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Research Paper

FORMAL CONCEPT RING, AN ALGEBRAIC MACHINE TO EQUIP THE FORMAL CONCEPT ANALYSIS WITH MORE CONSTRUCTIVE OPERATIONS

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Abstract. Formal Concept Analysis (FCA) as a well-known method in data analysis, has been widely noticed by researchers in pure and applied fields. Despite the increasing development in the application area, more work needs to be done in the pure area. Designing a theoretical system based on an algebraic-analytical structure is a significant help in enhancing its power to meet the needs of researchers. Therefore, in this article, by designing a system equipped with addition and multiplication, the development of Galois lattices will be discussed as the main goal. In the shadow of such extension, we will be able to achieve a special category of partially ordered rings, by defining partial and complete approximations. Hence, some fundamental results in FCA will be developed as induced properties from ring theory intuition. Two significant results of this research will devote to provide the possibility of combining concepts with each other, as well as breaking the space into much distinguished components. As a result, this yields a new window in the subject of digital communication, which provides the possibility of joining concepts. The long-term prospective of this study is to assign a new observation in Big Data Analysis, based on FCA.

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1. Introduction

FCA is a method of deriving a concept hierarchy from a collection of objects and their properties. Commonly, FCA method makes the possibiltiy to find the real-world meaning of mathematical order theory. More specifically, the interpreted outcomes actually follow from the data tables that can be transformed into ordered algebraic structures (called complete lattices). A data table representing a heterogeneous relation between objects and attributes, forms a basic data set, (i.e. an object a has an attribute b), and is referred to as a formal context. The extent is a set A of elements called objects while the intent B includes subsets having the common properties induced by objects, and is also called the attributes set. Any formal concept, is a pair of the extent A and the intent B such that A consists of all objects that share the attributes in B and dually, B consists of all attributes participated by the objects in A. Formal concepts of any formal context can be ordered in a hierarchy, called the context's concept lattice. In fact, FCA combines the mathematical ideas of lattice theory with philosophical concepts (in particular, that of intent and extent), ordering and classification methods, and diverse algorithmic tools for applications, to generate the brilliant ideas not only in mathematical but also in various medley-mathematical areas, from Technical Engineering, Biology, Medicine, History and Social Sciences to Fine Arts and Music [6].
This modern, practice-oriented and mathematically founded theory has been initiated and propagated by Rudolf Wille and his coworkers since the seventies of the last century. A comprehensive source telling the story can be [8], containing also a complete list of references to see the research gates.

A long with many applications of FCA, there is a significant motivation to equip it with extra algebraic operations. This implementing strengthens its computational methods in applied areas. On the other hand, the closer a process is to theoretical models, the higher the possibility of development and accuracy in its estimating methods. Although the current structure (without addition and multiplication) has a good ability to compute, but appending the algebraic operations can enhance its performance to show the real capacity. Combining the concepts is a bit confusing but at the same time, it yields a significant progress. There are two algebraic structures known as partially ordered rings and lattice ordered rings with convincing properties (see [9], [**?**]) nearly close to our goal but as a drawback for this purpose, they have been presented completely abstract and do not have the necessary capacity to fulfill the practical objectives. So, we will choose an alternative model that has a constructive approach.

Therefore, in the main result section, we define the concept ring to achieve an algebraic machine, converting the items into words, and assigning related codes to add and multiply them as digital objects. In a completely premeasured way, formal objects can be obtained by completing the approximant sequences from the objects of this basic collection. Performing this goal, naturally adds topological capabilities which in turn will have a more serious impact on the study. The formation of such a ring gives a set of ideals which can be viewed as a simple graph. In the application section, we develop a fully structured theorem, named as the existence result in FCA (see proposition 2.8). Also, we prove one of the most distinguished results in attribute implications, based on the presented studies. Eventually, the last theorem deals with the introducing of a computational algorithm with extra potential of accuracy and speed. The consequence will save a lot of time and energy. In total, the general purpose of this study can be summarized in two parts. First, adding speculative and constructive algebraic operations to the body of Galois lattices providing the possibility of combined elements and secondly, focusing on the discussion of approximation and showing how to achieve maximal elements in the case of infinite order sets. This, can open a new window to Big Data Analysis.

2. Preliminaries

In this section, some preliminaries will be reminded to follow the other topics. Most of the contents are extracted from [4].

A poset (partially ordered set) is a pair $(\mathcal{P}, \preceq)$ where $\mathcal{P}$ is a set and $\preceq$ is a partial order (i.e. $\preceq$ is reflexive, transitive and anti-symmetric). A poset $\mathcal{P}$ along with a partial order $\preceq$ is called a linear poset if every two distinct elements p and q of $\mathcal{P}$ are comparable (i.e. $p \preceq q$ or $q \preceq p$).

Let $(\mathcal{P}, \preceq)$ be a poset. An infinite sequence

$$p_1 \preceq p_2 \preceq \cdots \preceq p_n \preceq \ldots,$$

of elements in $\mathcal{P}$ is called a chain. Such a chain is called stationary if there is some positive integer n such that $p_m = p_{m+1}$ for all $m \geq n\cdot$.

For $S \subseteq \mathcal{P}$, an element $p \in \mathcal{P}$ is called an upper bound (lower bound) of S if and only if $x \preceq p$ $(p \preceq x)$, for each $x \in S$. Furthermore, $p \in \mathcal{P}$ is called the least upper bound (the greatest lower bound) of S and denoted by lub or sup (glb or inf) if and only if p is an upper (lower) bound and for any other upper (lower) bound q of S, it is the case that $p \preceq q$ $(q \preceq p)$. In this paper, we denote by $\bigvee S$ and $\bigwedge S$, the lub and glb of S, respectively. If $S = \mathcal{P}$, they are represented by $\top$ and $\bot$, respectively.

We remind that, if $p, q \in \mathcal{P}$ and $\mathcal{P}$ be a poset, p is said to cover q, if $p \prec q$ ($p \preceq q$ and $p \neq q$), and $p \preceq z \prec q$ implies that $z = p$. the lattar condition is demanding that there be no element z of $\mathcal{P}$ with $p \prec z \prec q$.

Definition 2.1. A poset is called a complete partial order (CPO), if and only if any of its chains has a lub.

A preorder is like partial order, but without anti-symmetry. A direction on a set S is a preorder in which any finite subset has an upper bound.

Definition 2.2. A directed subset is a nonempty subset S of a poset $\mathcal{P}$, with property that every pair of elements has an upper bound.

Definition 2.3. A directed complete partial order (DCPO), is a poset in which every directed subset has a supremum.

Definition 2.4. For a partial order $\preceq$ on a ring R, $(R, \preceq)$ is a partially ordered ring if, for all $x \in R$, $a \preceq b$ implies $a + x \preceq b + x$ and $a \preceq b$ implies $ax \preceq bx$, where $0 \preceq x$.

