Algebraic Structures and Their Applications Vol. 13 No. 1 (2026) pp 17-34.

Research Paper

THE SPACE OF PRIME $\pi-$FILTERS OF ALMOST DISTRIBUTIVE LATTICES

RAFI NOORBHASHA* AND MUKKAMALA SAMBASIVA RAO

Abstract. The concepts have been presented in Almost Distributive Lattices (ADLs), namely, regular filters and π-filters. A set of conditions has been identified that are equivalent to becoming an $\mathcal{D}$-filter into a regular filter. Moreover, it has been shown that for any $\mathcal{D}$-filter, there is a homomorphism with a dense kernel, which is itself a regular filter. The characterization of π-filters in relation to congruences and regular filters has been established. Additionally, equivalent conditions have been derived to show that the space containing all prime filters forms a Hausdorff space.

1. Introduction

An Almost Distributive Lattice (ADL) was originated by Swamy and Rao in [8], as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an

DOI: 10.22034/as.2025.22800.1774

MSC(2010): Primary: 06D99, 06D15.

Keywords: Almost distributive lattices, Hausdorff space, Minimal prime $\mathcal{D}$-filter, π-filter, Regular filter.

Received: 14 February 2025, Accepted: 06 June 2025.

*Corresponding author

ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set $\mathcal{PI}(\mathcal{L})$ of principal ideals of an ADL, forms a distributive lattice. This enables us to extend many existing concepts from the class of distributive lattices to the class of ADLs. Furthermore, "$w-$filters" are introduced in ADLs and [4] investigates their properties. In [3], the concept of $\mathcal{D}-$filters is introduced in an ADL and its properties are discussed. In [7], the concept of regular filters is introduced by M.S. Rao in distributive lattices, and studied their properties. The aim of this work is to investigate the characteristics of $\mathcal{D}$-filters and dense elements within ADLs. The study establishes an equivalent set of conditions that determine if a $\mathcal{D}$-filter may be converted into a regular filter. For a $\mathcal{D}$-filter of an ADL, it is demonstrate that there is a homomorphism whose dense kernel is a regular filter. Moreover, the study derives a necessary condition, stated in terms of regular filters, for every ADL to become relatively complemented. Additionally, give equivalent conditions that allow an ADL to become a Boolean algebra. This provides clarity on the algebraic qualities of ADLs and the conditions in which they show properties of Boolean algebras. Additionally, topological studies are done on a few characteristics of the space containing all prime π-filters of ADLs.

2. Preliminaries

The definitions and significant results from [5, 8] are gathered and given in this part; these will be needed during the entire document.

Definition 2.1. [8] An algebraic structure $(\mathcal{L}, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is an ADL with zero if it satisfies the conditions given below:

(1) $(\theta \vee \vartheta) \wedge \sigma = (\theta \wedge \sigma) \vee (\vartheta \wedge \sigma)$;

(2) $\theta \wedge (\vartheta \vee \sigma) = (\theta \wedge \vartheta) \vee (\theta \wedge \sigma)$;

(3) $(\theta \vee \vartheta) \wedge \vartheta = \vartheta$;

(4) $(\theta \vee \vartheta) \wedge \theta = \theta$;

(5) $\theta \vee (\theta \wedge \vartheta) = \theta$;

(6) $0 \wedge \theta = 0$, for any $\theta, \vartheta, \sigma \in \mathcal{L}$.

To define a partial order $\leq$ on $\mathcal{L}$, consider the condition $\theta = \theta \wedge \vartheta$ or equivalently $\theta \vee \vartheta = \vartheta$ for every $\theta, \vartheta \in \mathcal{L}$. This condition ensures that $\theta \leq \vartheta$, establishing $\leq$ as a partial order on $\mathcal{L}$. When $m \in \mathcal{L}$ is maximal with respect to this partial order, it is referred to as *maximal.* The collection of all such maximal elements in $\mathcal{L}$ is indicated by $\mathcal{M}_{\text{Max.elts}}$.

