Algebraic Structures and Their Applications

Yazd University

Algebraic Structures and Their Applications Vol. 13 No. 1 (2026) pp 35-47.

Research Paper

$\mathcal{S}$-MINIMAXNESS AND LOCAL-GLOBAL PRINCIPLE OF LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M a finitely generated R-module and $\mathfrak{a}$, $\mathfrak{b}$ two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. By using the concept of $\mathcal{S}$-minimax modules w ve define $\mathcal{S}^{\mathfrak{b}}$-minimaxness dimension $\mathcal{S}_{\mathfrak{a}}^{\mathfrak{b}}(M)$ of M relative to $\mathfrak{a}$ by $\mathcal{S}_{\mathfrak{a}}^{\mathfrak{b}}(M) := \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}_{\mathfrak{a}}^i(M)$ is not $\mathcal{S} -$ minimax for all $t \in \mathbb{N}\}$. Also, we say that the local global principle for the $\mathcal{S}$-minimaxness of local cohomology modules holds at level r if, for every choice of ideals $\mathfrak{a}$, $\mathfrak{b}$ of R with $\mathfrak{b} \subseteq \mathfrak{a}$ and for every choice of finitely generated R-module M, it is the case that $\mathcal{S}_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Leftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}$. In this paper, we investigate the local-global principle concerning the $\mathcal{S}$-minimaxness of local cohomology modules. Among other things, we will show that this principle holds at level 1 over an arbitrary commutative Noetherian ring R and at all levels whenever $\dim R \leq 2$. Then by using the obtained results for some specific Serre classes of R-modules we get some main results concerning the local global principle of local cohomology modules.

DOI: 10.22034/as.2025.20376.1660

MSC(2010): Primary: 13D45.

Keywords: Local cohomology, Serre subcategory, $\mathcal{S}$-minimax modules.

Received: 26 July 2023, Accepted: 28 July 2025.

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1. Introduction

Throughout, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of R and M is a non-zero R-module. For an R-module M, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For basic facts about commutative algebra see [11]; for local cohomology refer to [5].

Let $\mathcal{S}$ be a class of R-modules. Recall that $\mathcal{S}$ is a Serre subcategory of the category of R-modules, when it is closed under taking submodules, quotients and extensions. $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$ if for every $\mathfrak{a}$-torsion R-module M such that $0 :_M \mathfrak{a}$ is in $\mathcal{S}$, then M is in $\mathcal{S}$ (see [1, Definition 2.1]. It is easy to see that, if $\mathcal{S}$ is closed under taking injective hulls, then $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$.

In [14] we introduced the concept of $\mathcal{S}$-minimax modules and we investigated $\mathcal{S}$-minimaxness of local cohomolgy modules. In this note, we define $\mathcal{S}^{\mathfrak{b}}$-minimaxness dimension $\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M)$ of M relative to $\mathfrak{a}$ by

$$\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t\,\mathrm{H}^i_{\mathfrak{a}}(M) \text{ is not } \mathcal{S} - \text{minimax for all } t \in \mathbb{N}\}.$$

where $\mathfrak{b}$ is a second ideal of R. Also, we say that the local-global principle for the $\mathcal{S}$-minimaxness of local cohomology modules holds at level r if, for every choice of ideals $\mathfrak{a}$, $\mathfrak{b}$ of R with $\mathfrak{b} \subseteq \mathfrak{a}$ and for every choice of finitely generated R-module M, it is the case that

$$\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \mathrm{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}.$$

In this paper, we investigate the local-global principle concerning the $\mathcal{S}$-minimaxness of local cohomology modules.

Recall that the $\mathfrak{b}$-finiteness dimension of M relative to $\mathfrak{a}$ is defined by

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t\,\mathrm{H}^i_{\mathfrak{a}}(M) \neq 0 \text{ for all } t \in \mathbb{N}\}.$$

Brodmann et al. in [4] defined and studied the concept of the local-global principle for annihilation of local cohomology modules at level $r \in \mathbb{N}$ for the ideals of $\mathfrak{a}$ and $\mathfrak{b}$ of R. We say that the local-global principle for the annihilation of local cohomology modules holds at level r if, for every choice of ideals $\mathfrak{a}$, $\mathfrak{b}$ of R with $\mathfrak{b} \subseteq \mathfrak{a}$ and for every choice of finitely generated R-module M, it is the case that

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \mathrm{Spec}(R).$$

It is shown in [4] that the local -global principle for the annihilation of local cohomology modules holds at levels 1,2 over an arbitrary commutative Noetherian ring and at all levels whenever $\dim R \leqslant 4$.

