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Research Paper

ON THE GRADED GORENSTEIN HOMOLOGICAL DIMENSIONS

ZAHRA HEIDARIAN*

Abstract. This paper explores the intriguing relationships between graded homological dimensions and standard homological dimensions. We present an overview of the concept of graded Gorenstein homological dimensions for modules associated with commutative graded rings and derive key properties in this context. This framework provides a natural foundation for comparing graded Gorenstein homological dimensions with conventional Gorenstein homological dimensions.

1. Introduction

Suppose $R = \bigoplus_{i\in\mathbb{Z}} R_i$ is graded commutative ring that is nontrivial. In works like [1] and [2], Asensio, Ramos, and Torrecillas investigated Gorenstein graded-injective, graded-projective, and graded-flat modules, comparing these with their ungraded counterparts. This paper aims to expand upon their findings by applying the theory to graded modules, drawing connections

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*Corresponding author

between the graded case and the ungraded one. We also extend some results from [13] to the graded category.

In Section 2, we introduce gr-totally reflexive modules and define the gr-G-dimension for finitely generated modules providing a graded analog to the G-dimension theory discussed in [13].

In Section 3, we define gr-Gorenstein projective modules and describe them via their complete gr-projective resolutions. We denote the category of gr-Gorenstein projective R-modules by gr-Gp(R) and extend several theorems concerning gr-Gp(R) from [18] to the graded setting.

Finally, we introduce gr-Gorenstein injective modules and their respective dimensions, represented by gr-Gi(R) for gr-Gorenstein injective R-modules. We show that certain theorems regarding gr-Gi(R) from [18] hold in the graded setting. As a main result,We demonstrate that for a Noetherian graded ring R and a graded R-module M with a finite gr-Gorenstein injective dimension, the following inequality holds:

$$\mathrm{Gid}_R M \leq \text{gr-}\mathrm{Gid}_R M + 1,$$

where gr-$\mathrm{Gid}_R M$ signifies the graded Gorenstein injective dimension of M.

2. Prerequisites

In this study, let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ represent a non-trivial commutative graded ring. We will start by revisiting essential definitions and concepts related to graded rings and modules. The symbol R–Mod denotes the category of all R-modules, while R–gr represents the category of all graded R-modules.For two graded R-modules, $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and $N = \bigoplus_{i\in\mathbb{Z}} N_i$, a morphism $f : M \to N$ in R–gr is defined as an R-module homomorphism that satisfies $f(M_i) \subseteq N_i$ for every $i \in \mathbb{Z}$,referred to as a homogeneous morphism. The set of such morphisms is denoted:

$$\mathrm{Hom}_{R\text{–gr}}(M, N) = \{f : M \to N \mid f \text{ is a homogeneous } R\text{–homomorphism}\}.$$

An R-module homomorphism $f : M \to N$ is termed homogeneous of degree i if $f(M_n) \subseteq N_{n+i}$ for every $n \in \mathbb{Z}$. The collection of degree i homogeneous homomorphisms is written as $\mathrm{Hom}_R(M, N)_i$. Therefore, $\mathrm{Hom}_R(M, N)_0 = \mathrm{Hom}_{R\text{–gr}}(M, N)$. Together, the $\mathbb{Z}$-submodules $\mathrm{Hom}_R(M, N)_i$ of $\mathrm{Hom}_R(M, N)$ form a direct sum, leading to:

$${}^{\mathrm{gr}}\mathrm{Hom}_R(M, N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(M, N)_i,$$

which is thus a graded R-submodule within $\mathrm{Hom}_R(M, N)$. In general, ${}^{\mathrm{gr}}\mathrm{Hom}_R(M, N) \neq \mathrm{Hom}_R(M, N)$, though equality holds when M is fg module.

For any graded R-module N, we define ${}^{\text{gr}}\text{Ext}^i_R(M,N)$ as the i-th right derived functor of ${}^{\text{gr}}\text{Hom}_R(-,N)$ in R–gr. Thus, if $\mathbf{P}.$ is a projective resolution of M in R–gr, we have:

$$ {}^{\text{gr}}\text{Ext}^i_R(M,N) \cong H_i({}^{\text{gr}}\text{Hom}_R(\mathbf{P}.,N)), $$

for each $i \geq 0$. By definition, It directly follows that ${}^{\text{gr}}\text{Ext}^i_R(M,N) = \text{Ext}^i_R(M,N)$ when R is a Noetherian ring and M is a fg graded R-module.

The graded Abelian group $M \otimes_R N$ is constructed by defining $(M \otimes_R N)_m$ for $m \in \mathbb{Z}$ as the subgroup generated by elements $x \otimes y$, where $x \in M_i$ and $y \in N_j$ with $i + j = m$. Since graded modules have graded free resolutions, this results in naturally graded modules $\text{Tor}^R_i(M,N)$.

For a graded R-module M and an integer i, we denote by $M(i)$ the graded R-module with grading defined by $(M(i))_m = M_{i+m}$.

In R–gr, the injective (resp. projective, flat) objects are known as gr-injective (resp. gr-projective, gr-flat) modules. We refer to projective (resp. flat) objects within objects of R–gr simply as projective (resp. flat) graded modules, since M is gr-projective (resp. gr-flat) if and only if is a projective (resp. flat) graded module (cf. [22, CA, sec I.2]). The gr-injective dimension of a graded module M is denoted by ${}^{\text{gr}}\text{id}_R\, M$; similarly, $\text{pd}_R\, M$ and $\text{fd}_R\, M$ indicate the projective and flat dimensions of M, respectively.

For a family $(M_\alpha)_{\alpha \in I}$ of graded R-modules,the direct sum and product in in R–gr exist and are given by:

$$ \bigoplus_{\alpha \in I} M_\alpha = \bigoplus_{i \in \mathbb{Z}} \left(\bigoplus_{\alpha \in I} (M_\alpha)_i \right) \quad \text{and} \quad \prod_{\alpha \in I} M_\alpha = \bigoplus_{i \in \mathbb{Z}} \left(\prod_{\alpha \in I} (M_\alpha)_i \right), $$

as per [15, page 289].

We abbreviate Gorenstein projective module as Gor proj module, Gorenstain injective module as Gor inj, finitely generated module as fg module R-homomorphism as R-hom.

Proposition 2.1. *Suppose $\mathbf{X}_.$ is a complex of R–modules and R–homs.*

(i) *Suppose N is a direct summand of an R–module M. If the complex $Hom_R(\mathbf{X}_., M)$ is exact, then $Hom_R(\mathbf{X}_., N)$ is also exact.*

(ii) *For any projective R–module P, the complex $Hom_R(\mathbf{X}_., P)$ is exact if and only if the complex $Hom_R(\mathbf{X}_., F)$ is exact for every free R–module F.*

The graded version of this proposition also holds.

Remark 2.2. Let R and S be commutative graded rings.

(i) [22, Proposition I.2.14] For a graded a graded R–module M, a graded S–module E and a graded R-S–bimodule N, the following natural graded isomorphism exists:

$$ {}^{\text{gr}}\text{Hom}_S(M \otimes_R N, E) \cong {}^{\text{gr}}\text{Hom}_R(M, {}^{\text{gr}}\text{Hom}_S(N,E)). $$

If N is flat R–module and E is a gr-injective S–module, then ${}^{\mathrm{gr}}\mathrm{Hom}_S(N, E)$ is a gr-injective R–module.

(ii) [1, Lemma 2.3] Let M be a graded R–module, E a graded S–module and N an R-S–bimodule. If M is fg module and E is gr-injective, then there is an isomorphism:

$$M \otimes_R {}^{\mathrm{gr}}\mathrm{Hom}_S(N, E) \cong {}^{\mathrm{gr}}\mathrm{Hom}_S({}^{\mathrm{gr}}\mathrm{Hom}_R(M, N), E).$$

(iii) [15, page 289] For a family of graded R–modules, $(M_\alpha)_{\alpha \in I}$, The following bijections are valid:

$$ {}^{\mathrm{gr}}\mathrm{Hom}_R(\bigoplus_{\alpha\in I} M_\alpha, -) \xrightarrow{\cong} {}^{gr}\prod_{\alpha\in I} {}^{\mathrm{gr}}\mathrm{Hom}_R(M_\alpha, -),$$
$$ {}^{\mathrm{gr}}\mathrm{Hom}_R(-, \prod_{\alpha\in I} M_\alpha) \xrightarrow{\cong} {}^{gr}\prod_{\alpha\in I} {}^{\mathrm{gr}}\mathrm{Hom}_R(-, M_\alpha).$$

Lemma 2.3. *Let M be a graded R–module, E a graded S–module and N a R-S–bimodule.*

(i) *If E is an gr-injective S–module, then for all $n \geq 0$, there exists a natural graded isomorphism:*

$$ {}^{\mathrm{gr}}\mathrm{Ext}^n_R(M, {}^{\mathrm{gr}}\mathrm{Hom}_S(N, E)) \cong {}^{\mathrm{gr}}\mathrm{Hom}_S(\mathit{Tor}^R_n(M, N), E),$$

(ii) *If M is fg module and E is gr-injective, then for all $n \geq 0$, we have an isomorphism:*

$$\mathit{Tor}^R_n(M, {}^{\mathrm{gr}}\mathrm{Hom}_S(N, E)) \cong {}^{\mathrm{gr}}\mathrm{Hom}_S({}^{\mathrm{gr}}\mathrm{Ext}^n_R(M, N), E).$$

Proof. The result follows from Remark 2.2 (i), (ii) and the fact the homology functors $\mathrm{H}_n(-)$ commutes with the exact functor ${}^{\mathrm{gr}}\mathrm{Hom}_S(-, E)$. □

Definition 2.4. A graded ring $R = \bigoplus_{i\in\mathbb{Z}} R_i$ is referred to as a strongly graded ring if $R_iR_j = R_{i+j}$ holds for all $i, j \in \mathbb{Z}$. In other words, R is strongly graded if and only if $RR_1 = R$ (cf.[22, I.3]).

The forgetful functor $U : R{-}gr \to R{-}Mod$ maps each graded R-module M to its underlying ungraded R-module, denoted by $\underline{M}$(typically written as $\underline{M}$ instead of $U(M)$). This functor has a right adjoint F that assigns to $M \in R$–Mod the graded R-module $F(M) = \bigoplus_{i\in\mathbb{Z}} M_i$, where each M_i is a copy of M with the R-module structure defined by $rx_i = (rx)_{i+j}$ for $r \in R_j$. For an R–homomorphism $f : M \to N$ the corresponding map

$$F(f) : F(M) \to F(N),$$

is a homogeneous morphism by $F(f)(x_i) = (f(x))_i$.

Lemma 2.5. *(*[23, Lemma 1.3]*) For $M \in R - gr$, we have $F(\underline{M}) = \bigoplus_{i\in\mathbb{Z}} M(i)$.*

Definition 2.6. Suppose $\mathcal{F}$ is a collection of graded R–modules. An $\mathcal{F}$ -precover of a graded R–module M is defined as a homogeneous homomorphism $\varphi : F \to M$ with $F \in \mathcal{F}$ such that the sequence ${}^{\mathrm{gr}}\mathrm{Hom}_R(G, F) \to {}^{\mathrm{gr}}\mathrm{Hom}_R(G, M) \to 0$ is exact for every $G \in \mathcal{F}$. If, any homogeneous endomorphism $f : F \to F$ satisfying $\varphi f = \varphi$ is an automorphism, then $\varphi : F \to M$ is referred to as an $\mathcal{F}$-cover.