Definition 2.5. A map $f : (R_1, \preceq_1) \longrightarrow (R_2, \preceq_2)$ of partially ordered rings is order-preserving, if $x \preceq_1 y$, implies $f(x) \preceq_2 f(y)$ and is order-reversing if $x \preceq_1 y$, implies $f(y) \preceq_2 f(x)$.

Lattices as having the efficient structure to mathematical modelling, describe the patterns that do not have the usual algebraic and analytical formations. The core of this study is basically on complete lattices. We recall that the nonempty poset $(\mathcal{P}, \preceq)$, is called a lattice, if for any $x, y \in \mathcal{P}$, $x \vee y$ and $x \wedge y$ exist. If for any arbitrary subset S of $\mathcal{P}$, $\bigvee S$ and $\bigwedge S$ exist then we say that $(\mathcal{P}, \preceq)$ is a complete lattice.

Definition 2.6. A formal context is a triple (G, M, I) consisting of a set G, a set M and an incidence relation $I \subseteq G \times M$. The elements of G and M are called objects and attributes respectively. As usual, instead of writing $(g, m) \in I$ we write gIm and say "the object g has the attribute m".

For $A \subseteq G$ and $B \subseteq M$, define

$$A' = \{m \in M | (\forall g \in A) gIm\},$$
$$B' = \{g \in G | (\forall m \in B) gIm\}.$$

So A' is the set of attributes common to all the objects in A and B' is the set of objects possessing the attributes in B.

Definition 2.7. A formal concept of the formal context (G, M, I) is defined to be a pair (A, B) where $A \subseteq G, B \subseteq M, A' = B$ and $B' = A$. The extent of the formal concept (A, B) is A while its intent is B.

Note that a subset A of G is the extent of some formal concept if and only if $A'' = (A')' = A$. The set of all formal concepts of the formal context (G, M, I) is denoted by $\mathcal{B}(G, M, I)$. Let (G, M, I) be a formal context. For any formal concepts (A_1, B_1) and $(A_2, B_2) \in \mathcal{B}(G, M, I)$ we write $(A_1, B_1) \preceq (A_2, B_2)$, if $A_1 \subseteq A_2$.

Also, $A_1 \subseteq A_2$ implies that $A_1' \supseteq A_2'$, and the reverse implication is valid too, because $A_1'' = A_1$ and $A_2'' = A_2$. We therefore have

$$(A_1, B_1) \preceq (A_2, B_2) \Longleftrightarrow A_1 \subseteq A_2 \Longleftrightarrow B_1 \supseteq B_2.$$

One can easily show that the relation $\preceq$ is an order on $\mathcal{B}(G, M, I)$. As we see in Proposition 2.8, $(\mathcal{B}(G, M, I), \preceq)$ is a complete lattice; it is known as the concept lattice of the formal context (G, M, I).

Proposition 2.8. *Let (G, M, I) be a formal context. Then $(\mathcal{B}(G, M, I), \preceq)$ is a complete lattice in which join and meet are given by*

$$\bigvee_{j\in J}(A_j, B_j) = ((\bigcup_{j\in J} A_j)'', \bigcap_{j\in J} B_j),$$
$$\bigwedge\nolimits_{j\in J}(A_j, B_j) = (\bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)''),$$

respectively.

Definition 2.9. A *Galois connection* is an ordered quadruple

$$(f, (\mathcal{P}, \preceq_{\mathcal{P}}), (\mathcal{Q}, \preceq_{\mathcal{Q}}), g)$$

such that $(\mathcal{P}, \preceq_{\mathcal{P}})$ and $(\mathcal{Q}, \preceq_{\mathcal{Q}})$ are posets, $f : \mathcal{P} \longrightarrow \mathcal{Q}$ and $g : \mathcal{Q} \longrightarrow \mathcal{P}$ are order-reversing functions such that for each $p \in \mathcal{P}$ and $q \in \mathcal{Q}$:

$$p \preceq_{\mathcal{P}} gf(p) \text{and} q \preceq_{\mathcal{Q}} fg(q).$$

3. Main results

Among the miscellaneous requisitions, FCA has a huge capacity to play in a common ground between Mathematics and Computer Science. Since 2014, its outstanding role in Data Mining and Knowledge Discovery should be kept in mind, where its techniques has appealed to analyze complex data sets, such as Medical Records and Social Media, by determining the hidden patterns. Moreover, some of its tracks can be detected in developing new algorithms for Text Mining, Natural Language Processing (NLP), Software Engineering and Programming (in modeling software system and testing their functionality), Database Design and Management (helping to structure and organize data). To see more details, please refer to [4], [3], [11], [1], and [14].

Undoubtedly any theoretical development in the body of FCA will greatly contribute to its usable capabilities in the field of computer science. What brings us to this section, is relying on the evolution of its notion and algebraic mathematical models. So far presented operations on FCA, has been performed on pertained complete lattice and include union, intersection, complementation and closure. The leading works can be followed in [4], [3], [11], [1], [14]. But to extend the power of system, adjoining addition and multiplication is strongly felt as a necessity. This endowing can lift the domination of computational mathematics and logic used in the subject. In contrast to costume representation, applying dynamical models require to have formal contexts with infinite order or sets of attributes. As an example, consider a context where the objects are real numbers and the attributes are sets of real numbers. The creation of such a data matrix can be reproduced by a dynamic data generation system (for instance in designing the active web pages). According to our new extension, this is an infinite context which should still be analyzed using FCA techniques. Really it can be done by breaking the big data context into finite frames and eventually analyzing the sectional data in the framework of available tools. Similar to these circumstances, for example, can happen when the data flow in social networks or deliver as output results form an active real-time switcher. Therefore, in this section, the main goal will be enhancing the FCA to a more flexible algebraic machine. As the first measure, let us to consider any formal context as a triple $\mathcal{C} = (\mathcal{O}, \mathcal{A}, \mathcal{R})$ with infinite sets and the given relation.

Definition 3.1. Suppose that $\mathcal{O} = \{O_i | i \in I\}$ and $\mathcal{A} = \{A_j | j \in J\}$ are two indexed sets and $\mathcal{C} = (\mathcal{O}, \mathcal{A}, \mathcal{R})$ a context with a given relation $\mathcal{R}$. Any word on $\mathcal{C}$ can be considered as a net in $\mathcal{A}$. For any word W we say that it is a K-letter word, if the objects in $\mathcal{O}$ having the common relation $\mathcal{R}$ in terms of W, be a subset of $\mathcal{O}$ with cardinal number K. In the case of finite subsets in $\mathcal{O}$, we refer to W as a finite word.

In finite contexts, a K-letter can be described as the outcome rate in comparison with the whole transection, which is a number between zero and one. Here we intend to give the words, an orderly ring structure. The advantage of the decision is expanding the more algebraic operations on words.