Also the authors in [8] defined the $\mathfrak{b}$-minimaxness dimension of M relative to $\mathfrak{a}$ by

$$\mu_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}_{\mathfrak{a}}^i(M) \text{ is not minimax for all } t \in \mathbb{N}\}.$$

They defined that the local-global principle for minimaxness of local cohomology modules holds at level $r \in \mathbb{N}$ for the ideals of $\mathfrak{a}$ and $\mathfrak{b}$ of R if, for every choice of ideals $\mathfrak{a}$, $\mathfrak{b}$ of R with $\mathfrak{b} \subseteq \mathfrak{a}$ and for every choice of finitely generated R-module M, it is the case that

$$\mu_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Leftrightarrow \mu_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

It is shown in [8] that the local -global principle for the minimaxness of local cohomology modules holds at levels 1,2 over an arbitrary commutative Noetherian ring and at all levels whenever $\dim R \leqslant 3$. Recently Naghipour et al. in [13] introduced the notation of $h_{\mathfrak{a}}^{\mathfrak{b}}(M)^n$ by

$$h_{\mathfrak{a}}^{\mathfrak{b}}(M)^n = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}_{\mathfrak{a}}^i(M) \text{ is not in dimension} < n \text{ for all } t \in \mathbb{N}\},$$

and they showed that for $r = 1, 2$ and also for all $r \in \mathbb{N}$ over an arbitrary commutative Noetherian ring R whenever $\dim R \leqslant 3$ we have

$$h_{\mathfrak{a}}^{\mathfrak{b}}(M)^n > r \Leftrightarrow h_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Here, by using the concept of $\mathcal{S}$-minimaxness we generalize the above local-global principles for some Serre calsses of R-modules. In fact, we show that under certain conditions the local-global principle for the $\mathcal{S}$-minimaxness of local cohomology modules holds over any commutative Noetherian ring R at a level $r \in \mathbb{N}$. Among other things, we prove that if r is a positive integer such that $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}_{\mathfrak{a}}^i(M))$ is $\mathcal{S}$-minimax for all $i \leq r$ then

$$\mathcal{S}_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Leftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}.$$

and we show that the local-global principle for the $\mathcal{S}$-minimaxness of local cohomology modules holds at level 1 over an arbitrary commutative Noetherian ring R and at all levels whenever $\dim R \leq 2$. By using the obtained results for some certain classes of R-modules we get some results about the local-global principle for local cohomology modules in special cases.

2. Main results

Recall that an R-module L is called $\mathfrak{a}$-cofinite if $\operatorname{Supp}_R(L) \subseteq \operatorname{V}(\mathfrak{a})$ and $\operatorname{Ext}_R^j(R/\mathfrak{a}, L)$ is finitely generated for all $j \geq 0$. The R-module M is said to be a minimax module if there is a finitely generated submodule N of M, such that M/N is Artinian. The R-module M is said to be an FSF module if there is a finitely generated submodule N of M such that the support of the quotient module M/N is finite (see [15]). It is clear that, if M is FSF, then $\dim \operatorname{Supp} M/N \leq 1$. The authors in [2] introduced the class of *in dimension*$<$ n modules. Let n be a non-negative integer. An R-module M is said to be *in dimension*$<$ n, if there is a finitely generated submodule N of M such that $\dim \operatorname{Supp} M/N < n$.

In [14] we introduced the following definition of $\mathcal{S}$-minimax modules which is a generalization of the above definitions to Serre subcategories of the category of R-modules.

Definition 2.1. [14, Definition 2.1] Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. An R-module M is said to be $\mathcal{S}$-minimax if there exists a finitely generated submodule N of M such that M/N is in $\mathcal{S}$.

Example 2.2. The following classes of modules are Serre subcategories of the category of R-modules and satisfy the condition $C_{\mathfrak{a}}$ for all ideals $\mathfrak{a}$ of R:

i) ${}_0\mathcal{S}$:= The class of zero modules;

ii) ${}_A\mathcal{S}$:= The class of artinian R-modules;

iii) ${}_F\mathcal{S}$:= The class of R-modules with finite support;

iv) ${}_n\mathcal{S}$:= The class of all R-modules M with $\dim \operatorname{Supp} M < n$, where n is a non-negative integer.

Thus, the class of ${}_0\mathcal{S}$-minimax modules is equal to the class of finitely generated modules, the class of ${}_A\mathcal{S}$-minimax modules is equal to the class of minimax modules, the class of ${}_F\mathcal{S}$-minimax modules is equal to the class of FSF modules and the class of ${}_n\mathcal{S}$-minimax modules is equal to the class of *in dimension*$< n$ modules.