Definition 2.7. For a graded R–module M, We say that M is gr-faithfully flat if, for any short exact sequence of graded R–module and homogeneous R-hom

$$\mathbf{N} : 0 \to N_0 \to N_1 \to N_2 \to 0,$$

the sequence $\mathbf{N} \otimes_R M$ is exact if $\mathbf{N}$ is exact and conversely.

Example 2.8. (**Examples of $\mathcal{F}$ -cover and gr-faithfully flat Modules**)

- **Example of $\mathcal{F}$ -cover Module:** Let $R = k[x]$, the polynomial ring over a field k, and consider the graded module $M = k[x]/(x^2)$. The natural projection $\pi : R \to M$ is an example of an *-cover since any endomorphism of R that factors through π must be an automorphism of R.
- **Example of gr-faithfully flat Module:** Consider the graded ring $R = k[x, y]$ with the standard grading, where $R_0 = k$. The module $R[x^{-1}]$, obtained by inverting x, is a gr-faithfully flat module over R because tensoring with it preserves exact sequences and it does not annihilate any nonzero graded module.

Proposition 2.9. *Let R be a commutative graded ring and let M be a graded R–module. The following conditions are equivalent.*

(i) *M is gr-faithfully flat R-module.*
(ii) *M is flat R–module and $N \otimes_R M \neq 0$ for any non-zero graded R–module N.*
(iii) *M is flat R–module and $\mathfrak{m}M \neq M$ for every gr-maximal ideal $\mathfrak{m}$ of R.*

Proof. If φ is a morphism in R–gr, then both $\mathrm{Ker}\,\varphi$ and $\mathrm{Im}\,\varphi$ possess graded R-module structure. Additionally, if x is a non-zero homogeneous element in a graded R–module N, then $\mathrm{Ann}\,_R(x)$ is a graded ideal of R. Therefore the argumentis analogous to [20, Theorem 7.2]. □

Remark 2.10. Let $(R, \mathfrak{m})$ be graded local Noetherian ring with a unique gr-maximal ideal $\mathfrak{m}$. The gr–$\mathfrak{m}$–adic completion ${}^{gr}\widehat{R}$ of R is defined as the graded inverse limit of the system $\{R/\mathfrak{m}^n | n \in \mathbb{N}\}$ (cf.[21, Definition 6.3]). According to [21, 6.10], we have ${}^{gr}\widehat{R}/{}^{gr}\widehat{\mathfrak{m}}^n \cong R/\mathfrak{m}^n$ for every $n \geq 1$. Hence by Proposition 2.9 and [21, 6.10], ${}^{gr}\widehat{R}$ is a Noetherian gr-faithfully flat R–module. If $E := {}^{gr}E_R(R/\mathfrak{m})$ represents the gr-injective envelope of $R/\mathfrak{m}$, then by [21, Proposition 6.14], we have ${}^{\mathrm{gr}}\mathrm{Hom}_R(E, E) \cong {}^{gr}\widehat{R}$.

3. gr-G-dimension of finitely generated graded modules

In this section, let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a graded R–module. Consider a complex $\mathbf{L}$ of graded modules and homogeneous morphisms, represented by an infinite sequence

$$\mathbf{L} : \cdots \xrightarrow{\delta_{i+1}} L_i \xrightarrow{\delta_i} L_{i-1} \xrightarrow{\delta_{i-1}} L_{i-2} \longrightarrow \cdots ,$$

where $\delta_i\delta_{i+1} = 0$ for all $i \in \mathbb{Z}$. The i-th homology module of $\mathbf{L}$ is given by $\mathrm{H}_i(\mathbf{L}) = \mathrm{Ker}\,\delta_i/\mathrm{Im}\,\delta_{i+1}$, forming a graded R–module. The complex $\mathbf{L}$ is referred to as acyclic if for all $i \in \mathbb{Z}$, $\mathrm{H}_i(\mathbf{L}) = 0$.

Lemma 3.1. *Let $\mathbf{L}$ be an acyclic complex of fg graded projective R–module. The following statements concerning $\mathbf{L}$ are equivalent:*

(i) *The complex ${}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{L}, R)$ is an acyclic complex.*
(ii) *For all graded flat R–modules Q, the complex ${}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{L}, Q)$ is acyclic.*
(iii) *For any gr-injective R–module E, the complex $E \otimes_R \mathbf{L}$ is acyclic.*

Proof. Let Q be a graded flat and E a gr-injective R-module. Since $\mathbf{L}$ is composed of fg projective modules, by Remark 2.2 (i), we have an isomorphism of complexes

$${}^{\mathrm{gr}}\mathrm{Hom}_R({}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{L}, Q), E) \cong {}^{\mathrm{gr}}\mathrm{Hom}_R(Q, E) \otimes_R \mathbf{L}.$$

$(ii) \Rightarrow (i)$ This implication is immediate.

$(i) \Rightarrow (iii)$ Apply the isomorphism above with $Q = R$.

$(iii) \Rightarrow (ii)$ Suppose E be a faithfully gr-injective module. Since ${}^{\mathrm{gr}}\mathrm{Hom}_R(Q, E)$ is an gr-injective, the complex $\mathrm{Hom}\,_R(Q, E) \otimes_R \mathbf{L}$ is acyclic. Thus, the assertion follows from the above isomorphism. □

Definition 3.2. A complex $\mathbf{L}$ that fulfills the condition in Lemma 3.1 is referred to as gr-totally acyclic. A graded R-module M is termed gr-totally reflexive if there exists a gr-totally acyclic complex $\mathbf{L}$ for which M is isomorphic to the cokernel of the morphism $L_1 \to L_0$.

Note: Observe that every fg graded projective module L is gr-totally reflexive as shown by the complex $0 \to L \xrightarrow{\cong} L \to 0$.

By Lemma 3.1, we canreadily deduce the following Proposition.

Proposition 3.3. *Let S be an graded ring and let $\varphi : R \to S$ be a ring homomorphism such that $\varphi(R_i) \subseteq S_i$, for all $i \in \mathbb{Z}$ and $\mathrm{fd}\,_R S < \infty$. For any gr-totally reflexive R–module G, the module $S \otimes_R G$ is a gr-totally reflexive S–module.*

A fg R–module N is defined as gr-reflexive if the canonical morphism $M \to {}^{\mathrm{gr}}\mathrm{Hom}_R({}^{\mathrm{gr}}\mathrm{Hom}_R(N, R), R)$ is an isomorphism.

Lemma 3.4. *If R is a Noetherian graded ring, a fg graded R–module G is gr-totally reflexive if and only if it is gr-reflexive and satisfies* ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(G,R) = 0 = {}^{\mathrm{gr}}\mathrm{Ext}^i_R({}^{\mathrm{gr}}\mathrm{Hom}_R(G,R),R)$ *for each* $i \geq 1$.

Proof. The result follows in a similar manner to the ungraded case as presented in [3, Proposition 4.11]. □

Definition 3.5. The exact sequence $\cdots \to G_i \to G_{i-1} \to \cdots \to G_0 \to M \to 0$, is a gr-G-resolution for a fg graded module M, if each module G_i is gr-totally reflexive and each homomorphism in the sequence is homogeneous.

Definition 3.6. The gr-G-dimension of a non-zero fg graded R–module M, denoted by ${}^{gr}\text{G-dim}_R M$, is defined as the smallest integer $m \geq 0$, for which there exists a gr-G-resolution of M where $G_j = 0$, for all $j > m$. If no such m exists, then ${}^{gr}\text{G-dim}_R M$ is infinite. We set ${}^{gr}\text{G-dim}_R 0 = -\infty$.

Remark 3.7. Suppose R is a Noetherian graded ring and M a fg graded R–module. Since, for all graded modules N the functors $\mathrm{Hom}_R(-,N)$ and ${}^{\mathrm{gr}}\mathrm{Hom}_R(-,N)$ coincide on category of fg graded R–modules, and for each $i \geq 1$, ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(-,N) = \mathrm{Ext}^i_R(-,N)$ on the same category. Thus, by Lemma 3.4, M is totally reflexive if and only if M is gr-totally reflexive(in the graded sense). Consequently, ${}^{gr}\text{G-dim}_R M = \text{G-dim}_R M$ holds.

By [3, Corollary 3.16], [9, Theorem 1.2.7, Proposition 1.2.10] and Remark 3.7, directly derive the following theorem and corollary:

Theorem 3.8. *Let R be a Noetherian graded ring and M a finitely generated graded R–module of finite gr-G-dimension. For any integer $n \geq 0$ the following conditions are equivalent:*

(i) ${}^{gr}\textit{G-dim}_R M \leq n$.

(ii) ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(M,R) = 0$, *for all* $i > n$.

(iii) ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(M,N) = 0$, *for all $i > n$ and for all graded R–modules N, with* $pd_R N < \infty$.

(iv) *In any gr-G-resolution*

$$\cdots \to G_i \to G_{i-1} \to \cdots \to G_0 \to M \to 0,$$

the module $Coker(G_{n+1} \to G_n)$ *is gr-totally reflexive.*

Corollary 3.9. *Let R be a Noetherian graded ring. For every fg graded module M of finite gr-G-dimension we have:*

$${}^{gr}\textit{G-dim}_R M = sup\{i \in \mathbb{Z} | \, {}^{\mathrm{gr}}\mathrm{Ext}^i_R(M,R) \neq 0\}.$$

Proposition 3.10. *Let R be a Notherian graded ring and let M be a fg graded R–module. Then the following conditions are equivalent:*

(i) *M is gr-totally reflexive.*

(ii) *$M_{\mathfrak{p}}$ is totally reflexive $R_{\mathfrak{p}}$–module for all $\mathfrak{p} \in {}^{gr}Spec\,(R)$.*

(iii) *$M_{(\mathfrak{p})}$ is gr-totally reflexive $R_{(\mathfrak{p})}$–module for all $\mathfrak{p} \in {}^{gr}Spec\,(R)$.*

Proof. $(i) \Rightarrow (iii)$ This follows from proposition 3.3.

$(iii) \Rightarrow (ii)$ Let $\mathfrak{p} \in {}^{gr}\mathrm{Spec}\,(R)$ and suppose $M_{(\mathfrak{p})}$ is gr-totally reflexive $R_{(\mathfrak{p})}$–module. Then by [3, Corollary 4.15] and Remark 3.7, $M_{\mathfrak{p}} \cong (M_{(\mathfrak{p})})_{\mathfrak{p}R_{(\mathfrak{p})}}$ is totally reflexive $R_{\mathfrak{p}}$–module.

$(ii) \Rightarrow (i)$ For each R–module N let $\delta_M : N \to \mathrm{Hom}_R(\mathrm{Hom}_R(N, R), R)$ denote the canonical map. Then $(\delta_M)_{\mathfrak{p}} \cong \delta_{M_{\mathfrak{p}}}$, for all $\mathfrak{p} \in {}^{gr}\mathrm{Spec}\,(R)$. By assumption, $(\mathrm{Ker}\,\delta_M)_{\mathfrak{p}} = \mathrm{Ker}\,\delta_{M_{\mathfrak{p}}} = 0$, for all $\mathfrak{p} \in {}^{gr}\mathrm{Spec}\,(R)$. Since, $Ass_R(\mathrm{Ker}\,\delta_M) \subseteq {}^{gr}\mathrm{Spec}\,(R)$, we conclude $Ass_R(\mathrm{Ker}\,\delta_M) = \emptyset$, implying $\mathrm{Ker}\,\delta_M = 0$.