An ADL $\mathcal{L}$ exhibits many properties of a distributive lattice [1, 2], with the exception of commutativity of $\vee$ and $\wedge$ and lack of right distributivity of $\vee$ over $\wedge$, as highlighted in Swamy's work[8]. If either of these properties held, $\mathcal{L}$ would be classified as a distributive lattice. We define a non-void subset $\mathcal{I}$ of $\mathcal{L}$ as an ideal(a filter) if it satisfies that for any elements $\theta, \vartheta \in \mathcal{I}$ and $\mu \in \mathcal{L}$, the subset $\mathcal{I}$ must include $\theta \wedge \mu$ and $\theta \vee \vartheta$ ($\mu \vee \theta$ and $\theta \wedge \vartheta$). A maximal ideal (filter) contains every proper ideal (filter) of $\mathcal{L}$. The smallest ideal containing a subset $\mathcal{S}$ of $\mathcal{L}$ is defined as $(\mathcal{S}] := \{(\bigvee_{i=1}^{n} \theta_i) \wedge \mu \mid \theta_i \in \mathcal{S}, \mu \in \mathcal{L}, n \in \mathbb{N}\}$. A principal ideal generated by an element θ is denoted as $(\theta]$. Similarly, for each subset $\mathcal{S}$ of $\mathcal{L}$, the smallest filter containing $\mathcal{S}$ is defined as $[\mathcal{S}) := \{\mu \vee (\bigwedge_{i=1}^{n} \theta_i) \mid \theta_i \in \mathcal{S}, \mu \in \mathcal{L}, n \in \mathbb{N}\}$. A principal filter generated by an element θ is denoted as $[\theta)$. It is established that $(\theta] \vee (\vartheta] = (\theta \vee \vartheta]$ and $(\theta] \cap (\vartheta] = (\theta \wedge \vartheta]$ for any $\theta, \vartheta \in \mathcal{L}$. Represented all principal ideals of $\mathcal{L}$ by the set $(\mathcal{PI}(\mathcal{L}), \vee, \cap)$, this brings out a sublattice of the distributive lattice $(\mathcal{I}(\mathcal{L}), \vee, \cap)$ of all ideals of $\mathcal{L}$. Furthermore, the set $(\mathfrak{F}(\mathcal{L}), \vee, \cap)$ of all filters of $\mathcal{L}$ forms a bounded distributive lattice. In an ADL [6], a prime ideal $\mathcal{Q}$ of $\mathcal{L}$ exists if and only if $\mathcal{L} \setminus \mathcal{Q}$ is a prime filter of $\mathcal{L}$.

Theorem 2.2. [8] *Suppose $\mathcal{G}$ is a filter and $\mathcal{I}$ is an ideal in $\mathcal{L}$, with the condition that $\mathcal{G} \cap \mathcal{I} = \emptyset$. Then, there is a prime filter $\mathcal{Q}$ in $\mathcal{L}$ such that $\mathcal{G} \subseteq \mathcal{Q}$ and $\mathcal{Q} \cap \mathcal{I} = \emptyset$.*

An ADL $\mathcal{L}$ is called relatively complemented [5] if, for any elements $\mu, \pi \in \mathcal{L}$ where $\mu \leq \pi$, the interval $[\mu, \pi]$ forms a complemented distributive lattice. For any non-empty subset $\mathcal{S}$ of an Almost distributive lattice (ADL) $\mathcal{L}$, the set $\mathcal{S}^* = \{\mu \in \mathcal{L} \mid \theta \wedge \mu = 0 \text{ for all } \theta \in \mathcal{S}\}$ forms an ideal of $\mathcal{L}$. Specifically, for any $\theta \in \mathcal{L}$, it holds that $\{\theta\}^* = (\theta)^*$, where $(\theta) = (\theta]$. The set $(\theta)^* = \{\mu \in \mathcal{L} \mid \mu \wedge \theta = 0\}$ is referred to as the *annihilator* of θ. An element $e \in \mathcal{L}$ is called *dense* if $(e)^* = \{0\}$. The collection of all dense elements in $\mathcal{L}$ is denoted by $\mathcal{D}$. If $\mathcal{D}$ is non-empty, it forms a filter in $\mathcal{L}$.

