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Research Paper

ON DIAMETER AND DISTANCE ENERGY OF COMPLEMENT OF REGULAR GRAPHS

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Abstract. In this paper, we investigate the diameter and distance energy of the complement of a regular graph G. We improve and extend earlier results on the diameter and distance energy of graph complements obtained by Indulal [Algebr. Struct. Appl. 4 (2017) 53-58], removing the restrictions on the relationship between the degree r and the order n of the graph. We also derive a formula for the distance energy of the complement of a regular graph in terms of its adjacency energy and eigenvalues. This formula facilitates the characterization and construction of families of distance equienergetic graphs from adjacency equienergetic ones. Furthermore, we enhance some findings related to an open problem concerning adjacency and distance equienergetic graphs.

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1. Introduction

Let G be a simple graph of order n and size m. The number of edges that are incident to the vertex v is the vertex's degree. An *r-regular* graph is defined as a graph in which every vertex has degree r. The distance between two vertices v_j and v_k is equal to the length of the shortest path between them and is denoted by d_{jk} and the *diameter* of G is the maximum distance between any two vertices of G and denoted as $\delta(G)$.

Let $A(G)$ denote the *adjacency matrix* of graph G, defined as $A(G) = [a_{jk}]$, where $a_{jk} = 1$ if v_j is adjacent to v_k and $a_{jk} = 0$ otherwise. The *distance matrix* of a connected graph G, denoted by $D(G)$, is defined as $D(G) = [d_{jk}]$, where d_{jk} denotes the distance between the given vertices v_j and v_k. The eigenvalues of the adjacency (or distance) matrix of a graph G are referred to as the *A (or D)-eigenvalues* of G.

The concept of *adjacency energy* or *A-energy* of a graph G denoted by $\mathcal{E}_A(G)$, was introduced by I. Gutman in 1978. It is defined as the absolute sum of the eigenvalues associated with the adjacency matrix of graph G. If $\mathcal{E}_A(G_1) = \mathcal{E}_A(G_2)$, then two graphs G_1 and G_2 of same order are *A-equienergetic* or *adjacency equienergetic* graphs. The *distance energy* or *D-energy* of graph G was introduced by Indulal et al. [10] is defined as absolute sum of distance eigenvalues of graph G and is denoted by $\mathcal{E}_D(G)$. If $\mathcal{E}_D(G_1) = \mathcal{E}_D(G_2)$, then two graphs G_1 and G_2 of same order are *D-equienergetic* or *distance equienergetic* graphs. Indulal [8, 9] presented an open problem regarding the characterization or construction of graph families that share equal energy properties concerning both their adjacency and distance matrices. An affirmative answer to this problem is provided in the work by Ramane et al. [15].

The *complement* $\overline{G}$ of a graph G shares the same set of vertices, but two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *line graph* $\mathcal{L}(G)$ [6] associated with a graph G is constructed by using the edge set of G as its vertex set, in which two vertices are considered adjacent if the corresponding edges in $\mathcal{L}(G)$ share a common vertex. The k-th iterated line graph of G, defined as $\mathcal{L}^k(G) = \mathcal{L}(\mathcal{L}^{k-1}(G))$; $k = 1, 2, \ldots$, where $\mathcal{L}^0(G) = G$ and $\mathcal{L}^1(G) = \mathcal{L}(G)$. Let n_G^+, n_G^- and n_G^0 represent the counts of positive, negative and zero eigenvalues of a graph G respectively. The study of distance energy and A-equienergetic graphs can be seen in [8, 9, 7, 10, 11, 18, 14, 12, 13, 15, 16, 17] and references cited therein. For undefined notation and terminology, we follow [3].

Proposition 1.1. [2] *Let G be a graph of order n with a minimum degree of at least $\frac{n-1}{2}$. Then the diameter $\delta(G) = 2$ and G is connected.*

Proposition 1.2. [13] *Let G_1 be an r_1-regular graph and G_2 be an r_2-regular graph, both having the same order n and same A-energy with no A-eigenvalues in the interval $(-1, 0)$. Then the graphs $\overline{G_1}$ and $\overline{G_2}$ are A-equienergetic if and only if $r_1 + n_{G_1}^- = r_2 + n_{G_2}^-$.*

Let n_k and r_k represent the order and size of $\mathcal{L}^k(G)$.