Furthermore, for all $i \geq 1$ and $\mathfrak{p} \in {}^{gr}\mathrm{Spec}\,(R)$,

$$\mathrm{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0 = \mathrm{Ext}^i_{R_{\mathfrak{p}}}(\mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}), R_{\mathfrak{p}}).$$

Since $\mathrm{Ass}\,_R(\mathrm{Ext}^i_R(M, R))$ and $\mathrm{Ass}\,_R(\mathrm{Ext}^i_R(\mathrm{Hom}_R(M, R), R))$ are subsets of ${}^{gr}\mathrm{Spec}\,(R)$, it follows that $\mathrm{Ext}^i_R(M, R) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, R), R)$, for all $i \geq 1$. Thus, by Lemma 3.4, M is gr-totally reflexive. □

From Proposition 3.10, we can readily deduce the following proposition.

Proposition 3.11. *Consider R as a Noetherian graded ring and M as a fg graded R–module. The following statements are equivalent for any integer $n \geq 0$, :*

(i) *${}^{gr}G\text{-}dim\,_R M \leq n$.*

(ii) *$G\text{-}dim\,_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq n$ for all $\mathfrak{p} \in {}^{gr}Spec\,(R)$.*

(iii) *${}^{gr}G\text{-}dim\,_{R_{(\mathfrak{p})}} M_{(\mathfrak{p})} \leq n$ for all $\mathfrak{p} \in {}^{gr}Spec\,(R)$.*

Thus it holds,

$G\text{-}dim\,_R M = sup\{G\text{-}dim\,_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in {}^{gr}Spec\,(R)\} = sup\{G\text{-}dim\,_{R_{(\mathfrak{p})}} M_{(\mathfrak{p})} \mid \mathfrak{p} \in {}^{gr}Spec\,(R)\}.$

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a graded local ring, and let M be a fg graded R–module. Then ${}^{gr}G\text{-}dim\,_R M = G\text{-}dim\,_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$.*

Proof. Assuming $\mathrm{G\text{-}dim}\,_{R_{\mathfrak{m}}} M_{\mathfrak{m}} < \infty$, by Proposition 3.11, we have

$$\mathrm{G\text{-}dim}\,_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \mathrm{G\text{-}dim}\,_{(R_{\mathfrak{m}})_{\mathfrak{p}R_{\mathfrak{m}}}} (M_{\mathfrak{m}})_{\mathfrak{p}R_{\mathfrak{m}}},$$

for all $\mathfrak{p} \in {}^{gr}\mathrm{Spec}\,(R)\}$.

Proposition 3.11 then implies that $\text{G-dim}_R M < \infty$. Consequently, if $\text{G-dim}_R M = \infty$, then $\text{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m} = \infty$. We may therefore assume that $\text{G-dim}_R M < \infty$. Using Proposition 3.11, it follows that $\text{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m} \leq \text{G-dim}_R M$.

By [5, Proposition 1.5.15], the functor $- \otimes_R R_\mathfrak{m}$, is faithfully flat and exact on R-gr. Hence, from Remark 3.7,

$$\begin{aligned} {}^{gr}\text{G-dim}_R M = \text{G-dim}_R M &= sup\{i \in \mathbb{Z} |\, \text{Ext}^i_R(M,R) \neq 0\} \\ &= sup\{i \in \mathbb{Z} |\, \text{Ext}^i_R(M,R) \otimes_R R_\mathfrak{m} \neq 0\} \\ &= sup\{i \in \mathbb{Z} |\, \text{Ext}^i_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}) \neq 0\} \\ &= \text{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m}. \end{aligned}$$

□

Theorem 3.13. *Consider $(R,\mathfrak{m})$ as a graded local ring with ${}^{gr}dim R = d$. The following statements are equivalent:*

(i) *R is Gorenstein.*

(ii) *$gr\, G\text{-}dim_R(R/\mathfrak{m}) \leq d$.*

(iii) *For every fg graded R–module M, we have ${}^{gr}\, G\text{-}dim_R M \leq d$.*

Proof. The implication $(iii) \Rightarrow (ii)$ is immediate.

For $(ii) \Rightarrow (i)$, suppose ${}^{gr}\text{G-dim}_R(R/\mathfrak{m}) < \infty$. Then by Proposition 3.11, $\text{G-dim}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) < \infty$ which implies $R_\mathfrak{m}$ is a Gorenstein ring by [9, Theorem 1.4.9]. Consequently, by [5, Exercise 3.6.20], R is also Gorenstein ring.

For $(i) \Rightarrow (iii)$, this follows directly from [5, Exercise 3.6.20] and Corollary 3.12. □

Theorem 3.14. *Let $(R,\mathfrak{m})$ be a graded local ring. For any fg graded R–module M of finite gr-G-dimension the following equality holds:*

$${}^{gr}\, G\text{-}dim_R M = grade\,(\mathfrak{m}, R) - grade\,(\mathfrak{m}, M).$$

Proof. Using Corollary 3.12 and [3, Theorem 4.13], it follows that

$${}^{gr}\text{G-dim}_R M = \text{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m} = \text{depth}\, R_\mathfrak{m} - \text{depth}_{R_\mathfrak{m}} M_\mathfrak{m}.$$

Additionally, by [5, Proposition 1.5.15], we have

$$\text{grade}\,(\mathfrak{m}, R) = \text{depth}\, R_\mathfrak{m},$$

$$\text{grade}\,(\mathfrak{m}, M) = \text{depth}_{R_\mathfrak{m}} M_\mathfrak{m}.$$

Therefore, the assertion is established. □

Proposition 3.15. *Suppose $R = \bigoplus_{i\geq 0} R_i$ is positively graded ring with subring R_0 being local ring with maximal ideal $\mathfrak{m}_0$. Then R will be a local ring with graded structure and unique gr-maximal ideal $\mathfrak{m} = \mathfrak{m}_0 \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots$. If $\widehat{R_\mathfrak{m}}$ represents the completion of the local ring $R_\mathfrak{m}$ and $\hat{R}^\mathfrak{m}$ denotes $\mathfrak{m}$-adic completion of R with respect to $\mathfrak{m}$ then:*

$$^{gr}G\text{-}dim_{\,R}M = G\text{-}dim_{\,R_\mathfrak{m}} M_\mathfrak{m} = G\text{-}dim_{\,\hat{R}^\mathfrak{m}}(\hat{R}^\mathfrak{m} \otimes_R M).$$

Proof. From [7, 13.1.16], we have $\widehat{R_\mathfrak{m}} \cong \hat{R}^\mathfrak{m}$, so:

$$\begin{aligned}\text{G-dim}_{\,\widehat{R}^\mathfrak{m}}(\widehat{R}^\mathfrak{m} \otimes_R M) &= \text{G-dim}_{\,\widehat{R_\mathfrak{m}}}(\widehat{R_\mathfrak{m}} \otimes_R M) \\ &= \text{G-dim}_{\,\widehat{R_\mathfrak{m}}}(\widehat{R_\mathfrak{m}} \otimes_{R_\mathfrak{m}} (R_\mathfrak{m} \otimes_R M)) = \text{G-dim}_{\,R_\mathfrak{m}} M_\mathfrak{m} \\ &= {}^{gr}\text{G-dim}_{\,R}M.\end{aligned}$$

□

4. gr-Gorenstein projective module

The definitions introduced here are based on concepts from [1].

Definition 4.1. Consider a complex $0 \to M \to P^0 \to P^1 \to \cdots$ in R-gr. This structure is termed a gr-projective resolvent of M if each P_i is graded projective and if the functor $\text{Hom}_{\,R-gr}(.,P)$ ensures exactness for any graded projective module P,.

When a complete gr-projective resolution of M is constructed, the resulting complex

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots,$$

is referred to as a complete gr-projective resolution of M.

Definition 4.2. A graded R–module M is called gr-Gor proj if and only if it possesses a complete gr-projective resolution

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots,$$

that remains exact and retains its exactness when $\text{Hom}_{\,R-gr}(-,P)$ is applied for any graded projective module P.

We denote the collection of gr-Gor proj R–modules, as $gr - Gp(R)$.

Proposition 4.3. *A graded R–module M is gr-Gor proj if and only if it has a complete gr-projective resolution*

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots,$$

that is exact and remains exact under the functor ${}^{\text{gr}}\text{Hom}_R(.,P)$, for any projective graded module P.

Proof. Suppose M is gr-Gor proj with a complete gr-projective resolution,

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots .$$

We need to demonstrate that the functor ${}^{\mathrm{gr}}\mathrm{Hom}_R(-, P)$ maintains the exactness of this complex. Notice that ${}^{\mathrm{gr}}\mathrm{Hom}_R(-, P) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(-, P)_i$. Moreover, since $\mathrm{Hom}_R(-, P)_0 = \mathrm{Hom}_{R-gr}(-, P)$ we know from [5, Exercise 1.5.19] that $\mathrm{Hom}_R(-, P)_i = \mathrm{Hom}_{R-gr}(-, P(i))$. For any homogenous morphism $f : M \to N$ be a of graded R–modules, we obtain the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(N, P)_i & \to & \mathrm{Hom}_R(M, P)_i \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_{R-gr}(N, P(i)) & \to & \mathrm{Hom}_{R-gr}(M, P(i)). \end{array}$$

($\Longrightarrow$) Since $P(i)$ is projective graded R–module, $\mathrm{Hom}_{R-gr}(-, P(i))$ preserves the exactness of the complex,meaning that $\mathrm{Hom}_R(-, P)_i$ does as well. Therefore, ${}^{\mathrm{gr}}\mathrm{Hom}_R(-, P) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(-, P)_i$ guarantees the exactness of the complex as required.

($\Longleftarrow$) For the reverse implication, if $\mathrm{Hom}_R(-, P(-i))$ preserves the exactness for all $i \in \mathbb{Z}$, then

$$\mathrm{Hom}_R(-, P(-i))_i = \mathrm{Hom}_{R-gr}(-, P(-i)(i)) = \mathrm{Hom}_{R-gr}(-, P).$$

confirms that $\mathrm{Hom}_{R-gr}(-, P)$ maintains exactness, and thus M is gr-Gor proj module.

□

Definition 4.4. A graded R-module M is termed strongly gr-Gorenstein projective if there exists an exact sequence of graded projective R–module P, such that,

$$\mathbf{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \to \cdots ,$$

with $M \cong \mathrm{Ker}\, f$ and if ${}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{P}, Q)$ is exact for any graded projective R–module Q.

Remark 4.5. (**Relationship Between Strongly gr-Gorenstein Projective and gr-Gorenstein Projective Modules**)

The concept of gr-Gorenstein projective modules extends Gorenstein projective modules to the graded category. A module is gr-Gorenstein projective if it admits a complete gr-projective resolution, meaning an exact sequence of graded projective modules that remains exact when applying ${}^{\mathrm{gr}}\mathrm{Hom}_R(-, P)$ for any graded projective module P.

A module is strongly gr-Gorenstein projective if it has a periodic resolution of graded projective modules that remains exact under ${}^{\mathrm{gr}}\mathrm{Hom}_R(-, P)$ for any graded projective module P. This stronger condition ensures that every syzygy in the resolution is also gr-Gorenstein projective.

Key Differences:

- Every strongly gr-Gorenstein projective module is gr-Gorenstein projective, but the converse is not necessarily true.
- Strongly gr-Gorenstein projective modules are preserved under syzygies, whereas general gr-Gorenstein projective modules may not satisfy this property in every resolution.
- Strongly gr-Gorenstein projective modules are particularly useful in relative homological algebra because they behave more predictably in homotopy categories.

Example 4.6. Let $R = k[x]/(x^2)$ be a graded local ring where $\deg(x) = 1$. Consider the graded module $M = R$. M is gr-Gorenstein projective because it has a complete resolution of graded projective modules:

$$\cdots \to R \xrightarrow{x} R \xrightarrow{x} R \to M \to 0.$$

However, M is not necessarily strongly gr-Gorenstein projective, since its syzygies may not remain gr-Gorenstein projective under all conditions.