Suppose that $\mathcal{W} = \{W_k | k \in K\}$ is the set of words on $\mathcal{C}$. By considering two operations on $\mathcal{W}$, we turn it into a ring. For any W_i and W_j in $\mathcal{W}$, let O_{W_i} and O_{W_j} be the corresponding subsets of $\mathcal{O}$ respectively. Consider, $O_{W_i} \cap O_{W_j}$ and $O_{W_i} \cup O_{W_j}$ as having the minimum and maximum cardinals respectively, and define:

$$W_i + W_j = W_{O_{W_i} \cap O_{W_j}}, \qquad W_i.W_j = W_i W_j = W_{O_{W_i} \cup O_{W_j}}. \tag{1}$$

A straightforward calculation shows that $(\mathcal{W}, +, .)$ is an ordinary ring because its algebraic properties go back to the corresponding operations on cardinal numbers. It needs to equip this ring with an order. We say that $W_i \preceq W_j$, if $O_{W_j} \subseteq O_{W_i}$ and $W_i \prec W_j$, when $O_{W_j} \subset O_{W_i}$.

Under such operations, $(\mathcal{W}, +, ., \preceq)$ is called a *Concept Ring* under $\mathcal{O}$ and denoted by $CR_{\mathcal{O}}$. Each member of $\mathcal{W}$ will be called a *concept word.*

Definition 3.2. Let $\mathcal{W}$ be a set of words. The subset B of $\mathcal{W}$ is called a finite-word base, if it is countable and every bounded finite subset $X \subseteq B$ has a least upper bound in B.

Since $\mathcal{W}$ is in fact a poset, B can be considered as a poset with induced order by $\mathcal{W}$. Noting $\mathcal{W}$ (as the net of all elements) is a finite word, since its inverse image is empty, it is the least upper bound which is denoted by $\perp_B$. This shows that B is a poset with bottom element $\perp_B$. To simplify understanding the presented definitions, consider the following example.

Example 3.3. Let $\mathcal{O} = \{a, b, c\}$, equiped with lexicographic order, be our objects. Take at most three-letter words on $\mathcal{O}$. In the following, you see the table of $\mathcal{W}$:

Table 1. $CR_{\mathcal{O}}$ of Example 3.3

	0-word	1-word	2-word	3-word
$\emptyset$	×			
a		×		
b		×		
c		×		
ab			×	
ac			×	
bc			×	
abc				×

Also as a directed graph, finite-word base can be illustrated follows:

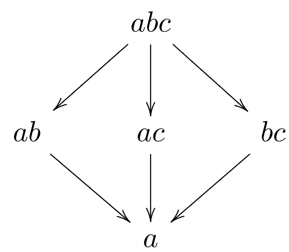

Diagram 2. Hasse diagram of a finite-word base on $\mathcal{O}$.

A more beautiful way to describe the shape of this ring is to encode members of $\mathcal{O}$. On applying the use of maximum three-word combinations, we put in at most three-digits of 0 and 1, for each combination.

This method of representation is a vital tool in *digital communication*. Take $a = 100$, $b = 010$ and $c = 001$. Then we have

$$a + a = aa = a, b + b = bb = b, c + c = cc = c.$$

Furthermore

$$a + b = a, a + c = a, b + c = b, ab = b, ac = c, bc = c.$$

A close observation by numerical coding method, trains us very simple rules in the construction:

$$\begin{aligned} a + b &= 100 + 010 = 100, \\ a + c &= 100 + 001 = 100, \\ b + c &= 010 + 001 = 010, \\ ab &= (100)(010) = 010, \\ ac &= (100)(001) = 001, \\ bc &= (010)(001) = 001. \end{aligned}$$

The result of these calculations shows that, addition always selects a notation which is in the left most possible position, while the multiplication does the opposite operation. Therefore, the sum must be thought as a *contracting operator* and multiplication as a *developing* one. Some significant relations occur directly in the shadow of the fact that $\mathcal{O}$ is ruled by the linear structure $a \preceq b \preceq c$. In order to discover a much more important ability, in numerical method, let us show $\perp_B$ by 000. A simple computation shows that for each word W, $W \perp_B = \perp_B W = W$. Then $\perp_B$ plays the role of a neutral element. This encourages us to define a net like $W_1 \perp_B W_2$ as a *term* on $\mathcal{O}$ for each $W_1, W_2 \in B$, which opens the way to define sentences. Finally, due to special circumstance, the two ends of this diagram are closed from top and bottom. In general, this does not necessarily happen (for example when $\mathcal{O}$ is an infinite set).

Formal Concept Ring (FCR)

A concept ring has the advantage of owning algebraic operations on concepts while maintaining the order. When the objects set $\mathcal{O}$ is infinite (even a countable set), the set of generated words can increase dramatically. Therefore, one of the best key points is looking for an idea to produce such a ring with a subset by smaller cardinal number. Maybe The leading option should be the finite-word base. Another outstanding point, is the discussion of optimization. Naturally ideals of a ring might be the best choice by considering the approximation induced by its order, but the way to make it must be quite clear.

On the other hand, any cluster of words may not be necessarily ended in one point. In computational processing, it is a very significant necessity to end a set of decisions at one or more final points, i.e. any sub process has the start and end points. The next try is looking for convincing answers in such a case. So, the initial objection is starting with finite-word base and the sequential operations include a processing called *approximating*, to achieve a new building which will be named the *Formal Concept Ring (FCR)* or *Galois Concept Ring (GCR).* Graphically, each FCR will be a simple and directional closed graph with starting and ending points if it exists, along with how to compound edges together. The approximation process may be performed with a sequence of iterated operations. In the case of a finite set, there will always be a FCR related to W. But infinite cases, require special conditions to reach the solution through the *fixed point* or *approximate fixed point* theory.

Definition 3.4. Suppose that $(W, +, ., \preceq)$ is a Concept Ring (CR) and B is a finite-word base in W. An implication on B will denote by $a \Rightarrow b$, for $a, b \in B$, and $a \leq b$ iff $a \Rightarrow b$ or $a = b$. A complete-word subset of W related to B, will be denoted by CW_B and will be made as follows: a bounded subset $X \subseteq B$ is an element of CW_B iff
i) For every $x \in X$ and $b \in B$, if $b \preceq x$ then $b \in X$.
ii) For every $x, y \in X, xy \in X$ (X is closed under multiplication).

This means that for any finite-word base B, CW_B is constructed by all bounded subsets of B that are closed under multiplication. For any $x_0 \in B$, take

$$I_{x_0} = \bigvee\{x \in B : x \preceq x_0\}.$$

This element is unique in CW_B. Moreover, CW_B contains elements corresponding to the *limits* of all developing subsets of B. This yields a clear perspective to construct CW_B from B. There are two categories of members that are made up of approximations with finite or infinite stages. We refer to this final products as *ideals* of B. So for any finite-word base B, we complete B by adding limit elements of all developing directed subsets of B. It is clear that B is in fact a subring of W with $|B| \preceq |W|$. This gives us a constructive method starting from B and its final product can be W. Here, one can observe that ideals have a relatively more computational formation than a normal ring.

Definition 3.5. For any finite-word base B, a subset $I \subseteq B$ is an ideal in B iff
i) If $i \in I$ then for each $b \in B$ and $b \preceq i$ we have $b \in I$.
ii) For any two finite-words $W_1, W_2 \in I, \sup\{W_1, W_2\} \in I$.