Proposition 1.3. [1] *Let G be an r-regular graph of order n. Then $\mathcal{L}^k(G)$ is a $(2^k r - 2^{k+1} + 2)$-regular graph of order $\frac{n}{2^k}\prod_{j=0}^{k-1}(2^j r - 2^{j+1} + 2)$.*

Theorem 1.4. [4] *Let G be an r-regular graph of order n with diameter $\delta(G) \leq 2$. If $r = \lambda_1$ and $\lambda_j; 2 \leq j \leq n$ are the A-eigenvalues of G, then the D-eigenvalues of G are $2n - r - 2$ and $-(\lambda_j + 2)$; $2 \leq j \leq n$.*

2. Results

It is interesting to characterize the particular class of graphs with diameter 2. The following provides insight into regular graphs.

Proposition 2.1. *If G is an r-regular graph of order n, then the diameter of G or $\overline{G}$ is at most 2.*

Proof. If G or $\overline{G}$ is a complete graph $K_n; n \geq 1$, then $\delta(G) = 1$ or $\delta(\overline{G}) = 1$. In case of $r \geq \frac{n-1}{2}$, by Proposition 1.1, it is clear that the diameter $\delta(G) = 2$. If $r \ngeq \frac{n-1}{2}$, that is, in case of $r < \frac{n-1}{2}$, we have $n - r - 1 > n - \frac{n-1}{2} - 1 = \frac{n-1}{2}$, which shows that the graph $\overline{G}$ satisfies the condition in Proposition 1.1. Therefore, the diameter $\delta(\overline{G}) = 2$. Hence, $\delta(G) \leq 2$ or $\delta(\overline{G}) \leq 2$. □

Remark 2.2. Theorem 2.1 of [7] states that for an r-regular graph G with $r \leq \frac{n-1}{2}$, the diameter $\delta(\overline{G}) = 2$. It is noted that this result is a direct consequence of Proposition 1.1, based on the fact that if $r \leq \frac{n-1}{2}$, then $n - r - 1 \geq \frac{n-1}{2}$.

For an r-regular graph, the following Theorem is same as Theorem 2.2 in [7], but it doesn't impose restrictions on r in terms of n. Here, we provide its proof for completeness.

Theorem 2.3. *If G is an $r(> 1)$-regular graph of order $n \geq 8$, then the diameter $\delta(\overline{\mathcal{L}^k(G)}) = 2$ for all $k \geq 1$.*

Proof. To prove $\delta(\overline{\mathcal{L}^k(G)}) = 2$ for all $k \geq 1$, it is enough to prove that $r_k \leq \frac{n_k - 1}{2}$ by Proposition 1.1. By Proposition 1.3, we have, $n_k = \frac{n}{2^k}\prod_{j=0}^{k-1}(2^j r - 2^{j+1} + 2) = \frac{n}{2^{k-1}}\prod_{j=0}^{k-2}(2^j r - 2^{j+1} + 2)\frac{1}{2}(2^{k-1}r - 2^k + 2) = n_{k-1}(2^{k-2}r - 2^{k-1} + 1)$. Therefore $n_k - 1 - 2r_k = n_{k-1}(2^{k-2}r - 2^{k-1} + 1) - 1 - 2(2^k r - 2^{k+1} + 2) = n_{k-1}(2^{k-2}r - 2^{k-1} + 1) - 1 - 8(2^{k-2}r - 2^{k-1} + 1 - \frac{1}{2}) = n_{k-1}p - 1 - 8(p - \frac{1}{2})$, where $p = 2^{k-2}r - 2^{k-1} + 1$. This implies $n_k - 1 - 2r_k = (n_{k-1} - 8)p + 3 > 0$ as $p > 0$ if $r > 1$ and $n_{k-1} \geq n \geq 8$, which completes the proof. □

Proposition 2.1 guarantees that examining a regular graph with a complement diameter $\delta(\overline{G}) \leq 2$ covers all regular graphs. In light of this, the following result is obtained.