This distinction is crucial in graded homological dimensions, particularly when studying gr-Gorenstein projective precovers or Gorenstein derived categories in the graded setting.

Proposition 4.7. *If $(P_\alpha)_{\alpha\in I}$ is a collection of strongly gr-Gorenstein projective modules, then $\bigoplus P_\alpha$ is also strongly gr-Gorenstein projective.*

Proof. The proof is similar to the one provided in the ungraded case in [6, Proposition 2.2]. □

Proposition 4.8. *Any graded projective R–module is also strongly gr-Gorenstein projective R–module.*

Proof. Since the functor ${}^{\mathrm{gr}}\mathrm{Hom}_R(-,-)$ is additive and left exact similar manner to $\mathrm{Hom}_R(-,-)$, the result follows directly, analogous to the reasoning found in [6, Proposition 2.3]. □

Theorem 4.9. *A module is gr-Gor proj if and only if it is a direct summand of a strongly gr-Gorenstein projective module.*

Proof. According to Remark 2.2 (iii), we have $\mathrm{Hom}_R(\bigoplus -, L) \cong {}^{gr}\prod {}^{\mathrm{gr}}\mathrm{Hom}_R(-, L)$ for any graded R–module L. Since it is evident that ${}^{gr}\prod(-)$ is an exact functor on R-gr, we can directly apply the proof in [6, Theorem 2.7]. □

Theorem 4.10. *Let R be a Noetherian graded ring. A fg graded R-module is gr-Gor proj if and only if it is gr-totaly reflexive.*

Proof. This result aligns with the ungraded case, as discussed in [9, Theorem 4.2.6]. □

Corollary 4.11. *A fg graded R–module is gr-Gor proj if and only if it is Gor proj R–module.*

Proof. The result follows immediately from Remark 3.7 and Theorem 4.10. □

Definition 4.12. For any collection, $\mathcal{X}$ of graded R–modules, we introduce several key terms, based on their ungraded counterparts.

(a) We say $\mathcal{X}$ is called gr-projectively resolving(resp. gr-injectively resolving) if it contains all graded projective(resp. graded injective) R–modules, and if in every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$(resp. $X' \in \mathcal{X}$) we have $X' \in \mathcal{X}$(resp. $X'' \in \mathcal{X}$) if and only if $X \in \mathcal{X}$.

(b) The associated left and right gr-orthogonal classes are defined as follows:

$$^{\perp}\mathcal{X} = \{M \in R - gr| \, ^{\mathrm{gr}}\mathrm{Ext}^j_R(M,Y) = 0 \quad \forall Y \in \mathcal{X} \quad \text{and} \quad \forall j > 0\},$$

correspondingly,

$$\mathcal{X}^{\perp} = \{N \in R - gr| \, ^{\mathrm{gr}}\mathrm{Ext}^j_R(Y,N) = 0 \quad \forall Y \in \mathcal{X} \quad \text{and} \quad \forall j > 0\}.$$

(c) A right $\mathcal{X}$–resolution of M is an exact sequence

$$\mathbf{X} = 0 \to M \to X^0 \to X^1 \to \cdots ,$$

with $X^n \in \mathcal{X}$ for all $n \geq 0$. A left $\mathcal{X}$–resolution of M is an exact sequence

$$\mathbf{X} = \cdots \to X_1 \to X_0 \to M \to 0,$$

with $X_n \in \mathcal{X}$ for all $n \geq 0$.

(d) For any (right or left) $\mathcal{X}$–resolution of M, denoted $\mathbf{X}$, we say that $\mathbf{X}$, is gr-co-proper(resp. gr-proper) if the sequence $^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{X}, Y)$(resp. $^{\mathrm{gr}}\mathrm{Hom}_R(Y, \mathbf{X})$) is exact for all $Y \in \mathcal{X}$.

Remark 4.13. Suppose $\mathcal{X}$ is a collection of graded R–module.

(i) For a family of graded R-modules $\{M_i\}_{i \in I}$, if the class $\mathcal{X}$ is closed under direct sums and each module M_i has a gr-co-proper right resolution within $\mathcal{X}$, then their direct sum $\bigoplus M_i$ also admits a gr-co-proper right resolution within $\mathcal{X}$.

(ii) The left gr-orthogonal class $^{\perp}\mathcal{X}$ is projectively resolving and stable under arbitrary direct sums.

(iii) The right gr-orthogonal class $\mathcal{X}^{\perp}$ is gr-injectively resolving and closed under arbitrary gr-direct product.

Using these definitions, we can characterize gr-Gor proj modules as follows:

Proposition 4.14. *A graded R–module M is gr-Gor proj if and only if it belongs to the left gr-orthogonal class $^{\perp gr}P(R)$ and has a gr-co-proper right resolution by graded projective R-modules.*

Proof. This characterization follows by analogy with the ungraded case presented in [18, Proposition 2.3]. □

Proposition 4.15. *If a class of graded R-modules $\mathcal{X}$ is either gr-projectively or gr-injectively resolving, and is also closed under either countable direct sums or countable graded direct products, then $\mathcal{X}$ is closed under direct summands.*

Proof. If X belongs to the class $\mathcal{X}$ and can be expressed as a direct sum $X = Y \oplus Z$ for some R-module Z, then Y is a direct summand of X.

Case 1: Suppose the class $\mathcal{X}$ is closed under countable direct sums. Let $W = Y \oplus Z \oplus Y \oplus Z \oplus \cdots$, so $W \cong X \oplus X \oplus X \oplus \cdots$, which implies $W \in \mathcal{X}$. Since $W \cong Y \oplus W$, we have $Y \oplus W \in \mathcal{X}$. For a gr-projectively resolving class X, consider the split exact sequence $0 \to Y \to X \to W \to 0$ where $X = Y \oplus W$, and for a gr-injectively resolving class X, consider the split exact sequence $0 \to W \to X \to Y \to 0$ where $X = Y \oplus W$. In either case, it follows that Y belongs to $\mathcal{X}$.

Case 2: Suppose $\mathcal{X}$ be closed under countable graded direct products. Let $W = {}^{gr}(Y \times Z \times Y \times Z \times \cdots) = {}^{gr}(X \times X \times \cdots)$, where $W_n = ({}^{gr}\prod X)_n = \prod X_n$ for all $n \in \mathbb{Z}$. Thus

$$\begin{aligned} W &= \bigoplus_{n\in\mathbb{Z}}(X_n \times X_n \times \cdots) = \bigoplus_{n\in\mathbb{Z}}(Y_n \times Z_n \times Y_n \times Z_n \times \cdots) \\ &= \bigoplus_{n\in\mathbb{Z}}(Y_n \times X_n \times X_n \times \cdots) = \bigoplus_{n\in\mathbb{Z}}(Y_n \times W_n) = Y \bigoplus W. \end{aligned}$$

Using reasoning similar to Case 1, we conclude that $Y \in \mathcal{X}$.

□

Proposition 4.16. *The class $^{gr}Gp(R)$ is closed under arbitrary direct sums and direct summands.*

Proof. Let $\{M_i\}_{i\in I}$ be a family of gr-Gorenstein projective R–modules. By Proposition 4.14, each M_i is in $^{\perp gr}P(R)$ and has a gr-co-proper right $^{gr}P(R)$–resolution for each $i \in I$. Since $^{gr}P(R)$ is closed under direct sum, Remark 4.13(i) and (ii) imply that $\bigoplus M_i \in {}^{\perp gr}P(R)$ and that $\bigoplus M_i$ has a gr-co-proper right $^{\perp gr}P(R)$–resolution. Thus, by proposition 4.14, we conclude $\bigoplus M_i \in {}^{gr}Gp(R)$.

Moreover, since ${}^{\perp gr}P(R)$ is projectively resolving(see Remark 4.13(ii)), it follows that ${}^{gr}Gp(R)$ is projectively resolving. Using Proposition 4.15, we conclude ${}^{gr}Gp(R)$ is closed under direct summands. □

Proposition 4.17. *If M is a gr-Gor proj module, then there exists a complete gr-projective resolution,*

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots,$$

consisting of graded free modules F_n and F^n such that $M \cong Im(F_0 \to F^0)$.

Proof. his follows analogously to the ungraded case presented in [18, Proposition 2.4]. □

Definition 4.18. A gr-Gor proj resolution of a graded module M is an exact sequence $\cdots \to G_i \to G_{i-1} \to \cdots \to G_0 \to M \to 0$, where each G_i is gr-Gor proj module.

Definition 4.19. The gr-Gor proj dimension of graded R-module $M \neq 0$ symbolized as ${}^{gr}\mathrm{Gpd}_R M$ is the smallest integer $n \geq 0$ for which gr-Gor proj resolution of M exists with $G_i = 0$ for all $i > n$. If no such integer n exists, then ${}^{gr}\mathrm{Gpd}_R M$ is considered infinite. By convention ${}^{gr}\mathrm{Gpd}_R 0 = -\infty$

Theorem 4.20. *For a graded R-module M of finite gr-Gor proj dimension, and some integer n, the following statements are equivalent:*

(i) ${}^{gr}Gpd_R M \leq n$

(ii) ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(M, L) = 0$*, for all $i > n$, where L is graded R–modules with $pd_R L < \infty$*

(iii) ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(M, Q) = 0$*, for all $i > n$, where Q is d graded projective R–module.*

(iv) *Whenever we have an exact sequence of the form*

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

with $G_0, \ldots, G_{n-1}$ being gr-Gor proj, then K_n is also gr-Gor proj. As a result, we have: ${}^{gr}Gpd_R M = sup\{i \in \mathbb{N}_0 |^i_R(M, L) \neq 0$ for any graded R–module L with finite projective dimension. $= sup\{i \in \mathbb{N}_0 | {}^{\mathrm{gr}}\mathrm{Ext}^i_R(M, Q) \neq 0$ for some graded projective R-module$Q\}$.

Proof. The proof mirrors the approach for the ungraded case found in [18, Theorem 2.20]. □

Lemma 4.21. *For a gr-Gor proj R-module M, it holds that ${}^{\mathrm{gr}}\mathrm{Ext}^m_R(M, T) = 0$ for all $m > 0$ and for any graded R-module T with finite projective dimension or gr-injective dimension.*

Proof. The proof parallels that of the ungraded case in [14, Lemma 2.1]. □

Proposition 4.22. *Consider a graded R–module M and two exact sequence:*

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

$$0 \to \widetilde{K}_n \to \widetilde{G}_{n-1} \to \cdots \to \widetilde{G}_0 \to M \to 0,$$

where $G_0, \ldots, G_{n-1}$, and $\widetilde{G}_0, \ldots, \widetilde{G}_{n-1}$ are gr-Gor proj modules. Then K_n is gr-Gor proj if and only if $\widetilde{K}_n$ is gr-Gor Proj.

Proof. This follows directly from Propositions 4.14 and 4.16 and [3, Lemma 3.12]. □

Proposition 4.23. *In an exact sequence $0 \to M' \to M \to M'' \to 0$, if any two of the modules have finite gr-Gor proj dimension,then the third one also has finite gr-Gor proj dimension.*

Proof. The proofs of [4, Proposition 3.4] and [18, Theorem 2.5] imply that this proposition is a consequence of Proposition 4.22. □

Remark 4.24. Let R be a graded ring and M, P be graded R–modules. Then the following properties hold.