In definition 3.5, property (i) is called the *downward closed property.* So, by definition 3.5, an ideal of B is a subring that has the property of downward closeness.

Now, there is productive method to form the CW_B.

Definition 3.6. Suppose that B is a finite-word base of W. Then CW_B generated by B, includes $(\mathcal{I}, \preceq_{\mathcal{I}})$, where $\mathcal{I}$ is the ideals set of B and $\preceq_{\mathcal{I}}$ is the inclusion order.

An elementary and basic fact can be introduced by the following theorem.

Theorem 3.7. *For any finite-word base B on W the followings hold:*
i) If $I_1, I_2 \in \mathcal{I}$ and $\sup\{I_1, I_2\}$ exist then it is an ideal of B.
ii) $(\mathcal{I}, \preceq_{\mathcal{I}})$is a CPO.

Proof. i) Suppose that $\alpha = \{I_1, I_2\}$ and

$$I_3 = I_1 I_2 = \{x_1 x_2 : x_1 \in I_1, x_2 \in I_2\}.$$

We claim that I_3 is an ideal of B and $\alpha = I_3$. With the help of (1) and a straight forward calculation, one can prove that:

$$\bigcap_{x \in I_3} O_x = O_{I_3} = O_{I_1 \cup I_2} = \bigcap_{y \in I_1 \cup I_2} O_y.$$

Let $i \in I_3$, $b \in B$ and $b \preceq i$, then $O_i \subseteq O_b$. But for some $i_1 \in I_1$ and $i_2 \in I_2$ we have $O_i = O_{i_1 \cup i_2}$. So $O_{i_1} \subseteq O_i$ and $O_{i_2} \subseteq O_i$. This means that $O_{i_1} \subseteq O_b$ and $O_{i_2} \subseteq O_b$ and so $O_{i_1 \cup i_2} \subseteq O_b$. But this implies that $b \in I_3$, since $\sup\{I_1, I_2\}$ exists. Also by hypothesis, I_3 is closed under least upper bounds on finite subsets. We claim that $\alpha = I_3$. Since $I_1 \preceq I_3$, $I_2 \preceq I_3$ then $\alpha \preceq I_3$. On the other hand $O_\alpha \subseteq O_{I_1}$ and $O_\alpha \subseteq O_{I_2}$. Then $O_\alpha \subseteq O_{I_1 \cup I_2} = O_{I_3}$. The final two inequalities show that $\alpha = I_3$.

ii) Since $\preceq_{\mathcal{I}}$ is the inclusion order, then $\mathcal{I}$ is a poset. According to i) and by induction property, for ideals $I_1, I_2, \ldots, I_n$ one can show that $I_1 I_2 \ldots I_n$ is an ideal of B and

$$\sup\{I_1, I_2, \ldots, I_n\} = I_1 I_2 \ldots I_n.$$

Now, suppose that $\{I_n\}_{n=1}^{\infty}$ be a sequence of ideals on B. Then imagine $\mathcal{O}_{\cup_{i=1}^{\infty} I_i} = \prod_{i=1}^{\infty} I_i$.We must show that $\prod_{i=1}^{\infty} I_i$ exists and is an ideal on B. Let us to use the notation, $\prod_{i=1}^{\infty} I_i = \lim_{n \to \infty} \prod_{i=1}^{n} I_i$. Given that we have not yet defined any analytical structure on CR_W and FCR, this notation seems a little unfamiliar, but with help of Zorn's lemma, such a definition becomes objective. For any k, suppose that W_k be the set of at most k-words on B, i.e.

$W_k = \prod_{i=1}^{k} I_i$. Then by given order on CR and noting this fact that CR is a directed set, we implies that the set $y = \{W_1, W_2, \dots\}$, has a maximal element. We call this element by $\prod_{i=1}^{\infty} I_i$. On the other hand, a simple verification shows that $\mathcal{O}_{\cup_{i=1}^{n} I_i} = \prod_{i=1}^{n} I_i$ where $O_i = O_{I_i}$. Again, by using the Zorn's lemma on $P(O)$ (power set of O), we infer that the set $V = \{O_1, O_1 \cup O_2, \dots, \bigcup_{i=1}^{n} O_i, \dots\}$ has $\bigcup_{i=1}^{\infty} O_i$ as the maximal element. Moreover

$$\bigcup_{i=1}^{\infty} O_i = \lim_{n\to\infty} (\bigcup_{i=1}^{n} O_i) = \lim_{n\to\infty} (\prod_{i=1}^{n} I_i) = \prod_{i=1}^{\infty} I_i.$$

So it remains to show that $\prod_{i=1}^{\infty} I_i$ is an ideal. Suppose that $\prod_{i=1}^{\infty} I_{i_k} \in \prod_{i=1} I_i$ and $b \in B$ be such that $b \preceq \prod_{k=1}^{\infty} I_{i_k}$. By applying induction on i, one can show that for any $n \in \mathbb{N}$ and $b \preceq \prod_{i=k}^{n} I_{i_k}$. We have $\prod_{k=1}^{n} I_{i_k} \in \prod_{i=1}^{n} I_i$.

Now by using this fact,

$$\prod_{i=1}^{\infty} I_{i_k} = \lim_{n\to\infty} (\prod_{k=1}^{n} I_{i_k}) \in \lim_{n\to\infty} (\prod_{i=1}^{n} I_i) = \prod_{i=1}^{\infty} I_i.$$

Finally, suppose that $x = \prod_{i=1}^{\infty} x_i$ and $y = \prod_{i=1}^{\infty} y_i$ be two elements of $\prod_{i=1}^{\infty} I_k$. Then

$$xy = (\prod_i x_i)(\prod_j y_j) = \lim_{n\to\infty} \lim_{m\to\infty} (\prod_{i=1}^{n} x_i)(\prod_{i=1}^{m} y_i) \in \prod_{k=1}^{\infty} I_k,$$

and this completes the proof. □

In the following, we intend to point out the pivotal role of a particular category of ideals called the *principal concept ideals*.

What has done so far, is to constrain the clusters of a ring, by two end points and the connection of edges (which are the nodes of the diagram) are composed of the ideals in the ring. Now we want to emphasize the fact that every branch arises from the joining of more elementary ideals called the main concept ideals.

For any finite-word base B of the ring $CR_{\mathcal{O}}$ and any $x_0 \in B$, the set

$$I_{x_0} = \{y \in B : y \preceq x_0\},$$

is an ideal of B (it is easy to check). This gives us the following definition:

Definition 3.8. For any given finite-word base $(B, \preceq)$, and any $x_0 \in B$, I_{x_0} is called the principal concept ideal generated by x_0.