Theorem 2.4. *Let G be an $r(>0)$-regular graph of order n with $\delta(\overline{G}) \leq 2$. If $r = \lambda_1$ and $\lambda_j; 2 \leq j \leq n$ be the A-eigenvalues of G, then*

$$\mathcal{E}_D(\overline{G}) = 2n + \mathcal{E}_A(G) - 2n_G^+ - 2 \sum_{\lambda_j \in (0,1)} (\lambda_j - 1). \tag{1}$$

Proof. For any real number y, we have

$$|y-1| = \begin{cases} |y| - 1, & \text{if } y \geq 1, \\ |y| + 1, & \text{if } y \leq 0, \\ -|y| + 1, & \text{if } 0 < y < 1 \ . \end{cases} \tag{2}$$

If $r = \lambda_1$ and $\lambda_j; 2 \leq j \leq n$ are the A-eigenvalues of G, then the A-eigenvalues of $\overline{G}$ are $n - r - 1$ and $-1 - \lambda_j; 2 \leq j \leq n$. By Theorem 1.4, $\overline{G}$ has the D-eigenvalues $n + r - 1$ and $\lambda_j - 1; 2 \leq j \leq n$. Therefore, the D-energy of $\overline{G}$ is,

$$\begin{aligned} \mathcal{E}_D(\overline{G}) &= n + r - 1 + \sum_{j=2}^{n} |\lambda_j - 1| \\ &= n + \sum_{j=1}^{n} |\lambda_j - 1| \\ &= n + \sum_{\lambda_j \leq 0} (|\lambda_j| + 1) + \sum_{\lambda_j \in (0,1)} (-|\lambda_j| + 1) + \sum_{\lambda_j \geq 1} (|\lambda_j| - 1) \ \text{ by (2)} \\ &= n + \sum_{\lambda_j \leq 0} |\lambda_j| + n_\lambda([\lambda_n, 0]) - \sum_{\lambda_j \in (0,1)} |\lambda_j| + n_\lambda((0,1)) + \sum_{\lambda_j \geq 1} |\lambda_j| - n_\lambda([1, \lambda_1]), \end{aligned}$$

where $n_\lambda(\mathbf{I})$ represents the count of eigenvalues of a graph G that fall within a given interval $\mathbf{I}$ and $n_\lambda([\lambda, p]) = 0$ if $\lambda \geq p$. Additionally, we have

$$n = n_\lambda([\lambda_n, 0]) + n_\lambda((0,1)) + n_\lambda([1, \lambda_1]) = n_\lambda([\lambda_n, 0]) + n_G^+,$$

and

$$\mathcal{E}_A(G) = \sum_{\lambda_j \leq 0} |\lambda_j| + \sum_{\lambda_j \in (0,1)} |\lambda_j| + \sum_{\lambda_j \geq 1} |\lambda_j|.$$

Using these two facts, we arrive at

$$\begin{aligned} \mathcal{E}_D(\overline{G}) &= 2n + \mathcal{E}_A(G) - 2 \sum_{\lambda_j \in (0,1)} |\lambda_j| - 2n_G^+ + 2n_\lambda((0,1)) \\ &= 2n + \mathcal{E}_A(G) - 2n_G^+ - 2 \sum_{\lambda_j \in (0,1)} (\lambda_j - 1), \end{aligned}$$

which concludes the proof. □

For $\lambda_j \in (0,1)$, it's clear that

$$\sum_{\lambda_j \in (0,1)} (\lambda_j - 1) < 0, n_G^+ + \sum_{\lambda_j \in (0,1)} (\lambda_j - 1) > 0,$$

and

$$\sum_{\lambda_j \in (0,1)} (\lambda_j - 1) = 0 \text{ if and only if } \lambda_j \notin (0,1).$$

Using these facts, we derive the following from equality (1).