- (i) P is gr-projective if and only if ${}^{\mathrm{gr}}\mathrm{Ext}^1_R(P, N) = 0$, for all graded R–module N.
- (ii) ${}^{gr}\mathrm{pd}\,_R(M) \leq n$ if and only if ${}^{\mathrm{gr}}\mathrm{Ext}^{n+1}_R(M, N) = 0$, for all graded R–module N.
- (iii) $\mathrm{pd}\,_R M = sup\{i \in \mathbb{Z} |\, {}^{\mathrm{gr}}\mathrm{Ext}^i_R(M, F) \neq 0 \,\text{for some graded free}\, R\text{-module}\, F\}$.

Proposition 4.25. *For every graded R-module M, we have ${}^{gr}Gpd\,_R M \leq pd\,_R M$ with equality if M has finite projective dimension.*

Proof. The first part is clear. From Remark 4.24(iii) and Theorem 4.20, it follows that ${}^{gr}\mathrm{Gpd}\,_R M \geq \mathrm{pd}\,_R M$, thus ${}^{gr}\mathrm{Gpd}\,_R M = \mathrm{pd}\,_R M$. □

Remark 4.26. Considering Theorem 4.10, Propositions 4.23 and 4.25, following results can be inferred:

- (i) Suppose R is a Noetherian graded ring. Consider an exact sequence of fg graded R-modules, $0 \to M' \to M \to M'' \to 0$ If any two modules among M', M and M" have finite gr-G-dimension, then the remaining module also has finite gr-G-dimension.
- (ii) Over a Noetherian graded ring R, for any fg R–module M, ${}^{gr}\mathrm{G}\text{-}\mathrm{dim}\,_R M \leq \mathrm{pd}\,_R M$ with equality if $\mathrm{pd}\,_R M$ is finite.

Theorem 4.27. *If M is a graded R–module with finite gr-Gor proj dimension n, then M has a surjective gr-Gor proj precover $\varphi : G \to M$, where $pd\,_R Ker\varphi = {}^{gr}Gpd\,_R M - 1$ (interpreted as $K = 0$, if $n = 0$).*

Proof. The proof follows directly from Definition 2.6 of precover and proceeds similarly to the ungraded case presented in [18, Theorem 2.10]. □

Proposition 4.28. *Consider a graded R–module M with finite gr-injective dimension. Then, we have:* ${}^{gr}Gpd_R M = pd_R M$.

Proof. Since ${}^{gr}\mathrm{Gpd}_R M \leq \mathrm{pd}_R M$, it remains to establish that $\mathrm{pd}_R M \leq {}^{gr}\mathrm{Gpd}_R M$. It can be assumed that ${}^{gr}\mathrm{Gpd}_R M < \infty$. Consider first the scenario where M is gr-Gor proj. By definition of gr-Gor proj module, there is a complete resolution:

$$\mathbf{P}_. = \cdots \to P_1 \to P_0 \to P^0 \to \cdots ,$$

of graded projective R–modules where $M \cong \mathrm{Ker}\,(P^0 \to P^1)$ and ${}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{P}_., Q)$ remains exact for any graded projective R–module Q. Consequently, there exists a short exact sequence $0 \to M \to P \to M' \to 0$, where P is gr-projective and M' is gr-Gorenstein projective. Given that ${}^{gr}\mathrm{id}_R M < \infty$, Lemma 4.21, implies that, ${}^{\mathrm{gr}}\mathrm{Ext}^1_R(M', M) = 0$. This causes the sequence $0 \to M \to P \to M' \to 0$ to split, indicating that M is a direct summand of a graded projective R-module, thus rendering projective.

For the case where ${}^{gr}\mathrm{Gpd}_R M > 0$. Theorem 4.27, guarantees the existence of an exact sequence $0 \to C \to H \to M \to 0$, where H is gr-Gor proj and $\mathrm{pd}_R C = {}^{gr}\mathrm{Gpd}_R M - 1$. Additionally, there exists an exact sequence $0 \to H \to P \to H' \to 0$, whith P as a graded projective and H' as a gr-Gor proj.

Consider the pullback diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & H & \to & M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & P & \to & Q & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H' & = & H' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since P is graded projective and $\mathrm{pd}_R C = {}^{gr}\mathrm{Gpd}_R M - 1$, the projective dimension of Q satisfies $\mathrm{pd}_R Q \leq {}^{gr}\mathrm{Gpd}_R M$. Furthermore, as H' is gr-Gor proj and ${}^{gr}\mathrm{id}_R M < \infty$, Lemma 4.21 ensures ${}^{\mathrm{gr}}\mathrm{Ext}^1_R(H', M) = 0$. Consequently, the last column $0 \to M \to Q \to H' \to 0$ splits, resulting in $Q \cong M \oplus H'$. Hence, we conclude that $\mathrm{pd}_R M \leq \mathrm{pd}_R Q \leq {}^{gr}\mathrm{Gpd}_R M$. □

Proposition 4.29. *Suppose $S = \bigoplus_{i\in\mathbb{Z}} S_i$ is a graded R–algebra with finite projective dimension over R, denoted by $\mathrm{pd}_R S < \infty$. Then, for any gr-Gor proj R–module G, the graded S–module $G \otimes_R S$ will also exhibit gr-Gorenstein projectivity.*

Proof. Suppose G is gr-Gor proj R–module. Then, there exists a complete gr-projective resolution:

$$\mathbf{P}. = \cdots \to P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} P^0 \overset{d^0}{\to} P^1 \overset{d^1}{\to} \cdots,$$

where $G \cong \mathrm{Ker}\,(P^0 \to P^1)$. We will demonstrate that the complex

$$\mathbf{P}. \otimes_R S = \cdots \to P_1 \otimes_R S \to P_0 \otimes_R S \to P^0 \otimes_R S \to P^1 \otimes_R S \to \cdots,$$

forms a complete gr-projective resolution for $G \otimes_R S$ and remains exact upon applying ${}^{\mathrm{gr}}\mathrm{Hom}_S(-, L)$ for any graded projective S–module L. Clearly, $G \otimes_R S \cong \mathrm{Ker}\,(d^0 \otimes_R S)$.

Using Remark 2.2 (i), for any graded projective S–module L, we obtain

$${}^{\mathrm{gr}}\mathrm{Hom}_S(\mathbf{P}. \otimes_R S, L) \cong {}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{P}.,{}^{*}\,\mathrm{Hom}_S(S, L)) \cong {}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{P}., L).$$

According to [25, Theorem 4.31], $P(S) \subseteq P(R)$, ensuring that for any graded projective S–module L, ${}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{P}., L)$ is exact. Therefore, ${}^{\mathrm{gr}}\mathrm{Hom}_S(\mathbf{P}. \otimes_R S, L)$ is exact for any graded projective S–module L.

To confirm that $\mathbf{P}. \otimes_R S$ is exact, let E be a gr-injective cogenerator R-module. Using the isomorphism,

$${}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{P}. \otimes_R S, E) \cong {}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{P}.,{}^{*}\,\mathrm{Hom}_R(S, E)),$$

we only need to verify that ${}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{P}. \otimes_R S, E)$ is exact. Since $\mathrm{pd}_R S < \infty$, it follows that $\mathrm{fd}_R S < \infty$, and thus by Remark 2.2(i), ${}^{gr}\mathrm{id}_S\, {}^{\mathrm{gr}}\mathrm{Hom}_R(S, E) < \infty$. For each n, there exists an exact sequence $0 \to \mathrm{Im}\, d_{n+1} \to P_n \to \mathrm{Im}\, d_n \to 0$. Consequently, for any graded R–module K, we obtain an exact sequence

$0 \to {}^{\mathrm{gr}}\mathrm{Hom}_R(\mathrm{Im}\, d_n, K) \to {}^{\mathrm{gr}}\mathrm{Hom}_R(P_n, K) \to {}^{\mathrm{gr}}\mathrm{Hom}_R(\mathrm{Im}\, d_{n+1}, K) \to {}^{\mathrm{gr}}\mathrm{Hom}^1_R(\mathrm{Im}\, d_n, K).$

Since $\mathrm{Im}\, d_n$ is gr-Gor proj, Lemma 4.21 ensures that, ${}^{\mathrm{gr}}\mathrm{Ext}^1_R(\mathrm{Im}\, d_n, L) = 0$ for all graded R–module K with finite gr-injective dimension. Therefore, ${}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{P}., {}^{\mathrm{gr}}\mathrm{Hom}_R(S, E))$ is exact. □

Proposition 4.30. *Suppose R is a graded Noetherian ring of finite gr-dimension. For any graded R–module M and any graded prime ideal $\mathfrak{p}$ in R,*

$${}^{gr}\,Gpd_{R_{(\mathfrak{p})}} M_{(\mathfrak{p})} \leq {}^{gr}\,Gpd_R.$$

Proof. Since R is Noetherian with finite krull dimension, it follows that $P(R) = F(R)$, and the conclusion follows from Proposition 4.29. □

Theorem 4.31. *Let R be graded Noetherian ring of finite gr-dimension and let M be a graded fg R–module. If both ${}^{gr}Gpd_R M$ and ${}^{gr}id_R M$ are finite, then $R_{(\mathfrak{p})}$ is Gorenstein for every $\mathfrak{p} \in Supp_R M$.*

Proof. Since $6gr\mathrm{id}_R M < \infty$ if and only if $\mathrm{id}_R M < \infty$, the result follows from Propositions 4.28, 4.30, and [16, Corollaries 4.3, 4.4]. □

Lemma 4.32. *If Q is a graded projective R-module. Then $F(Q)$ is graded projective R–module.*

Proof. This is immediate from 2.5, as $F(Q) \cong \bigoplus_{\lambda\in\mathbb{Z}} Q(\lambda)$. □

Proposition 4.33. *If M is gr-Gor proj R–module. Then M is also Gor proj as an R–module.*

Proof. Assume M is a gr-Gor proj. Then by 4.17, there exists a complete exact graded free resolution:

$$\mathbf{F}_. = \cdots \to F_1 \to F_0 \to F^0 \xrightarrow{d} F^1 \to \cdots,$$

with $\mathrm{Ker}\, d = M$. For any graded free R–module: ${}^{\mathrm{gr}}\mathrm{Hom}_R(\mathbf{F}_., P')$ is exact. giving an exact complex of free R–modules,

$$U(\mathbf{F}_.) = \cdots \to U(F_1) \to U(F_0) \to U(F^0) \xrightarrow{U(d)} U(F^1) \to \cdots,$$

where $\mathrm{Ker}\, U(d) = U(M)$.