In the following, we introduce the notation of $\mathcal{S}^{\mathfrak{b}}-$minimaxness dimension $\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M)$ which is a generalization of $\mathfrak{b}$-finiteness dimension $f^{\mathfrak{b}}_{\mathfrak{a}}(M)$ [5, Definition 9.1.5] and $\mathfrak{b}$-minimaxness dimension $\mu^{\mathfrak{b}}_{\mathfrak{a}}(M)$ [8] of M relative to $\mathfrak{a}$.

Definition 2.3. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, M a finitely generated R-module and $\mathfrak{a}$, $\mathfrak{b}$ two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. We define the $\mathcal{S}^{\mathfrak{b}}$-minimaxness dimension $\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M)$ of M relative to $\mathfrak{a}$ by

$$\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not } \mathcal{S} - \text{minimax for all } t \in \mathbb{N}\}.$$

Example 2.4. Let R be a Noetherian ring, M be a finitely generated R-module and $\mathfrak{a}$, $\mathfrak{b}$ two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. For Serre subcategories given in 2.2 we have:

i) ${}_0\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M)$ is not finitely generated for all $t \in \mathbb{N}\}$;

ii) ${}_A\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M)$ is not minimax for all $t \in \mathbb{N}\}$;

iii) ${}_F\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M)$ is not FSF for all $t \in \mathbb{N}\}$;

iv) ${}_n\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M)$ is not *in dimension*$< n$ for all $t \in \mathbb{N}\}$.

Definition 2.5. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, r be a positive integer, We say that Faltings' local global principle for the $\mathcal{S}$-minimaxness of local cohomology modules holds at level r if, for every choice of ideals $\mathfrak{a}$, $\mathfrak{b}$ of R with $\mathfrak{b} \subseteq \mathfrak{a}$ and for every choice of finitely generated R-module M, it is the case that

$$\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}.$$

Proposition 2.6. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$, M be a finitely generated R-module and i a non-negative integer. Then the following conditions are equivalent:*

i) There exists an integer t such that $\mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M)$ is $\mathcal{S}$-minimax;

ii) There exists an integer t such that $\mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M)$ is in $\mathcal{S}$.

Proof. i) $\Rightarrow$ ii): By [14, Theorem 2.8].

ii) $\Rightarrow$ i): It is clear. □

By using the following notations and Proposition 2.6 we have the next result.

$f^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \neq 0 \text{ for all } t \in \mathbb{N}\}$,

$\mu^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not minimax for all } t \in \mathbb{N}\}$,

$w^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not FSF for all } t \in \mathbb{N}\}$,

$h^{\mathfrak{b}}_{\mathfrak{a}}(M)^n = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \text{ is not in dimension} < n \text{ for all } t \in \mathbb{N}\}$.

Corollary 2.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$, M be a finitely generated R-module and n a non-negative integer. Then for Serre subcategories given in 2.2 we have:*

i)

$$\begin{aligned} {}_0\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) &= f^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \neq 0 \textit{ for all } t \in \mathbb{N}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \textit{ is not finitely generated for all } t \in \mathbb{N}\}; \end{aligned}$$

ii)

$$\begin{aligned} {}_A\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) &= \mu^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \textit{ is not artinian for all } t \in \mathbb{N}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \textit{ is not minimax for all } t \in \mathbb{N}\}; \end{aligned}$$

iii)

$$\begin{aligned} {}_F\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) &= w^{\mathfrak{b}}_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \textit{ is not finite support for all } t \in \mathbb{N}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \textit{ is not FSF for all } t \in \mathbb{N}\}; \end{aligned}$$

iv)

$$\begin{aligned} {}_n\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) &= h^{\mathfrak{b}}_{\mathfrak{a}}(M)^n = \inf\{i \in \mathbb{N}_0 : \textit{dim}\, \textit{Supp}_R(\mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M)) \geq n \textit{ for all } t \in \mathbb{N}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^t \operatorname{H}^i_{\mathfrak{a}}(M) \textit{ is not in dimension} < n \textit{ for all } t \in \mathbb{N}\}. \end{aligned}$$

Proposition 2.8. *Let R be a Noetherian ring and M be a finitely generated R-module. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and let r be a positive integer. Then we have:*

i) If $f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$, then $f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $R\,/\mathfrak{p} \notin {}_0\mathcal{S}$.

ii) If $\mu_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$*, then* $f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$ *such that* $R/\mathfrak{p} \notin {}_{A}\mathcal{S}$.

iii) If $w_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$*, then* $f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$ *such that* $R/\mathfrak{p} \notin {}_{F}\mathcal{S}$*; whenever* R *is semilocal.*

iv) If $h_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^{n} > r$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$*, then* $f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$ *such that* $R/\mathfrak{p} \notin {}_{n}\mathcal{S}$.

