*

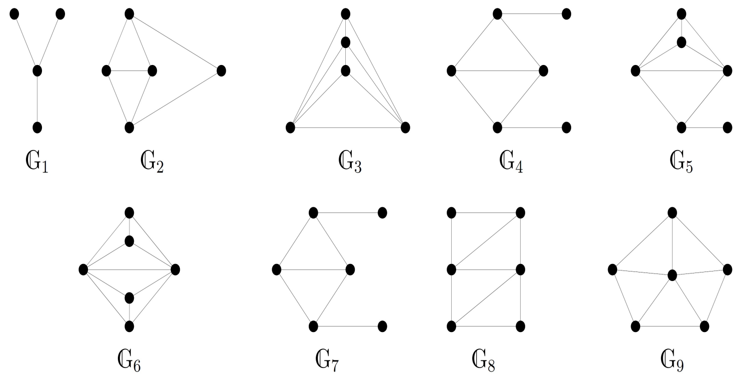

FIGURE 3. Forbidden induced subgraphs in a line graph.

Corollary 2.13. *The graph $\Gamma(D_n)$ is a line graph if and only if $n \leq 4$.*

Proof. This follows from Example 2.2 and Theorems 2.11 and 2.12. □

Proposition 2.14. *For $i, j \in [n]$, let $A_{ij} = \{f \in D_n | f(i) = j\}$. Then $|A_{ij}| = d_{n-2} + d_{n-1}$.*

Proof. Derangements $f \in A_{ij}$ can be divided into two types, those for which $f(j) = i$ and those for which $f(j) \neq i$. Counting derangements of the first type is easy: $f \in A_{ij}$ and $f(j) = i$ if and only if the restriction of f to $[n] - \{i, j\}$ is a derangement of order $n - 2$. So, we have $|D_{n-2}|$ derangements of this type. Also, there is a one-to-one correspondence between the set of derangements of the second type and D_{n-1}. In fact, for any derangement f of the second type, we can define the derangement $\overline{f} : [n] - \{j\} \to [n] - \{j\}$ by

$$\overline{f}(k) = \begin{cases} f(j), & k = i, \\ f(k), & k \neq i. \end{cases}$$

Therefore, $|A_{ij}| = d_{n-1} + d_{n-2}$. □

Proposition 2.15. *For $i, j \in [n]$, let $A_{ij} = \{f \in D_n | f(i) = j\}$. Then A_{ij} is a maximal independent set of $\Gamma(D_n)$.*

Proof. It is clear that A_{ij}s are independent sets. We prove that A_{ij} is a maximal independent set. To see this, suppose to the contrary $\alpha \in D_n - A_{ij}$ and $A_{ij} \cup \{\alpha\}$ is an independent set in $\Gamma(D_n)$. Now, consider the Latin rectangle

$$\begin{bmatrix} 1 & 2 & \dots & n \\ \alpha(1) & \alpha(2) & \dots & \alpha(n) \end{bmatrix}.$$

By Ryser's theorem (see [1, Theorem 7.5]), this Latin rectangle can be extended to an $n \times n$ Latin square, say L. Since j appears in the ith column exactly one time, we can deduce that the ith component of one of the rows of L is j. Represent this row as $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$. Since $\gamma_i = j$, so $\gamma \in A_{ij}$. Also, γ and α are two rows of the above Latin square and $\gamma \in N(\alpha)$, which is a contradiction. □

Conjecture 2.16. *For any positive integer n, any A_{ij} is a maximum independent set in $\Gamma(D_n)$ and so $\alpha(\Gamma(D_n)) = \frac{d_n}{n-1}$.*

Remark 2.17. If Conjecture 2.16 holds, then by Theorem 2.3, we have

$$\alpha(\Gamma(D_n))\chi(\Gamma(D_n)) = \alpha(\Gamma(D_n))\omega(\Gamma(D_n)) = d_n.$$

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