Now let Q be a projective R–module. Then there exists a free R-module F' such that Q is a direct summand of F'. We need to show that $\mathrm{Hom}_{R-Mod}(U(\mathbf{F}.), F')$ is exact. Since $F(-)$ is the right adjoint of $U(-)$, we have

$$\mathrm{Hom}_{R-Mod}(U(-), -) \simeq \mathrm{Hom}_{R-gr}(-, F(-)),$$

implying that $\mathrm{Hom}_{R-Mod}(U(\mathbf{F}.), Q) \cong \mathrm{Hom}_{R-gr}(\mathbf{F}_., F(F'))$. By Proposition 4.32, $F(F')$ is a graded free and therefore is a graded projective R–module. As a result, $\mathrm{Hom}_{R-Mod}(U(\mathbf{P}.), F')$ is exact. Consequently, by Proposition 2.1, $\mathrm{Hom}_{R-Mod}(U(\mathbf{F}.), Q)$ is also exact completing the proof. □

Lemma 4.34. *Let $f : M \to N$ be an R-homomorphism between graded R–modules. For each integer $\lambda \in \mathbb{Z}$, let $f(\lambda) : M(\lambda) \to N(\lambda))$ be defined by $f(\lambda)(x) = f(x)$. Then the following commutative diagram holds:*

$$\begin{array}{ccc} F(M) & \overset{F(f)}{\to} & F(N) \\ U(M)\downarrow & & \downarrow U(N) \\ \bigoplus_{\lambda\in\mathbb{Z}} M(\lambda) & \underset{\bigoplus_{\lambda\in\mathbb{Z}} f(\lambda)}{\to} & \bigoplus_{\lambda\in\mathbb{Z}} N(\lambda) \end{array}$$

Proof. For an element $x_i \in M_i = M$ write $x_i = \sum\limits_{j\in\mathbb{Z}} x_j$, with $f(x_j) = \sum\limits_{j\in\mathbb{Z}} y_{j\theta}$, for $(j \in \mathbb{Z})$. We then have

$$f(x)_i = \sum_{j\in\mathbb{Z}} f(x_j) = \sum_{j\in\mathbb{Z}} (\sum_{\theta\in\mathbb{Z}} y_{j\theta}).$$

Thus,

$$U(N)(f(x)_i) = \sum_{\eta\in\mathbb{Z}} (\sum_{j\in\mathbb{Z}} y_{j\theta}).$$

On the other hand

$$\bigoplus_{\lambda\in\mathbb{Z}} f(\lambda)(U(M)(x_i)) = \bigoplus_{\lambda\in\mathbb{Z}} f(\lambda)(\sum_{\eta\in\mathbb{Z}} x_\eta) = \sum_{\eta\in\mathbb{Z}} f(x_\eta) = \sum_{\eta\in\mathbb{Z}} \sum_{\theta\in\mathbb{Z}} y_{\eta\theta}.$$

□

Proposition 4.35. *For a graded R-module M, if M is Gor proj, then its associated graded module $F(M)$ is graded Gor proj.*

Proof. According to [18, Proposition 2.4], there exists a free complete resolution

$$\mathbf{F}_{.} = \cdots \to F_1 \to F_0 \to F^0 \xrightarrow{d} F^1 \to \cdots,$$

with $\operatorname{Ker} d = M$. Using lemma 4.34 we we obtain the following commutative diagram, where the vertical arrows are isomorphisms:

$$\begin{array}{ccccccccc}
F(\mathbf{F}_.) : \cdots \longrightarrow & F(F_1) & \xrightarrow{F(d_1)} & F(F_0) & \xrightarrow{F(d_0)} & F(F^0) & \xrightarrow{F(d^0)} & F(F^1) & \longrightarrow \cdots, \\
& U(F_1)\downarrow\cong & & \cong\downarrow U(F_0) & & U(F_1)\downarrow\cong & & \cong\downarrow U(F_0) & \\
\mathbf{G}_. \quad \cdots \longrightarrow & \bigoplus\limits_{\lambda\in\mathbb{Z}} F_1(\lambda) & \xrightarrow[\oplus_{\lambda\in\mathbb{Z}} d_1(\lambda)]{} & \bigoplus\limits_{\lambda\in\mathbb{Z}} F_0(\lambda) & \xrightarrow[\oplus_{\lambda\in\mathbb{Z}} d_0(\lambda)]{} & \bigoplus\limits_{\lambda\in\mathbb{Z}} F^0(\lambda) & \xrightarrow[\oplus_{\lambda\in\mathbb{Z}} d^0(\lambda)]{} & \bigoplus\limits_{\lambda\in\mathbb{Z}} F^1(\lambda) & \longrightarrow \cdots,
\end{array}$$

Suppose Q is a graded projective R–module then

$$\operatorname{Hom}_{R-gr}(F(\mathbf{F}_.), F(Q)) \cong \operatorname{Hom}_{R-Mod}(U(F(\mathbf{F}_.)), Q) \cong \operatorname{Hom}(\mathbf{G}_., Q).$$

This yields the following commutative diagram:

$$\begin{array}{cccc}
\operatorname{Hom}_R(\mathbf{G}_., Q) : \cdots \longrightarrow & \operatorname{Hom}_R(\bigoplus\limits_{\lambda\in\mathbb{Z}} F^0(\lambda), Q) & \xrightarrow{\operatorname{Hom}_R(\oplus_{\lambda\in\mathbb{Z}} d_0(\lambda), \mathrm{id}_Q)} & \operatorname{Hom}_R(\bigoplus\limits_{\lambda\in\mathbb{Z}} F_0(\lambda), Q) \longrightarrow \cdots, \\
& \downarrow & & \downarrow \\
\mathbf{L}_. : \quad \cdots \longrightarrow & \prod\limits_{\lambda\in\mathbb{Z}} \operatorname{Hom}_R(F^0(\lambda), Q) & \xrightarrow[\prod_{\lambda\in\mathbb{Z}} \operatorname{Hom}_R(d_0(\lambda), \mathrm{id}_Q)]{} & \prod\limits_{\lambda\in\mathbb{Z}} \operatorname{Hom}_R(F_0(\lambda), Q) \longrightarrow \cdots,
\end{array}$$

By assumption for each $\lambda \in G$, we have the exact sequence:

$$\cdots \longrightarrow \operatorname{Hom}_R(F^0(\lambda), Q) \xrightarrow{\operatorname{Hom}_R(d_0(\lambda), \mathrm{id}_Q)} \operatorname{Hom}_R(F_0(\lambda), Q) \longrightarrow \cdots,$$

As a result, the complex $\mathbf{L}_.$ is exact, which implies that the complex $\operatorname{Hom}_R(\mathbf{G}_., Q)$ is also exact. Consequently, $\operatorname{Hom}_{R-gr}(F(\mathbf{F}_., F(Q))$ is exact. By Lemma 4.32, Q is a direct summand

of $F(Q)$, and therefore by 2.1(i), the complex $\mathrm{Hom}_{R-gr}(F(\mathbf{F}_.), Q)$ is exact.
Since, F is an exact functor, and by Lemma 4.32, $F(F^n)$ and $F(F_n)$ are graded projective R–module, for all n, it follows that, the complex $\mathrm{Hom}_{R-gr}(F(\mathbf{F}_.), Q)$ is exact for any graded projective R–module Q. Therefore $F(M)$ is gr-Gor proj. □

Proposition 4.36. *If M is a graded Gor proj R–module. Then M is gr-Gor proj module.*

Proof. By Proposition 4.35, $F(M)$ is a gr-Gor proj and by 2.5, M is a direct summand of $F(M)$ (as a graded R–modules). Therefore, by Proposition 4.16, M is gr-Gor proj. □

5. gr-Gorenstein injective module

The following definitions build upon concepts from the work of Asensio and Torrecillas [1].

Definition 5.1. As outlined in [3, Proposition 3.3], a graded module N over R is termed a gr-Gorenstain injective module if there exists an exact complex:

$$\mathbf{E}^{\cdot} : \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots,$$

where each E_i is a gr-injective module, and N is defined as $\mathrm{Ker}\,(E^0 \to E^1)$. Moreover, this complex retains its exactness under the application of the functor $\mathrm{Hom}_{R-gr}(E, -)$ for any gr-injective module E. This sequence $\mathbf{E}^{\cdot}$, is referred to as a complete gr-injective resolution of N.

Proposition 5.2. *A graded R–module N qualifies as a gr-Gor inj module if and only if, there exists an exact complex*

$$\mathbf{E}^{\cdot} : \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots,$$

with each E_i being gr-injective and where $N = Ker(E^0 \to E^1)$. The complex remains exact when the functor $\mathrm{Hom}_R(E, -)$ is applied for any gr-injective module module E.

Proof. Suppose E is a gr-injective R-module. Then $E(?i)$ also qualifies as a gr-injective R-module for every integer $i \in \mathbb{Z}$. Based on [5, Exersise1.5.19(g)], we have the isomorphism $\mathrm{Hom}_{R-gr}(E, -) \cong \mathrm{Hom}_R(E, -)_0$ and similarly $\mathrm{Hom}_R(E, -)_i \cong \mathrm{Hom}_{R-gr}(E(-i), -)$ for all $i \in \mathbb{Z}$. This equivalence follows by a method analogous to that used in Proposition 4.3. □

Proposition 5.3. *Let $\mathcal{X}$ denote a collection of graded R–modules and $\{M_\alpha\}_{\alpha \in I}$ be a set of graded R–modules. If the class $\mathcal{X}$ is closed under arbitrary graded direct products and each M_α has a gr-proper left $\mathcal{X}$-resolution, then ${}^{gr}\prod M_\alpha$ also admits a gr-proper left $\mathcal{X}$-resolution.*

Proof. This result follows from the exact nature of the functor ${}^{gr}\prod_{\alpha\in I}(-)$ and the equivalent of functors ${}^{\mathrm{gr}}\mathrm{Hom}_R(-, {}^{gr}\prod M_\alpha) \cong {}^{gr}\prod {}^{\mathrm{gr}}\mathrm{Hom}_R(-, M_\alpha)$ (See Remark 2.2(iii)). □

Proposition 5.4. *A graded R–module M is classified as gr-Gore inj if and only if M in the left orthogonal class to the graded injective modules, denoted as ${}^{gr}\mathbf{I}(R)^{\perp}$ and has a gr-proper left ${}^{gr}\mathbf{I}(R)$–resolution.*

Theorem 5.5. *The class ${}^{gr}Gi(R)$ of all gr-Gor inj R–modules, is closed under both graded direct product and direct summands.*

Proof. It can be observed that for any $i \geq 0$, ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(-, {}^{gr}\prod N_\alpha) \cong {}^{gr}\prod {}^{\mathrm{gr}}\mathrm{Ext}^i_R(-, N_\alpha)$. Applying Propositions 5.3, 5.4, Remark 4.13(iii) and dual of Proposition 4.16, leads to the desired conclusion. □

The following theorem and propositions are dual to the statements provided in Theorem 4.27 and Proposition 4.21.

Theorem 5.6. *If a graded R–module N has finite gr-Gor inj dimension n. Then N has a gr-Gor inj preenvelope $\varphi : N \to H$, where $CK = Coker\varphi$ satisfies ${}^{gr}id_R CK = n - 1$.*

Proposition 5.7. *Let M be gr-Gor inj R-module. Then for all $j > 0$, ${}^{\mathrm{gr}}\mathrm{Ext}^j_R(T, M) = 0$, and for any graded R–module T of finite projective or gr-injective dimension.*

Proposition 5.8. *Consider an exact sequence of graded R–modules, $0 \to M \to G \to C \to 0$ where G is gr-Gor inj. If M is also gr-Gor inj, then C is gr-Gor inj. Other wise ${}^{gr}Gid_R C = {}^{gr}Gid_R M - 1$.*

The proof for the following theorem mirrors the dual argument reasoning employed in the proof of Theorem 4.20.

Theorem 5.9. *For a graded R–module N with finite gr-Gor inj dimension and an integer n. Then the following statements are equivalent:*

(i) ${}^{gr}Gid_R N \leq n$

(ii) ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(L, N) = 0$, *for all $i > n$, and for any graded R–modules L with with finite gr-injective dimension, ${}^{gr}id_R L < \infty$*

(iii) ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(Q, N) = 0$, *for all $i > n$, and for any graded gr-injective R–modules Q*

(iv) *For any exact sequence*

$$0 \to N \to H^0 \to \cdots \to H^{n-1} \to C^n \to 0,$$

where H^0,..., H^{n-1} are gr-Gor inj, then C^n is also gr-Gor inj module. As a result:

$$^{gr}Gid_RN = sup\{i \in \mathbb{N}_0 |\, {}^{\mathrm{gr}}\mathrm{Ext}^i_R(L,N) \neq 0 \, for\ some\ graded\, R\text{-}module\, L\ with\, {}^{gr}id_RL < \infty\}$$

$$= sup\{i \in \mathbb{N}_0 |\, {}^{\mathrm{gr}}\mathrm{Ext}^i_R(Q,N) \neq 0 \, for\ some\ gr\text{-}injective\, R\text{-}module\ Q.$$

If E is a injective R–module, it can be expressed as $E = \bigoplus {}^{gr}E_R(R/\mathfrak{p})$, where ${}^{gr}E_R(R/\mathfrak{p})$ represents the gr-injective hull $R/\mathfrak{p}$.This leads to the following corollary.