For any $\theta \in \mathcal{L}$, both $(\theta, \mathcal{D})$ and $(\{\theta\}, \mathcal{D})$ are well-defined, with $(\theta)^*$ forming the basis for these definitions. According to [3], a filter $\mathcal{G}$ of $\mathcal{L}$ is called a *$\mathcal{D}$-filter* if $\mathcal{D} \subseteq \mathcal{G}$. The smallest $\mathcal{D}$-filter in $\mathcal{L}$ is precisely $\mathcal{D}$. For any subset $\mathcal{S} \subseteq \mathcal{L}$, the set $(\mathcal{S}, \mathcal{D}) = \{\mu \in \mathcal{L} \mid \theta \vee \mu \in \mathcal{D} \text{ for all } \theta \in \mathcal{S}\}$ can be defined. It is observed that $(\mathcal{L}, \mathcal{D}) = \mathcal{D}$ and $(\mathcal{D}, \mathcal{D}) = \mathcal{L}$. Furthermore, for every subset $\mathcal{S}$ of $\mathcal{L}$, the inclusion $\mathcal{D} \subseteq (\mathcal{S}, \mathcal{D})$ holds. For each $\theta \in \mathcal{L}$, the set $(\{\theta\}, \mathcal{D})$ is denoted as $(\theta, \mathcal{D})$. In particular, if $m \in \mathcal{L}$ is a maximal element, then $(m, \mathcal{D}) = \mathcal{L}$. Importantly, for any subset $\mathcal{S}$ of $\mathcal{L}$, $(\mathcal{S}, \mathcal{D})$ forms a $\mathcal{D}$-filter.

Lemma 2.3. [3] *For all subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{L}$, we have:*

(1) *$\mathcal{S} \subseteq \mathcal{T}$ implies $(\mathcal{T}, \mathcal{D}) \subseteq (\mathcal{S}, \mathcal{D})$;*
(2) *$\mathcal{S} \subseteq ((\mathcal{S}, \mathcal{D}), \mathcal{D})$;*
(3) *$(((\mathcal{S}, \mathcal{D}), \mathcal{D}), \mathcal{D}) = (\mathcal{S}, \mathcal{D})$;*
(4) *$(\mathcal{S}, \mathcal{D}) = \mathcal{L} \Leftrightarrow \mathcal{S} \subseteq \mathcal{D}$.*

Proposition 2.4. [3] *For all filters $\mathcal{G}, \mathcal{U}, \mathcal{V}$ of $\mathcal{L}$, we have*

(1) *$(\mathcal{G}, \mathcal{D}) \cap ((\mathcal{G}, \mathcal{D}), \mathcal{D}) = \mathcal{D}$;*
(2) *$\mathcal{G} \cap \mathcal{U} \subseteq \mathcal{D}$ implies $\mathcal{G} \subseteq (\mathcal{U}, \mathcal{D})$;*
(3) *$((\mathcal{G} \vee \mathcal{U}), \mathcal{D}) = (\mathcal{G}, \mathcal{D}) \cap (\mathcal{U}, \mathcal{D})$;*
(4) *$(((\mathcal{G} \cap \mathcal{U}), \mathcal{D}), \mathcal{D}) = ((\mathcal{G}, \mathcal{D}), \mathcal{D}) \cap ((\mathcal{U}, \mathcal{D}), \mathcal{D})$.*

It is evident that $([\mu), \mathcal{D}) = (\mu, \mathcal{D})$. Then obviously $(0, \mathcal{D}) = \mathcal{D}$.