Proof. i) It is clear.

ii) Assume that $\mu_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Let $i \leqslant r$ be an integer and $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $R/\mathfrak{p} \notin {}_{A}\mathcal{S}$. Thus there exists $\mathfrak{m} \in \operatorname{Max}(R)$ such that $\mathfrak{p} \subsetneq \mathfrak{m}$. By assumption there exists an integer t such that $(\mathfrak{b}^{t}\operatorname{H}_{\mathfrak{a}}^{i}(M))_{\mathfrak{m}}$ is minimax. Recall that a module L which is minimax has the property that the localization $L_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module for each non-maximal prime ideal $\mathfrak{p}$. Hence

$$((\mathfrak{b}R_{\mathfrak{m}})^{t}\operatorname{H}_{\mathfrak{a}R_{\mathfrak{m}}}^{i}(M_{\mathfrak{m}}))_{\mathfrak{p}R_{\mathfrak{m}}} \simeq (\mathfrak{b}^{t}\operatorname{H}_{\mathfrak{a}}^{i}(M))_{\mathfrak{p}},$$

is a finitely generated $R_{\mathfrak{p}}$-module. It follows that $f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $R/\mathfrak{p} \notin {}_{A}\mathcal{S}$.

iii) Assume that $w_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Let $i \leqslant r$ be an integer and $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $R/\mathfrak{p} \notin {}_{F}\mathcal{S}$. Since R is semilocal, there exists $\mathfrak{m} \in \operatorname{Max}(R)$ such that $R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}} \notin {}_{F}\mathcal{S}$. By assumption there exists an integer t such that $R_{\mathfrak{m}}$- module $(\mathfrak{b}^{t}\operatorname{H}_{\mathfrak{a}}^{i}(M))_{\mathfrak{m}}$ is FSF. Hence there exists a finitely generated submodule N of $(\mathfrak{b}^{t}\operatorname{H}_{\mathfrak{a}}^{i}(M))_{\mathfrak{m}}$ such that $(\mathfrak{b}^{t}\operatorname{H}_{\mathfrak{a}}^{i}(M))_{\mathfrak{m}}/N$ is finite support. Since $R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}} \notin {}_{F}\mathcal{S}$ we have

$$((\mathfrak{b}R_{\mathfrak{m}})^{t}\operatorname{H}_{\mathfrak{a}R_{\mathfrak{m}}}^{i}(M_{\mathfrak{m}}))/N)_{\mathfrak{p}R_{\mathfrak{m}}} = 0,$$

and so

$$((\mathfrak{b}R_{\mathfrak{m}})^{t}\operatorname{H}_{\mathfrak{a}R_{\mathfrak{m}}}^{i}(M_{\mathfrak{m}}))_{\mathfrak{p}R_{\mathfrak{m}}} \simeq (\mathfrak{b}^{t}\operatorname{H}_{\mathfrak{a}}^{i}(M))_{\mathfrak{p}},$$

is a finitely generated $R_{\mathfrak{p}}$-module. Thus it follows that $f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $R/\mathfrak{p} \notin {}_{F}\mathcal{S}$.

iv) Assume that $h_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^{n} > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Let $i \leqslant r$ be an integer and $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $R/\mathfrak{p} \notin {}_{n}\mathcal{S}$. Thus there exists $\mathfrak{m} \in \operatorname{Max}(R)$ such that $\dim R/\mathfrak{p} = \dim R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}} \geqslant n$. By assumption there exists an integer t and a finitely generated submodule N of $(\mathfrak{b}^{t}\operatorname{H}_{\mathfrak{a}}^{i}(M))_{\mathfrak{m}}$ such that $\dim \operatorname{Supp}(\mathfrak{b}^{t}\operatorname{H}_{\mathfrak{a}}^{i}(M))_{\mathfrak{m}}/N) < n$. Since $\dim R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}} \geqslant n$ we have

$$((\mathfrak{b}R_{\mathfrak{m}})^{t}\operatorname{H}_{\mathfrak{a}R_{\mathfrak{m}}}^{i}(M_{\mathfrak{m}}))/N)_{\mathfrak{p}R_{\mathfrak{m}}} = 0,$$

and so

$$((\mathfrak{b}R_{\mathfrak{m}})^{t}\operatorname{H}_{\mathfrak{a}R_{\mathfrak{m}}}^{i}(M_{\mathfrak{m}}))_{\mathfrak{p}R_{\mathfrak{m}}} \simeq (\mathfrak{b}^{t}\operatorname{H}_{\mathfrak{a}}^{i}(M))_{\mathfrak{p}},$$

is a finitely generated $R_{\mathfrak{p}}$-module. Thus we conclude that $f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $R/\mathfrak{p} \notin {}_n\mathcal{S}$. □