Corollary 5.10. *For a commutative graded Noetherian ring R and R-module N with ${}^{gr}Gid_RN < \infty$, then,*

$$^{gr}Gid_RN = sup\{i \in \mathbb{N}_0 | there\ exists \mathfrak{p} \in {}^{gr}Spec(R); {}^{\mathrm{gr}}\mathrm{Ext}^i_R({}^{gr}E_R(R/\mathfrak{p}), N) \neq 0\}.$$

The following propositions are established using dual arguments similar to those used in Propositions 4.23, 4.25, and 4.28.

Proposition 5.11. *In any exact sequence $0 \to M' \to M \to M'' \to 0$ of graded R-modules, if any two of the modules have finite gr-Gor inj dimension, then so does the third.*

Proposition 5.12. *For any graded R–module M, we have ${}^{gr}Gid_RM \leq {}^{gr}id_RM$, with equality if M has finite gr-injective dimension.*

Proposition 5.13. *If a graded R–module M of finite projective dimension then ${}^{gr}Gid_RM = {}^{gr}id_RM$.*

Remarks and example.

Remark 5.14. In [1], Asensio, Lopez Ramos and Torrecillas, demonstrated that if M is a graded and Gor inj R–module, then M is gr-Gor inj. Their result in [1, Theorem 3.10] states that a graded R-module M over a Gorenstein ring graded by a finite group is Gor inj if and only if it is graded Gor inj.

The next example demonstrates that the result in [1, Theorem 3.10] fails to hold if the finiteness assumption on the group is dropped.

Example 5.15. Let k be a field and let x be an indeterminate with positive degree then $A = k[x, x^{-1}]$ is a graded Gorenstein ring with $\dim A = 1$. Thus, A is a both gr-injective A–module and gr-Gor inj A–module. However, $\mathrm{id}_A A < \infty$. Hence $\mathrm{Gid}_A A = \mathrm{id}_A A = 1$.

This example illustrates that, generally, not every gr-Gor inj R-module is necessarily Gor inj.

Lemma 5.16. *Let M and N be graded R–modules. If N is a graded direct summand of M, then ${}^{gr}Gid_RN \leq {}^{gr}Gid_RM$.*

Proof. The statement clear if ${}^{gr}\mathrm{Gid}_R M = \infty$.

Assume ${}^{gr}\mathrm{Gid}_R M = n < \infty$. We proceed by induction on n. If $n = 0$, then by Theorem 5.5, N is a gr-Gor inj R–module.

For $n > 0$, consider a graded R-module M as adirect sum as $M = N \oplus M'$ for some graded R–module M'. Assuming we have exact sequences $0 \to N \to G \to C \to 0$ and $0 \to M' \to G' \to C' \to 0$ such that G and G' are gr-injective, the split exact rows form a commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M' & \to & M & \to & N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G' & \to & G' \oplus G & \to & G & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C' & \to & C' \oplus C & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

By Proposition 5.8 applied to the middle column, we deduce that ${}^{gr}\mathrm{Gid}_R C' \oplus C = n - 1$. From the induction hypothesis, it follows that, ${}^{gr}\mathrm{Gid}_R C \leq n - 1$. Hence, the exact sequence $0 \to N \to G \to C \to 0$, implies ${}^{gr}\mathrm{Gid}_R N \leq n = {}^{gr}\mathrm{Gid}_R M$. □

Proposition 5.17. *Let M be a graded R–module. Then ${}^{gr}Gid_R M \leq Gid_R M$.*

Proof. This is evident if $\mathrm{Gid}_R M = \infty$.

Now, suppose $\mathrm{Gid}_R M = t < \infty$. Then there, exists a Gor inj resolution of length t:

$$0 \to M \to G_0 \to \cdots \to G_t \to 0,$$

Since the functor $F(-)$ is exact, and by [1, Theorem 3.6], each $F(G_i)$ is gr-Gor inj, we obtain a gr-Gorenstein resolution for $F(M) : 0 \to F(M) \to F(G_0) \to \cdots \to F(G_t) \to 0$ is a gr-Gorenstein resolution of $F(M)$. Hence ${}^{gr}\mathrm{Gid}_R F(M) \leq t$. By [23, Lemma 3.1], $F(M)$ has a direct summand isomorphic to M. Therefore, by Lemma 5.16, we conclude

$${}^{gr}\mathrm{Gid}_R M \leq {}^{gr}\mathrm{Gid}_R F(M) \leq t.$$

□

Proposition 5.18. *Assume that S is a graded R–Algebra with finite projective dimension. For any gr-Gor inj R-module B the S–module ${}^{\mathrm{gr}}\mathrm{Hom}_R(S, B)$ is also gr-Gor inj.*

Proof. Let P denote a graded projective S–module and I be an gr-injective R–module. According to Remark 2.2 (i), ${}^{\mathrm{gr}}\mathrm{Hom}_R(P, I)$ is a gr-injective S–module. Furthermore, for any graded S–module J, we have:

$$ {}^{\mathrm{gr}}\mathrm{Hom}_S(J, {}^{\mathrm{gr}}\mathrm{Hom}_R(P, I)) \cong {}^{\mathrm{gr}}\mathrm{Hom}_S(P, {}^{\mathrm{gr}}\mathrm{Hom}_R(J, I)). $$

The desired result follows directly from the proof of [10, Lemma 4.6 (ii)]. □

Lemma 5.19. *For a graded R–module M with finite gr-Gor inj dimension, We have a short exact sequence of graded R–modules:*

$$ 0 \to B \to H \to M \to 0, $$

such that B is gr-Gor inj and ${}^{gr}id_R H = {}^{gr}Gid_R M - 1$.

Proof. Suppose M is a gr-Gor inj consider the initial short exact sequence on the left of a complete gr-injective resolution of M,

$$ 0 \to B \to H \to M \to 0. $$

Assuming ${}^{gr}\mathrm{Gid}_R M = n > 0$. According to Theorem 5.6, an exact sequence $0 \to M \to B' \to K \to 0$ exists, where B' is gr-Gor inj and ${}^{gr}\mathrm{id}_R K = n - 1$. Since B' is gr-Gor inj, a short exact sequence $0 \to B \to E \to B' \to 0$ exists, where E is gr-injective and B is gr-Gor inj module. The pullback is given as follows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & M & \to & B' & \to & K & \to & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \to & H & \to & E & \to & K & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & B & = & B & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The desired sequence appears in the first column of this diagram. By Remark 4.13(iii), the class of gr-Gor inj module is gr-injectively resolving. If H were gr-injective R–module, then ${}^{gr}\mathrm{Gid}_R M = 0$, which is contradiction. Thus ${}^{gr}\mathrm{id}_R H > 0$, implying ${}^{gr}\mathrm{id}_R H = {}^{gr}\mathrm{id}_R K + 1 = n$.

□

Width of an R-module and its graded counterpart. The width of an R-module M over a local ring R with maximal ideal $\mathfrak{m}$ is defined by:

$$\operatorname{width}_R M = inf\{i \in \mathbb{Z} | \operatorname{Tor}_i^R(k, M) \neq 0\}.$$

For a commutative Noetherian ring R and ideal $\mathfrak{a}$ of R, the $\mathfrak{a}$–width of M, denoted $\operatorname{width}_R(\mathfrak{a}, M)$ is given by:

$$\operatorname{width}_R(\mathfrak{a}, M) = inf\{i \in \mathbb{Z} | \operatorname{Tor}_i^R(R/\mathfrak{a}, M) \neq 0\}.$$

We now define the graded width of a graded R–module over gr-local ring.

Definition 5.20. For a gr-local ring $(R, \mathfrak{m})$, the graded width of a graded R-module M, denoted by ${}^{gr}\operatorname{width}$, is defined by:

$${}^{gr}\operatorname{width}_R M = inf\{i \in \mathbb{Z} | \operatorname{Tor}_i^R(R/\mathfrak{m}, M) \neq 0\}.$$

Remark 5.21. This remark compares the the definitions of width and ${}^{gr}\operatorname{width}$ For a graded R–module M.

(i) For $\mathfrak{p} \in {}^{gr}\operatorname{Spec}(R)$,

$${}^{gr}\operatorname{width}_{R_{(\mathfrak{p})}} M_{(\mathfrak{p})} = inf\{i \in \mathbb{Z} | \operatorname{Tor}_i^{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \neq 0\}.$$

Additionally,

$$\begin{aligned} \operatorname{Tor}_i^{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \neq 0 &\iff (\operatorname{Tor}_i^{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}))_{\mathfrak{p}R_{(\mathfrak{p})}} \neq 0 \\ &\iff \operatorname{Tor}_i^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) \neq 0. \end{aligned}$$

Thus, ${}^{gr}\operatorname{width}_{R_{(\mathfrak{p})}} M_{(\mathfrak{p})} = \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p}$.

(ii) if $\mathfrak{p}$ is an ungraded prime ideal, then $(R_{(\mathfrak{p})}, \mathfrak{p}^* R_{(\mathfrak{p})})$ forms a gr-local ring. Observe that $R_{(\mathfrak{p})}/\mathfrak{p}^* R_{(\mathfrak{p})} \cong k[x, x^{-1}]$, where k is field and x is is a positively graded element that is transcendental over k. This implies $\mathfrak{p} = \alpha R + \mathfrak{p}^*$, for some $\alpha \in R - \mathfrak{p}^*$, yielding the exact sequence:

$$0 \to R/\mathfrak{p}^* \xrightarrow{\alpha} R/\mathfrak{p}^* \to R/\alpha R + \mathfrak{p}^* \to 0,$$

which produces the long exact sequence:

$$\cdots \to \operatorname{Tor}_i^R(R/\mathfrak{p}^*, M) \xrightarrow{\alpha} \operatorname{Tor}_i^R(R/\mathfrak{p}^*, M) \to \operatorname{Tor}_i^R(R/\mathfrak{p}, M) \to \cdots.$$

For each $i \geq 0$, $\operatorname{Tor}_i^R(R/\mathfrak{p}^*, M)$ is a graded module over $R/\mathfrak{p}^* \cong k[x, x^{-1}]$ and hence by [5, Exercise 1.5.20], is free over $R/\mathfrak{p}^*$. Since $\alpha \notin \mathfrak{p}^*$, the map, $\operatorname{Tor}_i^R(R/\mathfrak{p}^*, M) \xrightarrow{\alpha} \operatorname{Tor}_i^R(R/\mathfrak{p}^*, M)$ is injective for all $i \geq 0$. Therefore, we have:

$$\operatorname{Tor}_i^R(R/\mathfrak{p}, M) \cong \operatorname{Tor}_i^R(R/\mathfrak{p}^*, M)/a\operatorname{Tor}_i^R(R/\mathfrak{p}^*, M).$$

Since $\mathfrak{p} = \alpha R + \mathfrak{p}^*$, it follows that $\operatorname{Tor}_i^R(R/\mathfrak{p}, M)$ is a free $R/\mathfrak{p}$–module and it's rank is as same as the rank of free $R/\mathfrak{p}^*$–module $\operatorname{Tor}_i^R(R/\mathfrak{p}^*, M)$. Thus $\operatorname{Tor}_i^{R_{\mathfrak{p}^*}}(R_{\mathfrak{p}^*}/\mathfrak{p}^* R_{\mathfrak{p}^*}, M_{\mathfrak{p}^*}) \neq 0$ if and only if $\operatorname{Tor}_i^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) \neq 0$. Consequently $\operatorname{width}_{R_{\mathfrak{p}^*}} M_{\mathfrak{p}^*} = \operatorname{width}_{R_\mathfrak{p}} M\mathfrak{p}$.