Corollary 2.5. [3] *For $\theta, \vartheta, \sigma \in \mathcal{L}$, we have*

(1) $\theta \leq \vartheta$ *implies* $(\theta, \mathcal{D}) \subseteq (\vartheta, \mathcal{D})$;

(2) $((\theta \wedge \vartheta), \mathcal{D}) = (\theta, \mathcal{D}) \cap (\vartheta, \mathcal{D})$;

(3) $(((\theta \vee \vartheta), \mathcal{D}), \mathcal{D}) = ((\theta, \mathcal{D}), \mathcal{D}) \cap ((\vartheta, \mathcal{D}), \mathcal{D})$;

(4) $(\theta, \mathcal{D}) = \mathcal{L}$ *if and only if* θ *is dense.*

Suppose $\mathcal{G}$ is a $\mathcal{D}$-filter and $\mu \notin \mathcal{G}$. Then, there is a prime $\mathcal{D}$-filter $\mathcal{Q}$ such that $\mathcal{G} \subseteq \mathcal{Q}$ and $\mu \notin \mathcal{Q}$. A prime $\mathcal{D}$-filter $\mathcal{Q}$ in $\mathcal{L}$ is said to be minimal if no prime $\mathcal{D}$-filter $\mathcal{P}$ exists with $\mathcal{P} \subset \mathcal{Q}$.

Theorem 2.6. [3] *In ADL $\mathcal{L}$, a prime $\mathcal{D}$-filter $\mathcal{Q}$ is minimal if, for every $\mu \in \mathcal{Q}$, there exists some $\pi \notin \mathcal{Q}$ with $\mu \vee \pi \in \mathcal{D}$.*

Throughout this article, $\mathcal{L}$ denotes an ADL with maximal elements unless otherwise mentioned.

3. Regular filters of ADLs

This section introduces the notion of regular filters in ADL. It then examines certain properties of $\mathcal{D}$-filters. A set of equivalent conditions is established to determine when a $\mathcal{D}$-filter of an ADL becomes as a regular filter. The collection of all $\mathcal{D}$-filters, prime $\mathcal{D}$-filters and set of all minimal prime $\mathcal{D}$-filters of an ADL $\mathcal{L}$ is denoted by $\mathfrak{F}^{\mathcal{D}}(\mathcal{L})$, $Spec_F^{\mathcal{D}}(\mathcal{L})$ and $Min_F^{\mathcal{D}}(\mathcal{L})$ respectively.

Definition 3.1. A filter $\mathcal{G}$ of an ADL $\mathcal{L}$ is called *regular* if $\mathcal{G} = ((\mathcal{G}, \mathcal{D}), \mathcal{D})$.

Example 3.2. Let $\mathcal{L} = \{0, \theta, \vartheta, \sigma, e, \varphi, \rho\}$ and define $\vee$, $\wedge$ on $\mathcal{L}$ as follows:

$\wedge$	0	θ	ϑ	σ	e	φ	ρ
0	0	0	0	0	0	0	0
θ	0	θ	ϑ	σ	e	φ	ρ
ϑ	0	θ	ϑ	σ	e	φ	ρ
σ	0	σ	σ	σ	0	σ	σ
e	0	e	e	0	e	e	e
φ	0	φ	φ	σ	e	φ	φ
ρ	0	ρ	ρ	σ	e	φ	ρ

$\vee$	0	θ	ϑ	σ	e	φ	ρ
0	0	θ	ϑ	σ	e	φ	ρ
θ	θ	θ	θ	θ	θ	θ	θ
ϑ	ϑ	ϑ	ϑ	ϑ	ϑ	ϑ	ϑ
σ	σ	θ	ϑ	σ	φ	φ	ρ
e	e	θ	ϑ	φ	e	φ	ρ
φ	φ	θ	ϑ	φ	φ	φ	ρ
ρ	ρ	θ	ϑ	ρ	ρ	ρ	ρ