Corollary 2.9. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module. Let r be a positive integer such that*

$$\mathcal{S}_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Leftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec}(R) | R\ /\mathfrak{p} \notin \mathcal{S}\}.$$

Then

i) If $\mathcal{S} = {}_0\mathcal{S}$, then

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Longleftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R);$$

ii) If $\mathcal{S} = {}_A\mathcal{S}$, then

$$\mu_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Longleftrightarrow \mu_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R);$$

iii) If $\mathcal{S} = {}_F\mathcal{S}$, then

$$w_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Longleftrightarrow w_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R);$$

whenever R is semilocal.

iv) If $\mathcal{S} = {}_n\mathcal{S}$, then

$$h_{\mathfrak{a}}^{\mathfrak{b}}(M)^n > r \Longleftrightarrow h_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Proof. The result follows by assumption and Corollary 2.7 and Proposition 2.8. □

Theorem 2.10. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module. Let r be a positive integer. Then*

$$\mathcal{S}_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Rightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}.$$

Proof. Let i be an arbitrary non-negative integer such that $i \leq r$. By assumption there exists an integer t such that $\mathfrak{b}^t \operatorname{H}_{\mathfrak{a}}^i(M)$ is $\mathcal{S}$-minimax. Thus there exists a finitely generated submodule N of $\mathfrak{b}^t \operatorname{H}_{\mathfrak{a}}^i(M)$ such that $\mathfrak{b}^t \operatorname{H}_{\mathfrak{a}}^i(M)/N$ is in $\mathcal{S}$. Let $L := \mathfrak{b}^t \operatorname{H}_{\mathfrak{a}}^i(M)/N$. If $\mathfrak{p} \in \operatorname{Spec}(R)$ and $R/\mathfrak{p} \notin \mathcal{S}$, then since $L \in \mathcal{S}$ we have $L_{\mathfrak{p}} = 0$ and so $(\mathfrak{b}^t \operatorname{H}_{\mathfrak{a}}^i(M))_{\mathfrak{p}} = N_{\mathfrak{p}}$ is finitely generated. Thus $(\mathfrak{b}R_{\mathfrak{p}})^t \operatorname{H}_{\mathfrak{a}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$ is a finitely generated $R_{\mathfrak{p}}$-module and it follows that $f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$. □

Theorem 2.11. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module. Let r be a positive integer such that $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^i_{\mathfrak{a}}(M))$ is $\mathcal{S}$-minimax for all $i \leq r$. Then*

$$\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}.$$

Proof. $\Rightarrow$): By Theorem 2.10.

$\Leftarrow$): Assume that $j \leq r$ be an integer. It is enough to show that there exists an integer v such that $\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)$ is in $\mathcal{S}$. At first, we prove that there exists an integer v such that $\operatorname{Supp}_R(\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M) \mid R/\mathfrak{p} \in \mathcal{S}\}$.