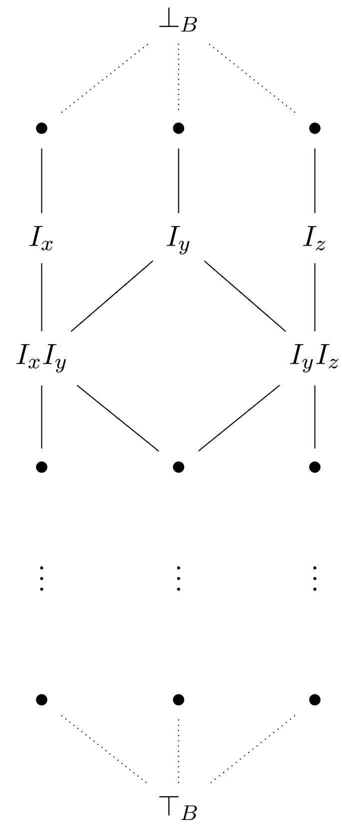

Diagram 3. The discrete topology of FCR generated by B and their concept ideals.

The diagram 3 tells us that the ring structure of a *decision process* based on a data context, starts from an evolutionary point ($\bot_B$) and (under decision making conditions) is broken down into several branches of the principal concept ideals. When we are facing a *distributive system* of *decision-making processes*, their interference leads to the combination of principal concept ideals, and ultimately forms an end point to complete the process ($\top_B$). Suppose that $\{S_i\}_{i=1}^{\infty}$ be an indexed arbitrary directed subsets of a given FCR, on a finite-word base B. Define

$$\prod_{i=1}^{\infty} S_i = \{\prod_{i=1}^{\infty} s_{i_j} : s_{i_j} \in S_i \ , i, j \in \mathbb{N}\},$$

and also, take

$$\mathcal{O}_{S_i} = \bigcap_{s \in S_i} \mathcal{O}_s \quad , \quad \mathcal{O}_{\prod_{i=1}^{\infty} S_i} = \bigcup_{i=1}^{\infty} \mathcal{O}_{S_i}.$$

Similar to the argument proposed in theorem 3.7, it can be proved that

$$\prod_{i=1}^{\infty} S_i = \sup\{S_i\}_{i=1}^{\infty}.$$

Correspondingly, for any subsets $\{S_i\}_{i=1}^{\infty}$, one can show that if

$$+_{i=1}^{\infty} S_i = \{+_{i=1}^{\infty} s_{ij} : s_{ij} \in S_i\} \text{and} \mathcal{O}_{+_{i=1}^{\infty} S_i} = \bigcap_{i=1}^{\infty} \mathcal{O}_{S_i},$$

then

$$+_{i=1}^{\infty} S_i \in FCR, \text{and} +_{i=1}^{\infty} S_i = \inf\{S_i\}_{i=1}^{\infty}.$$

These yield the summation and production of words and subsets of FCR. Moreover, based on definition 3.8 and theorem 3.7, there is a one-to-one corresponding between finite-words and principal concept ideals on B, i.e. $W \simeq I_W = \langle W \rangle$ and $W_1 W_2 \simeq I_{W_1} I_{W_2} = \langle W_1 \rangle \langle W_2 \rangle$, where for any set X, $\langle X \rangle$ is the ideal generated by X. So the natural result of this argument can be summarized as follows:

Theorem 3.9. *Let $\mathcal{I}_B$, be the set of principal concept ideals over B. Then $\mathcal{I}_B$ forms a finite-word base under the subset ordering on concept ideals.*

Theorem 3.9, will give us a lot of ability to create a new important ring called the *Item Set Ring* (ISR_B) or *Attribute Ring* (AR_B). The complete ring generated by this, will be called a *Closed Item Set Ring* ($CISR_B$) or a *Closed Attribute Ring* (CAR_B). In the next step, we see that CAR_B will play a main role in calculating the attributes implications. Here, let us to remind the definition of a *finite* element in a CPO. An element of any CPO, $(X, \preceq)$ is called finite iff for any directed subset S of X such that $f = \bigvee_{s \in S} S$ one can implies that $f \in S$. In ordered topological vector spaces, this can be imagined as the limit points of ascending sequences in a closed subsets, especially think of end points in a closed bounded subset $[a, b]$ in $\mathbb{R}$. For normal order on $\mathbb{R}, b$ is a finite point, while in the case of $[a, +\infty)$ there is no such point. But one can write $[a, +\infty) = \bigcup_{n \in \mathbb{N}} [a, n]$. So, $[a, +\infty) = \bigvee_{n \in \mathbb{N}} [a, n]$, at the same time $\bigvee_{n \in \mathbb{N}} [a, n]$ is not a closed interval. In the case of partially ordered cases, this situation can be thought of as a box of matchsticks that can be connected from top to bottom, under a given topology. More precisely, this intuition will help to design the *topology of attribute rings.*

Theorem 3.10. *All finite elements of $\mathcal{I}_B$ are principal concept ideals and vise versa.*

Proof. For any $x_0 \in B$ and $\langle x_0 \rangle = \{y \in B : y \preceq x_0\}$, as the principal ideal, a simple computation shows that $x_0 = \bigvee_{y \in \langle x_0 \rangle} y$. Conversely, for any $x_0 \in CW_B$ as a finite element, take $\langle x_0 \rangle = \{y \in B : y \preceq x_0\}$. By applying the definition 3.8, it can be proved that $\langle x_0 \rangle$ is a principal concept ideal. □

Next theorem clearly states how the principal ideals can be used as a tool to approximate (by finite or infinite process) other concept ideals. Put

$$pr\mathcal{I}_B = \{\mathcal{I}_a : \mathcal{I}_a \text{ is a principal ideal in } \mathcal{I}_B\}.$$

Theorem 3.11. *For any ideal $\mathcal{A}$ on CW_B,*

$$\mathcal{A} = \sup\{\mathcal{I}_a : \mathcal{I}_a \in pr\mathcal{I}_B \ and \ \mathcal{I}_a \preceq \mathcal{I}\}.$$

Proof. Take $\alpha = \sup\{\mathcal{I}_a : \mathcal{I}_a \in pr\mathcal{I}_B \ and \ \mathcal{I}_a \preceq \mathcal{I}\}$. By theorem 3.9, $\mathcal{I}_B$ is a poset and hence $\alpha \preceq \mathcal{A}$. On the other hand suppose that $\{I_n\}_{n=1}^{\infty}$ be a sequence of ideals in $pr\mathcal{I}_B$ such that $I_n \preceq \mathcal{A}$, $\forall n \in \mathbb{N}$. So $\bigvee_{n=1}^{\infty} I_n \preceq \mathcal{A}$. Also by theorem 3.7, $\bigvee_{n=1}^{\infty} I_n$ is an ideal on CW_B. If $I_n = \langle x_n \rangle$, then $\bigvee_{n=1}^{\infty} I_n = \langle \prod_{n=1}^{\infty} x_n \rangle$. Hence $\mathcal{A} = \bigvee_{n=1}^{\infty} I_n$ is a principal concept ideal and we have $\mathcal{A} \preceq \alpha$. This shows that $\mathcal{A} = \alpha$. □

The obtained results convince us to separate the elements with in CR_B in two basic categories:
1. Elements that are complete themselves and do not need any approximation.
2. Elements that are generated by finite or infinite iterations of partial elements.