Then $(\mathcal{L}, \vee, \wedge)$ is an ADL. Clearly, we have that $\mathcal{D} = \{\theta, \vartheta, \varphi, \rho\}$ is the dense set of $\mathcal{L}$. Consider the filters $\mathcal{G}_1 = \{\theta, \vartheta, \rho\}$, $\mathcal{G}_2 = \{\theta, \vartheta, e, \varphi, \rho\}$, $\mathcal{G}_3 = \{\theta, \vartheta, \sigma, \varphi, \rho\}$, $\mathcal{G}_4 = \{\theta, \vartheta\}$. Clearly we have that $((\mathcal{G}_2, \mathcal{D}), \mathcal{D}) = \mathcal{G}_2$ and hence $\mathcal{G}_2$ is a regular filter of $\mathcal{L}$. But $\mathcal{G}_1$ is not a regular filter of $\mathcal{L}$ because $((\mathcal{G}_1, \mathcal{D}), \mathcal{D}) = \mathcal{D} \neq \mathcal{G}_1$.

$\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$ represents the class of all regular filters of $\mathcal{L}$. The following result can be verified easily.

Lemma 3.3. *Let $\mathcal{L}$ be an ADL with dense set $\mathcal{D}$. Then we have the following:*

(1) *for any non-empty subset $\mathcal{S}$ of $\mathcal{L}$, $(\mathcal{S}, \mathcal{D}) \in \mathfrak{F}^{\mathcal{R}}(\mathcal{L})$;*

(2) *$\mathcal{D}$ is the smallest regular filter;*

(3) *each regular filter is $\mathcal{D}-$filter.*

Theorem 3.4. *$\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$ constitutes a complete Boolean algebra.*

Proof. It is observed that $(\mathfrak{F}^{\mathcal{R}}(\mathcal{L}), \subseteq)$ is a poset. Let $\mathcal{G}, \mathcal{U}$ be any two regular filters of $\mathcal{L}$. Then clearly $((\mathcal{G}, \mathcal{D}), \mathcal{D}) \cap ((\mathcal{U}, \mathcal{D}), \mathcal{D}) = (((\mathcal{G} \cap \mathcal{U}), \mathcal{D}), \mathcal{D})$ is the infimum of both $\mathcal{G}$ and $\mathcal{U}$ in $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$. Consider $\mathcal{G} \sqcup \mathcal{U} = (\mathcal{G}, \mathcal{D}) \cap ((\mathcal{U}, \mathcal{D}), \mathcal{D})$ as the binary operation $\sqcup$ on $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$. It is obvious that the supremum for $\mathcal{G}$ and $\mathcal{U}$ in $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$ is $(\mathcal{G}, \mathcal{D}) \cap ((\mathcal{U}, \mathcal{D}), \mathcal{D})$. In $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$, $\mathcal{D}$ and $\mathcal{L}$ have to be the least and largest elements, respectively. It gives $(\mathfrak{F}^{\mathcal{R}}(\mathcal{L}), \cap, \sqcup, \mathcal{D}, \mathcal{L})$ is a bounded distributive lattice. $(\mathcal{G} \cap (\mathcal{G}, \mathcal{D}) = ((\mathcal{G}, \mathcal{D}), \mathcal{D}) \cap (\mathcal{G}, \mathcal{D}) = \mathcal{D}$ and $\mathcal{G} \sqcup (\mathcal{G}, \mathcal{D}) = (\mathcal{G}, \mathcal{D}) \cap ((\mathcal{G}, \mathcal{D}), \mathcal{D}) = (\mathcal{D}, \mathcal{D}) = \mathcal{L}$ are obtained for any $\mathcal{G} \in \mathfrak{F}^{\mathcal{R}}(\mathcal{L})$. As a result, the unique complement of $\mathcal{G}$ in $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$ is $(\mathcal{G}, \mathcal{D})$. Hence, a complete Boolean algebra is $(\mathfrak{F}^{\mathcal{R}}(\mathcal{L}), \cap, \sqcup, \mathcal{D}, \mathcal{L}, \mathcal{D})$. □