Let $\mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M)$ and $R/\mathfrak{p} \notin \mathcal{S}$. By assumption there exists an integer t such that $(\mathfrak{b}^t \operatorname{H}^j_{\mathfrak{a}}(M))_{\mathfrak{p}}$ is finitely generated $R_{\mathfrak{p}}$-module. But $(\mathfrak{b}^t \operatorname{H}^j_{\mathfrak{a}}(M))_{\mathfrak{p}}$ is $\mathfrak{a}R_{\mathfrak{p}}$-torsion and so there is an integer $v \geqslant 1$ such that $(\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M))_{\mathfrak{p}} = 0$. On the other hand, by assumption $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^j_{\mathfrak{a}}(M))$ is $\mathcal{S}$-minimax, and so Lemma [14, Lemma 2.6] implies that the set $\{\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{H}^j_{\mathfrak{a}}(M)) | R/\mathfrak{p} \notin \mathcal{S}\}$ is finite. Let $\{\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{H}^j_{\mathfrak{a}}(M)) | R/\mathfrak{p} \notin \mathcal{S}\} = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$. By the hypothesis there is an integer $v_i \geqslant 1$ such that $(\mathfrak{b}^{v_i} \operatorname{H}^j_{\mathfrak{a}}(M))_{\mathfrak{p}_i} = 0$ for all $1 \leq i \leq k$. Let $v := \operatorname{Max}\{v_1, ..., v_k\}$. Then $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\} \cap \operatorname{Supp}_R(\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)) = \emptyset$. We claim that $\operatorname{Supp}_R(\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) \mid R/\mathfrak{p} \in \mathcal{S}\}$. If there exists a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(\mathfrak{a}^t \operatorname{H}^j_{\mathfrak{a}}(M))$ such that $R/\mathfrak{p} \notin \mathcal{S}$, then there exists $\mathfrak{q} \in \operatorname{Ass}_R(\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M))$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Since $R/\mathfrak{p} \simeq (R/\mathfrak{q})/(\mathfrak{p}/\mathfrak{q})$ and $R/\mathfrak{p} \notin \mathcal{S}$ we have $R/\mathfrak{q} \notin \mathcal{S}$. But $\mathfrak{q} \in \operatorname{Supp}_R(\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M))$ and so $\mathfrak{q} \in \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\} \cap \operatorname{Supp}_R(\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)) = \emptyset$ which is a contradiction. Thus $\operatorname{Supp}_R(\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M) \mid R/\mathfrak{p} \in \mathcal{S}\}$. But $\operatorname{Supp}_R(\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)) \subseteq \operatorname{Supp}_R(\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M))$ and thus

$$\operatorname{Supp}_R(\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M) \mid R/\mathfrak{p} \in \mathcal{S}\}.$$

On the other hand, since $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^j_{\mathfrak{a}}(M))$ is $\mathcal{S}$-minimax $\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M))$ is $\mathcal{S}$-minimax by [14, Lemma 2.3]. Thus by [14, Lemma 2.7] $\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)) \in S$. Since by the hypothesis, $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$ we have $\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)$ is in $\mathcal{S}$. Therefore $\mathfrak{b}^v \operatorname{H}^j_{\mathfrak{a}}(M)$ is in $\mathcal{S}$, as required. □

Theorem 2.12. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module. Let r be a positive integer such that $\operatorname{H}^i_{\mathfrak{a}}(M)$ is $\mathcal{S}$-minimax for all $0 \leq i < r$. Then*

$$\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Spec}(R) | R\ /\mathfrak{p} \notin \mathcal{S}\}.$$

Proof. It follows by [14, Corollary 2.10] and Theorem 2.11. □

Corollary 2.13. *The local-global principle for $\mathcal{S}$-minimaxness of local cohomology modules holds at level 1.*

Proof. For any finitely generated R-module M, $\mathrm{H}^0_{\mathfrak{a}}(M)$ is finitely generated and so is $\mathcal{S}$-minimax. Thus assertion follows by Theorem 2.12. □

Corollary 2.14. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module. Let r be a positive integer.*

i) If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is finitely generated for all $0 \leq i < r$, then

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Longleftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \mathrm{Spec}(R);$$

ii) If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is minimax for all $0 \leq i < r$, then

$$\mu^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Longleftrightarrow \mu^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \mathrm{Spec}(R);$$

iii) If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is FSF for all $0 \leq i < r$, then

$$w^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Longleftrightarrow w^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \mathrm{Spec}(R);$$

whenever R is semilocal.

iv) If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is in-dimension$< n$ for all $0 \leq i < r$, then

$$h^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > r \Longleftrightarrow h^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r \text{ for all } \mathfrak{p} \in \mathrm{Spec}(R).$$

Proof. It follows by Theorem 2.12 and Corollary 2.9. □

Theorem 2.15. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module and let r be a positive integer such that $\mathrm{H}^0_{\mathfrak{a}}(M), \cdots, \mathrm{H}^{r-1}_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite. Then*

$$\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \mathrm{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}.$$

Proof. Since $\mathrm{H}^0_{\mathfrak{a}}(M), \cdots, \mathrm{H}^{r-1}_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite, by [7, Theorem 2.1] we conclude that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ is finitely generated and so is $\mathcal{S}$-minimax for all $0 \leq i \leq r$. Now, the result follows by Theorem 2.11. □

Corollary 2.16. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module. Let r be a positive integer such that $\mathrm{H}^0_{\mathfrak{a}}(M), \cdots, \mathrm{H}^{r-1}_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite. Then*

i) $f^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Longleftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$;

ii) $\mu^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Longleftrightarrow \mu^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$;

iii) $w^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Longleftrightarrow w^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$; *whenever R is semilocal.*

iv) $h^{\mathfrak{b}}_{\mathfrak{a}}(M)^n > r \Longleftrightarrow h^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. The result follows by Theorem 2.15 and Corollary 2.9. □