Proposition 5.22. *Let R be a commutative Noetherian graded ring. For any gr-Gor inj R-module M and any prime ideal $\mathfrak{p} \in {}^{gr}Spec\,(R)$, we have:*

$$depth\, R_\mathfrak{p} - width_{R_\mathfrak{p}} M_\mathfrak{p} = depth\, R_{(\mathfrak{p})} - {}^{gr} width_{R_{(\mathfrak{p})}} M_{(\mathfrak{p})} \leq 0.$$

with equality if $\mathfrak{p}$ is maximal in ${}^{gr}Supp_R M$.

Proof. Suppose M is a gr-Gor inj R–module. Then there exists an exact sequence:

$$\mathbf{E} : \cdots \to E_1 \to E_0 \to M \to 0,$$

where each E_i is gr-injective graded R-module. Define $K_1 = \operatorname{Ker}(E_0 \to M)$ and for all $i \geq 2$, $K_i = \operatorname{Ker}(E_{i-1} \to E_{i-2})$. As a result, we obtain an exact sequence:

$$\mathbf{E}_{(\mathfrak{p})} : \cdots \to E_{1(\mathfrak{p})} \to E_{0(\mathfrak{p})} \to M_{(\mathfrak{p})} \to 0,$$

where each $E_{i(\mathfrak{p})}$ is a gr-injective module over $R_{(\mathfrak{p})}$, for each $\mathfrak{p} \in {}^{gr}\operatorname{Spec}(R)$. Now, consider T, a graded $R_{(\mathfrak{p})}$–module with finite proj dimension. Then, we get, ${}^{\mathrm{gr}}\operatorname{Ext}^i_{R_{(\mathfrak{p})}}(T, M_{(\mathfrak{p})}) \cong {}^{\mathrm{gr}}\operatorname{Ext}^{i+t}_{R_{(\mathfrak{p})}}(T, K_{t(\mathfrak{p})})$ for any two positive integers i and t. Therefore, ${}^{\mathrm{gr}}\operatorname{Ext}^i_{R_\mathfrak{p}}(T, M_{(\mathfrak{p})}) = 0$ for all $i > 0$. Let $n = \operatorname{grade}(\mathfrak{p}R_{(\mathfrak{p})}, R_{(\mathfrak{p})})$. As noted in [22, Example III.3.2(2)], there exists a homogeneous $R_{(\mathfrak{p})}$–sequence $\mathbf{x}$ in $\mathfrak{p}R_{(\mathfrak{p})}$, such that $R_{(\mathfrak{p})}/\mathbf{x}R_{(\mathfrak{p})}$ has finite projective dimension as a graded $R_{(\mathfrak{p})}$–module. Consequently, we get:

$$\begin{aligned}
0 &\geq sup\{i \geq 0 | \operatorname{Ext}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathbf{x}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \neq 0\} \\
&= sup\{i \geq 0 | \operatorname{Tor}_i^{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathbf{x}R_{(\mathfrak{p})}, (M_{(\mathfrak{p})})^\vee) \neq 0\} \\
&= sup\{i \geq | H_i(\mathbf{x}, (M_{(\mathfrak{p})})^\vee) \neq 0\} \\
&= n - \operatorname{grade}(\mathbf{x}R_{(\mathfrak{p})}, (M_{(\mathfrak{p})})^\vee) \\
&\geq n - \operatorname{grade}(\mathfrak{p}R_{(\mathfrak{p})}, (M_{(\mathfrak{p})})^\vee) = n - \operatorname{width}(\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}),
\end{aligned}$$

where the first and last equalities are justified by Lemma 2.3, and the second and third equalities follow from [24, Theorem 6.1.6 and page 103].

Now let $\mathfrak{p}$ be a maximal element of ${}^{gr}\operatorname{Supp}_R(M)$. Set $S = R_{(\mathfrak{p})}$ which is a Noetherian gr-local ring with $\operatorname{depth} S = n$, gr-maximal ideal $\mathfrak{n} = \mathfrak{p}R_{(\mathfrak{p})}$, $B = M_{(\mathfrak{p})}$ and $E = {}^{gr}E_S(S/\mathfrak{n})$. The result follows from ${}^{\mathrm{gr}}$Hom–evaluation 2.2, Remark 2.10 and an approach similar to that in [11, Lemma 1.2]. □

Proposition 5.23. *Suppose M is a graded R–module with ${}^{gr}id_R M = n < \infty$ and let $\mathfrak{p}$ be a homogeneous ideal in R that is maximal with respect to the condition:*

$$ {}^{\mathrm{gr}}\mathrm{Ext}^n_R(R/\mathfrak{p}, M) \neq 0. $$

Then I is a graded prime ideal, and ${}^{\mathrm{gr}}\mathrm{Ext}^n_R(R/\mathfrak{p}, M) \cong {}^{\mathrm{gr}}\mathrm{Ext}^n_R(R/\mathfrak{p}, M_{\mathfrak{p}})$.

Proof. This result can be derived by following an analogous approach to that of the ungraded case in [8, Proposition 3]. □

Corollary 5.24. *Let M be a graded R–module with finite graded injective dimension. Then:*

$$ {}^{gr}id_R M = sup\{depth\, R_{\mathfrak{p}} - width_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in {}^{gr}Spec\,(R)\}. $$

Proof. Using Proposition 5.23 and knowing that ${}^{gr}\mathrm{id}_R M_{(\mathfrak{p})} \leq {}^{gr}\mathrm{id}_R M < \infty$, it suffices to identify the largest n such that, ${}^{\mathrm{gr}}\mathrm{Ext}^n_R(R/\mathfrak{p}, M_{\mathfrak{p}}) \neq 0$ as $\mathfrak{p}$ varies over graded prime ideals. By construction, ${}^{\mathrm{gr}}\mathrm{Ext}^n_R(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \cong \mathrm{Ext}^n_R(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$, for all n. From Lemma 2.3, we have $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})^{\vee} = \mathrm{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. Applying [8, Proposition 1], and Remark 5.21 we get:

$$ \begin{aligned} {}^{gr}\mathrm{id}_R M &= sup\{\mathrm{grade}\,(\mathfrak{p}R_{(\mathfrak{p})}, R_{(\mathfrak{p})}) - \mathrm{depth}\,(M_{(\mathfrak{p})})^{\vee} | \mathfrak{p} \in {}^{gr}\mathrm{Spec}\,(R)\} \\ &= sup\{\mathrm{grade}\,(\mathfrak{p}R_{(\mathfrak{p})}, R_{(\mathfrak{p})}) - \mathrm{width}\, M_{(\mathfrak{p})} | \mathfrak{p} \in {}^{gr\,gr}\mathrm{Spec}\,(R)\} \\ &= sup\{\mathrm{depth}\, R_{\mathfrak{p}} - \mathrm{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in {}^{gr}\mathrm{Spec}\,(R)\}. \end{aligned} $$

□

Theorem 5.25. *For a graded Noetherian ring R and a graded R–module M with finite gr-Gorenstein injective dimension, we have:*

$$ {}^{gr}Gid_R M = sup\{depth\, R_{\mathfrak{p}} - width_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in {}^{gr}Spec\,(R)\}. $$

Proof. This follows directly from Lemmas 5.19, 5.22, Corollary 5.24, and the reasoning in the ungraded case as in [19, Theorem 2.3]. □

Corollary 5.26. *For a graded Noetherian ring R and a graded R–module M with finite gr-Gor inj dimension and $Gid_R M < \infty$, we have:*

$$ Gid_R M \leq {}^{gr}Gid_R M + 1. $$

Proof. From [5, Theorem 1.5.9], we know that $\operatorname{depth} R_{\mathfrak{p}} \leq \operatorname{depth} R_{\mathfrak{p}^*} + 1$ and by Remark 5.21(ii), $\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{width}_{R_{\mathfrak{p}^*}} M_{\mathfrak{p}^*}$, for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Then, using Theorem 5.25 we obtain:

$$\begin{aligned}
\operatorname{Gid}_R M &= \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\
&\leq \sup\{\operatorname{depth} R_{\mathfrak{p}^*} + 1 - \operatorname{width}_{R_{\mathfrak{p}^*}} M_{\mathfrak{p}^*} \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\
&= \sup\{\operatorname{depth} R_{\mathfrak{p}^*} - \operatorname{width}_{R_{\mathfrak{p}^*}} M_{\mathfrak{p}^*} \mid \mathfrak{p} \in \operatorname{Spec}(R)\} + 1 = {}^{gr}\operatorname{Gid}_R M + 1.
\end{aligned}$$

□

More explanation on relationships between different module types

In this paper, we define and study various graded module types, including:

- gr-Gorenstein projective modules
- gr-Gorenstein injective modules
- gr-faithfully flat modules

To better illustrate the relationships among them, let's explore how they interact within homological algebra.

Comparison Table:

Table 1. Comparison of different graded module types.

Property	**gr-Gorenstein Projective**	**gr-Gorenstein Injective**	**gr-Faithfully Flat**
Defined by	Complete gr-projective resolution	Complete gr-injective resolution	Tensoring preserves exact sequences
Key Example	Syzygies of modules over a Gorenstein graded ring	Injective envelope of a simple module	Localization modules like $R[x^{-1}]$
Stability	Closed under direct sums and summands	Closed under products and summands	Ensures non-trivial action on all modules
Relation to Each Other	Dual to gr-Gorenstein injective modules	Dual to gr-Gorenstein projective modules	Ensures homological control in exact sequences

How they relate in a homological framework.

(1) gr-Gorenstein projective and gr-Gorenstein injective modules are dual concepts, meaning that in a well-behaved category (e.g., over a Noetherian graded Gorenstein ring), their Ext-orthogonality characterizes homological dimensions.

- If M is gr-Gorenstein projective, then ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(M,N) = 0$ for all gr-Gorenstein injective N and $i > 0$.
- Conversely, if N is gr-Gorenstein injective, then ${}^{\mathrm{gr}}\mathrm{Ext}^i_R(G,N) = 0$ for all gr-Gorenstein projective G and $i > 0$.

(2) gr-faithfully flat modules provide a bridge between these concepts by ensuring that homological properties (e.g., injective/projective resolutions) are preserved when tensoring.

- If M is gr-faithfully flat, then any gr-Gorenstein projective module remains acyclic in exact sequences involving tensoring.
- If M is gr-faithfully flat and R is Noetherian, then M induces an equivalence between derived categories of gr-Gorenstein projective and gr-Gorenstein injective modules.

Example relating these concepts. Let $R = k[x,y]$ be a positively graded Noetherian ring, where $\deg(x) = \deg(y) = 1$.

- The free module R is trivially gr-Gorenstein projective and gr-faithfully flat.
- The injective envelope $E = {}^{\mathrm{gr}}\mathrm{Hom}_k(R/m, k)$ is an example of a gr-Gorenstein injective module.
- The localization $R[x^{-1}]$ is gr-faithfully flat but neither gr-Gorenstein projective nor gr-Gorenstein injective.

This shows that gr-faithfully flat modules enable homological constructions that preserve both gr-Gorenstein projective and injective structures, even when these are not naturally equivalent in a given module category.

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Zahra Heidarian

Department of Mathematics,

NT. C., Islamic Azad University,

Tehran, Iran.

zahraheidarian@gmail.com, zahraheidarian@iau.ac.ir