For any $\mu \in \mathcal{L}$, $(\mu, \mathcal{D}) \in \mathfrak{F}^{\mathcal{R}}(\mathcal{L})$ and hence supremum and infimum of $(\mu, \mathcal{D})$ and $(\pi, \mathcal{D})$ in $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$ are $(\mu, \mathcal{D}) \sqcup (\pi, \mathcal{D}) = ((\mu, \mathcal{D}), \mathcal{D}) \cap (((\pi, \mathcal{D}), \mathcal{D}), \mathcal{D}) = (((\mu \vee \pi, \mathcal{D}), \mathcal{D}), \mathcal{D}) = (\mu \vee \pi, \mathcal{D})$ are $(\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = (\mu \wedge \pi, \mathcal{D})$ respectively.

The following theorem is an immediate outcome of the preceding observation.

Theorem 3.5. *The set $\mathcal{RF}_{\bullet}(\mathcal{L})$, which consists of all regular filters of the form $(\mu, \mathcal{D})$ where $\mu \in \mathcal{L}$, forms a lattice under the operations $\cap$ and $\sqcup$. This lattice $\langle \mathcal{RF}_{\bullet}(\mathcal{L}), \cap, \sqcup \rangle$ is also a sublattice of the distributive lattice $\langle \mathcal{RF}(\mathcal{L}), \cap, \sqcup \rangle$, which includes all regular filters of $\mathcal{L}$. Additionally, $\mathcal{RF}_{\bullet}(\mathcal{L})$ has a greatest element, denoted by $\mathcal{L} = (e, \mathcal{D})$ for any $e \in \mathcal{D}$, and a smallest element, $(0, \mathcal{D})$, corresponding to $\mathcal{D}$.*

Theorem 3.6. *Consider a $\mathcal{D}$-filter $\mathcal{G}$ of an ADL $\mathcal{L}$. Then $\mathcal{G} \vee (\mathcal{G}, \mathcal{D}) = \mathcal{L}$ holds if and only if $\mathcal{G}$ is regular and $(\mathcal{G}, \mathcal{D}) \vee ((\mathcal{G}, \mathcal{D}), \mathcal{D}) = \mathcal{L}$.*

Proof. Assume that $\mathcal{G} \vee (\mathcal{G}, \mathcal{D}) = \mathcal{L}$. Then $((\mathcal{G}, \mathcal{D}), \mathcal{D}) = ((\mathcal{G}, \mathcal{D}), \mathcal{D}) \cap \mathcal{L} = ((\mathcal{G}, \mathcal{D}), \mathcal{D}) \cap (\mathcal{G} \vee (\mathcal{G}, \mathcal{D})) = ((\mathcal{G}, \mathcal{D}), \mathcal{D}) \cap \mathcal{G} \vee ((\mathcal{G}, \mathcal{D}), \mathcal{D}) \cap (\mathcal{G}, \mathcal{D}) = \mathcal{G} \vee \mathcal{D} = \mathcal{G}$. Hence $\mathcal{G}$ is regular. Also $(\mathcal{G}, \mathcal{D}) \vee ((\mathcal{G}, \mathcal{D}), \mathcal{D}) = (\mathcal{G}, \mathcal{D}) \vee \mathcal{G} = \mathcal{L}$. The converse is obvious. □

Equivalent conditions are identified for a prime $\mathcal{D}$-filter of $\mathcal{L}$ to become a minimal prime $\mathcal{D}$-filter.

Theorem 3.7. *In an ADL, the conditions listed below are equivalent:*

(1) *each prime $\mathcal{D}$-filter is minimal;*

(2) $[\mu) \vee (\mu, \mathcal{D}) = \mathcal{L}$, *for all* $\mu \in \mathcal{L}$*;*

(3) $[\mu) = ((\mu, \mathcal{D}), \mathcal{D})$ *and* $(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$, *for all* $\mu \in \mathcal{L}$.