Corollary 2.17. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let R be a Noetherian ring with $\dim R \leq 2$. Then the local global-principle for $\mathcal{S}$-minimaxness of local cohomology modules holds at all levels $r \in \mathbb{N}$.*

Proof. The result follows by [12, Theorem 7.10] and Theorem 2.15. □

Corollary 2.18. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module such that one of the following conditions is satisfied:*

i) $\dim M \leq 2$;

ii) $\dim M/\mathfrak{a}M \leq 1$;

iii) $\mathfrak{a}$ *is principal.*

Then for any integer r,

$$\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) > r \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \mathrm{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}.$$

Proof. By [6, Corollary 5.2], [3, Theorem 1.3] and [10, Theorem 1], in each of the above conditions the R-modules $\mathrm{H}^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite for all integers i. Thus the result follows by Theorem 2.15. □

Corollary 2.19. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module such that $M \neq \mathfrak{a}M$. Then*

$$\mathcal{S}^{\mathfrak{b}}_{\mathfrak{a}}(M) > \mathrm{grade}_M\, \mathfrak{a} \Leftrightarrow f^{\mathfrak{b}R_{\mathfrak{p}}}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > \mathrm{grade}_M\, \mathfrak{a} \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \mathrm{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}.$$

Proof. Since $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i < \mathrm{grade}_M\, \mathfrak{a}$, the result follows by Theorem 2.15. □

Corollary 2.20. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module such that one of the following conditions is satisfied:*

i) $\dim M \leq 2$;

ii) $\dim M/\mathfrak{a}M \leq 1$;

iii) $\mathfrak{a}$ is principal.

Then for any integer r,

i) $f_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Longleftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$;

ii) $\mu_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Longleftrightarrow \mu_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$;

iii) $w_{\mathfrak{a}}^{\mathfrak{b}}(M) > r \Longleftrightarrow w_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > r$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$; whenever R is semilocal.

iv) $h_{\mathfrak{a}}^{\mathfrak{b}}(M)^n > r \Longleftrightarrow h_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}})^n > r$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. The result follows by Corollary 2.18 and Corollary 2.9. □

In the following, we define the concept of $\mathfrak{b}$-closed Serre classes and then we obtain a main result concerning the local-global principle for $\mathcal{S}$-minimaxness of local cohomology modules under the additional assumption that $\mathcal{S}$ is an $\mathfrak{b}$-closed Serre subcategory of the category of R-modules.

Definition 2.21. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules and $\mathfrak{b}$ be an ideal of R. We say that $\mathcal{S}$ is $\mathfrak{b}$-closed, if $L \to M \to N$ is an exact sequence of R-homomorphisms and R-modules such that for two non-negative integers s, t we have $\mathfrak{b}^s L \in \mathcal{S}$ and $\mathfrak{b}^t N \in \mathcal{S}$ then there exists a non-negative integer l such that $\mathfrak{b}^l M \in \mathcal{S}$.

Example 2.22. By [5, Lemma 9.1.1], ${}_0\mathcal{S}$, the class of zero modules and by [9, Lemma 2.9], ${}_n\mathcal{S}$, the class of all R-modules M with $\dim \mathrm{Supp}\, M < n$, where n is a non-negative integer are $\mathfrak{b}$-closed Serre subcategory of the category of R-modules for any ideal $\mathfrak{b}$ of R.

Theorem 2.23. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and M be a finitely generated R-module. Let $\mathcal{S}$ be an $\mathfrak{b}$-closed Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Then the local-global principle for $\mathcal{S}$-minimaxness of local cohomology modules holds at level 2.*

Proof. We must show that

$$\mathcal{S}_{\mathfrak{a}}^{\mathfrak{b}}(M) > 2 \Leftrightarrow f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > 2 \text{ for all } \mathfrak{p} \in \{\mathfrak{p} \in \mathrm{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}.$$

By using Theorem 2.10, it is enough for us to show that, if $f_{\mathfrak{a}R_{\mathfrak{p}}}^{\mathfrak{b}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > 2$ for all $\mathfrak{p} \in \{\mathfrak{p} \in \mathrm{Spec}(R) | R/\mathfrak{p} \notin \mathcal{S}\}$, then $\mathcal{S}_{\mathfrak{a}}^{\mathfrak{b}}(M) > 2$. In view of Corollary 2.13, we need to show that there exists an integer t such that $\mathfrak{b}^t\, \mathrm{H}_{\mathfrak{a}}^2(M)$ is in $\mathcal{S}$.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $R\ /\mathfrak{p} \notin \mathcal{S}$. By assumption there exists an integer s such that $(\mathfrak{b}^s \operatorname{H}^i_{\mathfrak{a}}(M))_{\mathfrak{p}} = 0$ for all $0 \leq i \leq 2$.