The elements of the first category are called the *Complete elements* and those of the second type are called *approximate elements*. Remember that these elements can imagine as concepts and formal concepts respectively, but here thought as the members of a topological space.

The final conclusion of this section is a fundamental theorem based on the ring isomorphism. This will enable us to generalize the existence theorem of FCA and also it helps to find a constructive method in computing the attribute implications. By $\mathcal{I}_B = \{I_b : b \in B\}$, equip $\mathcal{I}_B$ with the following operations as adding and multiplication:

$$I_a + I_b = I_{W_{\mathcal{O}_a \cap \mathcal{O}_b}}, \quad I_a I_b = I_{W_{\mathcal{O}_a \cup \mathcal{O}_b}},$$

and construct $CR_A = \{\prod_{a \in B} I_a\}$. These operations, change CR_A as a ring which will be called the *attribute ring*. By theorem 3.9, $\mathcal{I}_B$ is a finite-word base for CR_A under the approximately ordering of inclusion. The following assertion is our final result in theoretical section.

Theorem 3.12. *CW_{CR_A} is isomorphic to $CW_{CR_\mathcal{O}}$.*

Proof. It is clear that there is a one-to-one correspondening,

$$B \longrightarrow \mathcal{I}_B,$$

by

$$x_0 \mapsto \langle x_0 \rangle, \forall x_0 \in B,$$

and consider the embedding $pr\mathcal{I}_B \longrightarrow CW_{CR_\mathcal{O}}$. Now define

$$\phi : CW_{CR_\mathcal{O}} \longrightarrow CW_{CR_A},$$

by

$$\phi(x) = \bigvee \{y \in pr\mathcal{I}_B : y \preceq x\}, \forall x \in CW_{CR_\mathcal{O}}.$$

Then ϕ is an isomorphism and the proof is complete. □

4. Applications to extend the results of FCA

So far, two main goals have been received:

1. Designing an algebraic-analytical machine that beside of maintaining a hierarchical relationship, can also perform operations such as addition and multiplication on concepts.
2. Development of the FCA notion by making a new ring that has been able to generalize all previous results even in innumerable infinite set of objects.

We have already considered the results of first perspective, so in this section our focus will be on the aftermaths of the second view.

As a basic fact, it should be noted that each finite-word base is essentially a lattice and $CR_{\mathcal{O}}$ is a complete lattice. So in this building, it practically starts from a lattice-like base (also countable) and ends with its completion, which can end to even uncountable cases. In the meantime, the ideals of the ring play an irreplaceable role, in fact, points filled by the process of completion, which in the language of Galois lattice are called formal concepts. In order to study the subject in more details, we will present the same digital tapes in finite dimension of up to two-digit cases. Therefore, as a context in FCA, we have

$$\mathcal{O} = \{0, 1\}, A = \{\bot = 0 - digits,\ 1 - digit,\ 2 - digits\},$$

and we assume that the relation R induces a finite-word base. Suppose that R be such that we have the following diagram (induced order is from down to up):

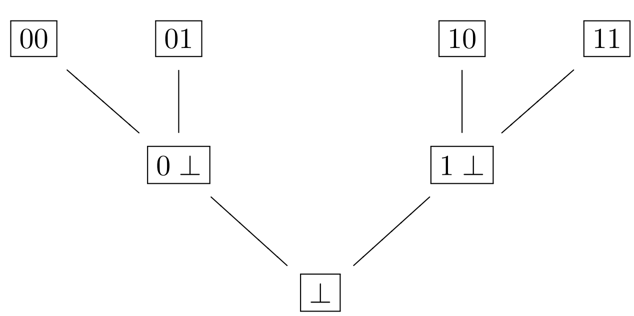

Diagram 4. A finite-word Hass diagram for digital strips.

The finite-word base of diagram 4 is actually a lattice, but not a complete lattice. Now, in order to achieve a complete lattice, we calculate the ideals of the ring induced by finite-word base of diagram 4.

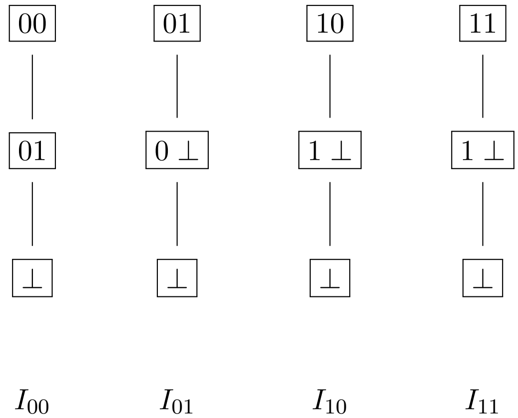

Diagram 5. Finite-word base of Diagram 4.

Our formal concepts are I_{00}, I_{01},I_{10} and I_{11}. Each concept ideals are principal with top points $00, 01, 10$ and 11, as formal concepts. For example, $I_{00} = \langle 00\rangle$ and as $\langle 00\rangle \simeq 00$, it is denoted by 00. Also we have

$$extent(00) = \{00\},$$

$$intent(00) = \{2 - words\}.$$

Each process is carried out in developing way, in the path of main concept ideals, starting from bottom, and ending at a saturation (top) point called the *generator* of principal concept ideal. The operations are performed by iteration method. Note that under multiplication, the process is incremental, and decreasing by sum. At the same time, the result of both operations ends at a fixed ordered pair. Another valuable inference bears that in the case of infinite order, this process is turned into the calculation of a limit point. Method of calculation can be explained by *fixed point* approach. If the goal is to be satisfied with a degree of accuracy, *approximate fixed point* methods give us a reasonable solution. So the key points in modeling the Galois lattice goe back to identify the principal concept ideals. At the end of this section, we will see how the proved theorems help us to generalize the basic results in FCA.

As seen so far, each CR induces a lattice and each FCR induces a complete lattice respectively. Despite more results, two types of applications in FCA will be discussed here, existential and computational results. One of the basic consequences is to extend the proposition 2.8. Based on a constructive development, it begins from a countable subset (finite-word base) and extends to a likely uncountable ring. The advantage of this method is to commence with basic elements. But first, it is necessary to rewrite it with new notations.