Corollary 2.5. *Let G be an $r(>0)$-regular graph of order n with $\delta(\overline{G}) \leq 2$. Let $r = \lambda_1$ and $\lambda_j; 2 \leq j \leq n$ be the A-eigenvalues of G. Then*

$$2n + \mathcal{E}_A(G) - 2n_G^+ \leq \mathcal{E}_D(\overline{G}) < 2n + \mathcal{E}_A(G).$$

The left-side equality is true if and only if $\lambda_j \notin (0,1)$ for any $j \in \{1, 2, \ldots, n\}$.

Now constructing distance equienergetic graphs is an easier task with the aid of Theorem 2.4 and adjacency equienergetic graphs.

Corollary 2.6. *Let G_1 be an r_1-regular graph and G_2 be an r_2-regular graph, both having the same order n and same A-energy, along with their eigenvalues $\lambda_1(G_1) \geq \lambda_2(G_1) \geq \cdots \geq \lambda_n(G_1)$ and $\lambda_1(G_2) \geq \lambda_2(G_2) \geq \cdots \geq \lambda_n(G_2)$, respectively. Let the complement graphs $\overline{G_1}$ and $\overline{G_2}$ both have diameter $\delta(\overline{G_1}) \leq 2$ and $\delta(\overline{G_2}) \leq 2$. Then the graphs $\overline{G_1}$ and $\overline{G_2}$ are distance equienergetic if and only if*

$$n_{G_1}^+ + \sum_{\lambda_j(G_1) \in (0,1)} (\lambda_j(G_1) - 1) = n_{G_2}^+ + \sum_{\lambda_j(G_2) \in (0,1)} (\lambda_j(G_2) - 1).$$

Especially, when neither G_1 nor G_2 have eigenvalues in the interval $(0,1)$, the graphs $\overline{G_1}$ and $\overline{G_2}$ are distance equienergetic if and only if $n_{G_1}^+ = n_{G_2}^+$.

Proof. The proof can be derived directly from Theorem 2.4 by considering two equienergetic graphs of the same order and utilizing the fact that $\sum_{\lambda \in (0,1)} (\lambda - 1) = 0$ if and only if G does not has any eigenvalue λ within the interval $(0,1)$. □

The Cartesian product $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u_i, v_j) and (u_h, v_k) are adjacent if and only if either (a) u_i is adjacent to u_h in G and $v_j = v_k$, or (b) $u_i = u_h$ and v_j is adjacent to v_k in H.

Example 2.7. It is observed that for all $k \geq 1$ and $n \geq 6$, the graphs $\mathcal{L}^k(K_{n,n} \Box K_{n-1})$ and $\mathcal{L}^k(K_{n-1,n-1} \Box K_n)$ are non-isomorphic regular A-equienergetic graphs. They are integral graphs which share identical counts of positive and negative eigenvalues, as well as having the same order and same degree [17]. Also, $\delta(\overline{\mathcal{L}^k(K_{n,n} \Box K_{n-1})}) = \delta(\overline{\mathcal{L}^k(K_{n-1,n-1} \Box K_n)}) = 2$ for all

$k \geq 1$ and $n \geq 6$. Therefore, by Corollary 2.6 and Proposition 1.2, the graphs $\overline{\mathcal{L}^k(K_{n,n}\square K_{n-1})}$ and $\overline{\mathcal{L}^k(K_{n-1,n-1}\square K_n)}$ are distance equienergetic as well as adjacency equienergetic for all $n \geq 6$ and $k \geq 1$.

Remark 2.8. The Theorem 2.3, Theorem 2.4 and Corollary 2.6 offer more generalized results than those in the paper [7]. Also, the findings concerning the open problem on adjacency and distance equienergetic graphs, as outlined in Propositions 3.15, 3.26 and 3.32 in [15], can be extended without limitations on the value of r in relation to n if G is an $r(>1)$-regular graph with order $n \geq 8$.

3. Conclusion

In this work, we examined the diameter and distance energy of the complement of a regular graph, improving and generalizing earlier results on the diameter and distance energy of complements of iterated line graphs of regular graphs. We derived a formula expressing the distance energy of the complement in terms of its adjacency energy and eigenvalues, providing a framework for constructing distance equienergetic graphs from adjacency equienergetic ones. This study can be extended to non-regular graphs, offering potential for further generalizations.

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