Let $\bar{M} = M/\Gamma_{\mathfrak{b}}(M)$. Now from the short exact sequence

$$0 \to \Gamma_{\mathfrak{b}}(M) \to M \to \bar{M} \to 0,$$

we have the following exact sequence

$$\cdots \to (\operatorname{H}^1_{\mathfrak{a}}(M))_{\mathfrak{p}} \to (\operatorname{H}^1_{\mathfrak{a}}(\bar{M}))_{\mathfrak{p}} \to (\operatorname{H}^2_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)))_{\mathfrak{p}} \to (\operatorname{H}^2_{\mathfrak{a}}(M))_{\mathfrak{p}} \to (\operatorname{H}^2_{\mathfrak{a}}(\bar{M}))_{\mathfrak{p}} \to \cdots .$$

But, there exists an integer k such that $\mathfrak{b}^k \operatorname{H}^i_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) = 0$ for all $i \geq 0$. Thus the above long exact sequence and [5, Lemma 9.1.1] implies that there exist integeres v and u such that $(\mathfrak{b}R_{\mathfrak{p}})^v(\operatorname{H}^1_{\mathfrak{a}}(\bar{M}))_{\mathfrak{p}} = 0$ and $(\mathfrak{b}R_{\mathfrak{p}})^u(\operatorname{H}^2_{\mathfrak{a}}(\bar{M}))_{\mathfrak{p}} = 0$. On the other hand, by [5, Lemma 2.1.1 (ii)] $\mathfrak{b}$ contains an element r which is a non-zerodivisor on $\bar{M}$. Thus the short exact sequence

$$0 \to \bar{M}_{\mathfrak{p}} \xrightarrow{r^v} \bar{M}_{\mathfrak{p}} \to \bar{M}_{\mathfrak{p}}/v\bar{M}_{\mathfrak{p}} \to 0,$$

induces the exact sequence $\operatorname{H}^0_{\mathfrak{a}R_{\mathfrak{p}}}(\bar{M}_{\mathfrak{p}}/r^v\bar{M}_{\mathfrak{p}}) \to \operatorname{H}^1_{\mathfrak{a}R_{\mathfrak{p}}}(\bar{M}_{\mathfrak{p}}) \to 0$ and so, it follows that $\operatorname{H}^1_{\mathfrak{a}R_{\mathfrak{p}}}(\bar{M}_{\mathfrak{p}})$ is a finitely generated $R_{\mathfrak{p}}$ module. Since $\operatorname{H}^0_{\mathfrak{a}R_{\mathfrak{p}}}(\bar{M}_{\mathfrak{p}})$ is also finitely generated $R_{\mathfrak{p}}$ module , [14, Theorem 2.12] implies that $\operatorname{H}^i_{\mathfrak{a}}(\bar{M})$ is $\mathcal{S}$-minimax R module for all $0 \leq i < 2$. Thus by [14, Theorem 2.9], $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^2_{\mathfrak{a}}(\bar{M}))$ is $\mathcal{S}$-minimax R module. Now, Theorem 2.11 implies that there exists an integer s such that $\mathfrak{b}^s \operatorname{H}^2_{\mathfrak{a}}(\bar{M})$ is in $\mathcal{S}$. Since there exists an integer k such that $\mathfrak{b}^k \operatorname{H}^2_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) = 0$ and by assumption $\mathcal{S}$ is $\mathfrak{b}$-closed, the exact sequence

$$\operatorname{H}^2_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) \to \operatorname{H}^2_{\mathfrak{a}}(M) \to \operatorname{H}^2_{\mathfrak{a}}(\bar{M}),$$

implies that there exists an integer t such that $\mathfrak{b}^t \operatorname{H}^2_{\mathfrak{a}}(M)$ is in $\mathcal{S}$, as required. □

Corollary 2.24. *i) The local-global principle for ${}_0\mathcal{S}$-minimaxness of local cohomology modules holds at level 2.*

ii) The local-global principle for ${}_n\mathcal{S}$-minimaxness of local cohomology modules holds at level 2.

Proof. It follows by Example 2.22 and Theorem 2.23. □

3. Acknowledgments

The author would like to thank the referee for his/her useful comments.

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