Theorem 4.1. *$FCR_{\mathcal{O}}$ is a complete ring, where B is a finite-word base related to the context $((\mathcal{O}, A, R), \preceq)$. In addition, each member of $FCR_{\mathcal{O}}$ can be written as $\{W^*, W_* : W_* \preceq W^*\}$,* *with*

$$W^* = \lim_{n\to\infty}(\prod_{i=1}^{n} W_i) \quad and \quad W_* = \lim_{n\to\infty}(+_{i=1}^{n} W_i), \forall i : W_i \in B.$$

Proof. For each $k \in \mathbb{N}$, suppose that W_k be the class of all finite-words that has the subsets of $\mathcal{O}$ with k members as its extent and take $B_{\mathbb{N}} = \{W_k : k \in \mathbb{N}\}$. One can see that $B_{\mathbb{N}}$ is a finite-word base by considering,

$$w_i w_j = extw_i \cup extw_j, \;\; w_i + w_j = extw_i \cap extw_j, \forall w_i, w_j \in B_{\mathbb{N}}.$$

By comparing subsets with a positive index to elements with even indices in B and subsets with a negative index to odd ones, the rang of indices in B can be extended to the integers. So, we write $B_{\mathbb{Z}} = \{W_k : k \in \mathbb{Z}\}$. Now, consider the quotient of $B \times B$ as:

$$quotient(B_{\mathbb{Z}} \times B_{\mathbb{Z}}) = \{(W_k, W_l) : k, l \in \mathbb{Z}, (k, l) = 1\}.$$

Just like the previous case, it can be seen that the recent set will be correspondent to the case where the indexes are chosen in $\mathbb{Q}$. Therefore, without losing the generality and replacing quotient $(B_{\mathbb{Z}} \times B_{\mathbb{Z}})$ with $B_{\mathbb{Q}}$, finally $B_{\mathbb{Z}}$ can be considered as:

$$B_{\mathbb{Q}} = \{W_q : q \in \mathbb{Q}\}.$$

But $\mathbb{Q}$ is a linear poset and as a subset of a topological space, $\mathbb{Q}$ is dense in $\mathbb{R}$. So each $\alpha \in \mathbb{R}$ can be written as $\alpha = \lim_{t\in\mathbb{Q}} \alpha_t$ and $\{\alpha_t\}$ is a monotone sequence in $\mathbb{Q}$. For $w_t \in B_{\mathbb{Q}}$, it is enough to see that:

$$W^* = \prod_{t\in\mathbb{Q}} \alpha_t = \bigcup_{t\in\mathbb{Q}} extw_t,$$
$$W_* = +_{t\in\mathbb{Q}} W_t = \bigcap_{t\in\mathbb{Q}} extw_t.$$

Now by using the Zorn's lemma, the proof is complete. □

Remark 4.2. In a rough language, from the previous theorem, we infer that in fact $B_{\mathbb{Q}}$ is simultaneously *addition-dense* and *multiplication-dense* subset of $CR_{\mathcal{O}}$ and practically $CR_{\mathcal{O}}$ is the set of limit points, for all monotone and bounded sequences.

Attribute Implications (AI)

An attribute implication can be defined as an ordered pair (X, Y) belongs to the subsets of attributes collection A. Generally, it will denote by $X \longrightarrow Y$. The validity of an implications means that in any context, the set of extents related to Y must be in that of X [13]. In the world of interpretation, this yields many wonderful results. A logical consequence of such an understanding can say that, " in data modeling, X is more valid than Y". Therefore, in terms of informative classification systems, it can be asserted that X contains information within Y.

The logical result of this view states that, X has a higher (considering the hierarchy) degree of information in comparison with Y. This new perception encourages us to arrive at a key generalization of the definition:

Definition 4.3. Any ordered pair $(W_1, W_2) \in CW_{CR_{\mathcal{O}}}$ is called an attribute implication. Also we say that $W_1 \longrightarrow W_2$ holds in $CW_{CR_{\mathcal{O}}}$ if $W_2 \preceq W_1$.

Definition 4.3 can be recounted as follows:
Considering W_1 as a more complete word, it is practically described (approximates) by W_2 and with a degree of more error.

In extending this idea, suppose that W_1 is obtained with a sequence of approximations $X_1, X_2, \cdots, X_n$ that each stage of describing be more complete, then one can depict this process by:

$$W_2 \longrightarrow X_1 \longrightarrow X_2 \cdots \longrightarrow X_n \longrightarrow W_1,$$

or

$$W_2 \preceq X_1 \preceq X_2 \cdots \preceq X_n \preceq W_1.$$

If this chain stops at some point, it means that the processing observation cannot be modified further, and we keep it in mind as a complete or *ideal attribute*. As the lattice-element items, we call the final entity as an *indecomposable* element, and the rest of the parts as *decomposable* cases. In Hasse diagram, indecomposable objects are at the head of branches.

Definition 4.4. Suppose that $x \in CW_{CR_{\mathcal{O}}}$ and define

$$\begin{aligned} x^* &= \{\prod_{i \in I} x_i : x_i \prec x\}, \\ x_* &= \{+_{i \in I} x_i : x_i \prec x\}. \end{aligned}$$

We say that x is *product-indecomposable* (*p-indecomposable*) if $x^* \neq x$ and dually, x is called *sum-indecomposable* (*s-indecomposable*) if $x_* \neq x$. Otherwise, x are called *product-decomposable* (*p-decomposable*) and *sum-decomposable (s-decomposable)*, respectively.
A subset $X \subseteq CW_{CR_{\mathcal{O}}}$ is called product-dense if for each $y \in CW_{CR_{\mathcal{O}}}$ there exists a net $\{x_i\}_{i \in I}$ of approximants in X such that $y = \prod_{i \in I} x_i$. Sum-dense subset $X \subseteq CW_{CR_{\mathcal{O}}}$ can be defined similarly .i.e. there exists $\{t_i\}_{i \in I} \subseteq X$ such that $y = +_{i \in I} t_i$.

The following theorem is so crucial, because indecomposable elements will play a key role in drawing the complete ring. In addition, rings with desired properties will be formed from finite rings.

Theorem 4.5. *In a finite $CW_{CR_{\mathcal{O}}}$:*
i) Any elements x_0 is product-decomposable iff the ideal (attribute ideal) generated by x_0 is

principal .i.e. $I = \langle x_0 \rangle = I_{x_0}$*. The similar case holds for sum-decomposable.*
ii) Any ideal can be formed as product and sum-decomposable principal ideals.
iii) As in Hass-diagram, an element of $CW_{CR_\mathcal{O}}$ *is product-indecomposable iff it covers exactly one elements and sum-decomposable iff it is covered by exactly on element.*
iv) The sets $p - CW_{CR_\mathcal{O}}$ *and* $s - CW_{CR_\mathcal{O}}$ *of p and s-decomposable elements of* $CW_{CR_\mathcal{O}}$ *are dense in* $CW_{CR_\mathcal{O}}$*.*

Proof. i) Suppose that x_0 is p-decomposable. then by definition 4.4, there exists a net $\{x_i\}_{i\in I}$ such that $x_0 = \prod_{i\in I} x_i$. As we know, according to the induced order by finite-word definition, if I_1 and I_2 be two subsets of I such that $|I_1| \preceq |I_2|$, then $\prod_{i\in I_1} x_i \preceq \prod_{i\in I_2} x_i$.
So by considering

$$\mathfrak{T} = \{\prod_{i\in I'} x_i : I' \subseteq I\},$$

we see that $\mathfrak{T}$ is an ideal on $CW_{CR_\mathcal{O}}$ and $\mathfrak{T} = \langle x_0 \rangle$, conversely let $I_{x_0} = \langle x_0 \rangle$. Then for each $x_i \in I_{x_0}$, $y \preceq x_0$ and then $\prod_{i\in I_1} x_i \preceq x_0$, for any subset $I_1 \subseteq I$. So $x^* \preceq x_0$. On the other hand $x_0 \in I_{x_0}$ and hence $x_0 \preceq x^*$. Then $x^* = x_0$ and so x_0 is product decomposable. The second part will be proved similarly.

ii) We assume that I is any ideal in $CW_{CR_\mathcal{O}}$. By Zorn's lemma, there is a maximal subset $X \subseteq CW_{CR_\mathcal{O}}$ such that I is generated by X, i.e. $I = \langle X \rangle$. Let $X = \{x_j : j \in J\}$, and $I_{x_j} = \langle x_j \rangle$. Then $I = \prod_{i\in J} I_{x_j}$. The same argument can be presented for sum-decomposable principal ideals by taking X as the minimal subset such that $I = \langle X \rangle$.

iii) First we note that $x \in CW_{CR_\mathcal{O}}$ is p-indecomposable iff $x^* \prec x$ since by definition 4.4, x is p-indecomposable iff $x \neq x^*$ and also x^* is the supremum of all elements $y \in CW_{CR_\mathcal{O}}$ such that $y \prec x$. So $x^* \preceq x$. Now let x be a p-indecomposable element in $CW_{CR_\mathcal{O}}$, and assume that m is another element covered by x with $m \neq x^*$. Then $m \prec x$ and hence $x^* \prec m$. So we have $x^* \prec m \prec x$ which is a contradiction. Hence x^* is the only element covered by x. Conversely assume that m is the unique element covered by x. Let

$$L = \{y \in CW_{CR_\mathcal{O}} : y \prec x\}.$$

Then m is an upper bound for L and hence $m \succeq x^*$. Since $m \prec x$ this means that $x^* \prec x$ and then x is p-indecomposable. The second part can be shown by the same way.

iv) One can easily prove that $\langle X_1 \rangle \langle X_2 \rangle = \langle X_1 \cup X_2 \rangle$ and $\langle X_1 \rangle + \langle X_2 \rangle = \langle X_1 \cap X_2 \rangle$, for any subsets X_1, X_2 of $CW_{CR_\mathcal{O}}$. Now, an element $x \in CW_{CR_\mathcal{O}}$ has two cases. If x is p-indecomposable then $x \in P(CW_{CR_\mathcal{O}})$, otherwise x is the product limit of strictly smaller elements. Again, any strictly smaller element is either p-indecomposable or the product limit of strictly smaller elements. Since $CW_{CR_\mathcal{O}}$ is finite and by what we mentioned at first, this process must end, and so x is a product limit of elements in $P(CW_{CR_\mathcal{O}})$. Second part of the proof is the same. □

The given results combined with theorem 3.12 yield two following valuable consequences:
a) Breaking a complete ring into subrings constructively,
b) Extension of some basic theorems in attribute implications.
According to theorem 4.5, each finite ring can be broken into a set of principal concept ideals. Each principal ideal is generated by an indecomposable elements when it has only one generator (if there is at least one generator it is enough to consider the smallest one). So the only thing to do is computing the indecomposable elements. In the case of infinite ring having a countable dense and finite-word base, the study is more exciting. This occasion reminds us as the method of studying integer rings by means of prime numbers. If S is a set of attribute implications, then closed item set containing S is $\langle S\rangle$. Moreover, if $\{P_i : i \in I\}$ is the indecomposable elements in S, again $\langle S\rangle = \prod_{i\in I}\langle P_i\rangle$. For any principal ideal $\langle x_0\rangle$, x_0 is indecomposable and the best element to generate the information in $\langle x_0\rangle$. Moreover, theorem 3.12 states that the ring of attribute implication is nothing but the ring of objects. So having $CW_{CR_{\mathcal{A}}}$ is equivalent to having $CW_{CR_{\mathcal{O}}}$ and vice versa. The more important message of this theorem is a useful generalization of the following theorem.

Theorem 4.6. *The set of all implications of a context is a complete lattice.*

Proof. See the results 3.3.22 and 3.3.23 in [10]. □

Our last try is deducing the extension of basic constructive theorem 2.2.4 in [10] by given consequences. First, we prove the following result.

Theorem 4.7. *Suppose that for any sets G and M there exist mapping $\lambda_1 : G \longrightarrow CW_{CR_A}, \lambda_2 : M \longrightarrow CW_{CR_{\mathcal{A}}}$ such that $\lambda_1(G)$ is p-dense and $\lambda_2(M)$ is s-dense in CW_{CR_A}. If we define $I \subseteq G \times M$ by $(g, m) \in I$ iff $\lambda_1(g) \leq \lambda_2(m)$, for all $g \in G$ and $m \in M$, then CW_{CR_A} is isomorphic to $CW_{CR_{\mathcal{G}}}$. In particular for any set of complete attribute implications CAI, we have $CAI \cong CW_{CR_{CAI}}$.*

Proof. Define $\varphi : CW_{CR_A} \longrightarrow CW_A \times CW_{I_A}$, by $\varphi(w) = (O_w, O_{I_w})$. Then by applying theorem 3.12, for product lattices, we see that φ is an isomorphism and hence the proof complete. □

Now, by mixing theorem 3.12 and 4.7, theorem 2.2.4 in [10] has been proved.

5. Conclusion and future research directions

In this paper, we have introduced the concept of the FCR as an innovative algebraic structure designed to enhance FCA through the integration of additional algebraic operations, specifically new addition and multiplication. By establishing a theoretical framework that combines the geometric interpretation of concept lattices with algebraic properties, we have demonstrated the potential of FCR to facilitate more constructive operations within FCA.

Our findings indicate that the introduction of algebraic operations not only enriches the computational capabilities of FCA but also allows for the effective manipulation of concepts and their relationships. The development of Galois lattices within this framework provides a powerful tool for data analysis, particularly in applications involving Big Data and digital communication. The ability to break complex data structures into manageable components through the use of approximations and ideals offers new avenues for exploration in data mining and knowledge discovery.

Future research can build upon the framework established in this study in several ways:

- Extension to Other Algebraic Structures: Investigating the application of additional algebraic structures, such as rings and fields, to further enhance the capabilities of FCR and explore new mathematical properties that can be utilized in FCA.
- Applications in Big Data: Conducting empirical studies to apply the FCR framework to real-world Big Data scenarios, particularly in fields such as healthcare, social media analysis, and complex network analysis, to assess its effectiveness in uncovering hidden patterns and insights.
- Algorithm Development: Developing algorithms that leverage the properties of FCR for efficient data processing and concept retrieval, focusing on optimizing computational resources while maintaining accuracy.
- Integration with Machine Learning: Exploring the integration of FCR with machine learning techniques to improve classification and prediction tasks, specifically in contexts where traditional methods face challenges due to high dimensionality or data sparsity.
- Theoretical Advancements: Further theoretical exploration of the implications of FCR in relation to existing concepts in lattice theory and order theory, potentially leading to the discovery of new relationships and properties that can inform future studies.

By addressing these areas, future research can significantly advance the understanding and applicability of FCA, ultimately leading to richer and more nuanced insights in data science and related fields.

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