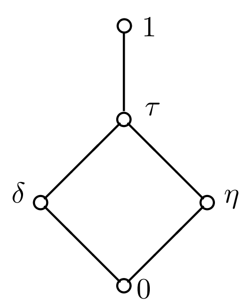
Proof. (1) $\Rightarrow$ (2) : Assume that every prime $\mathcal{D}$-filter is minimal. Let $\mu \in \mathcal{L}$. Suppose $[\mu) \vee (\mu, \mathcal{D}) \neq \mathcal{L}$. Hence there is a prime filter $\mathcal{Q}$ such that $[\mu) \vee (\mu, \mathcal{D}) \subseteq \mathcal{Q}$. Given that $(\mu, \mathcal{D})$ is a $\mathcal{D}$-filter, it follows that $\mathcal{Q} \in \mathfrak{F}^{\mathcal{D}}(\mathcal{L})$. According to the given hypothesis, $\mathcal{Q}$ is minimal. Since $(\mu, \mathcal{D}) \subseteq \mathcal{Q}$, we conclude that $\mu \notin \mathcal{Q}$, it leads to a contradiction. Therefore, it must be that $[\mu) \vee (\mu, \mathcal{D}) = \mathcal{L}$.

(2) $\Rightarrow$ (3) : It's obvious.

(3) $\Rightarrow$ (1) : Assume (3). Let $\mathcal{Q} \in Spec_F^{\mathcal{D}}(\mathcal{L})$. Suppose there is another $\mathcal{P} \in Spec_F^{\mathcal{D}}(\mathcal{L})$ such that $\mathcal{P} \subset \mathcal{Q}$. Now, select an element $\mu \in \mathcal{Q} \setminus \mathcal{P}$. Since $\mu \notin \mathcal{P}$, it follows that $(\mu, \mathcal{D}) \subseteq \mathcal{P}$. Given that μ is in $\mathcal{Q}$, applying the assumed condition yields $\mathcal{L} = ((\mu, \mathcal{D}), \mathcal{D}) \vee (\mu, \mathcal{D}) = [\mu) \vee (\mu, \mathcal{D}) \subseteq \mathcal{Q} \vee \mathcal{P} = \mathcal{Q}$, it gives a contradiction. Hence, $\mathcal{Q} \in Min_F^{\mathcal{D}}(\mathcal{L})$. □

Definition 3.8. A filter $\mathcal{G}$ of an ADL $\mathcal{L}$ is said to be condensed if it satisfies the condition $(\mathcal{G}, \mathcal{D}) = \mathcal{D}$.

Example 3.9. Consider a discrete ADL $\mathcal{C} = \{0, \theta\}$ and a distributive lattice $\mathcal{L}' = \{0, \delta, \eta, \tau, 1\}$ whose Hasse-diagram is given below



Clearly, $\mathcal{L} = \mathcal{C} \times \mathcal{L}' = \{(0,0), (0,\delta), (0,\eta), (0,\tau), (0,1), (\theta,0), (\theta,\delta), (\theta,\eta), (\theta,\tau), (\theta,1)\}$ is an ADL with zero element $(0,0)$. Take $\mathcal{L} = \{o, \upsilon, \phi, \theta, \chi, \psi, \omega, \pi, e, \xi\}$, where $o = (0,0), \upsilon = (0,\delta), \phi = (0,\eta), \theta = (0,\tau), \chi = (0,1), \psi = (\theta,0), \omega = (\theta,\delta), \pi = (\theta,\eta), e = (\theta,\tau), \xi = (\theta,1)$. Define $\wedge$, $\vee$ of $\mathcal{L}$ as

$\wedge$	o	υ	ϕ	θ	χ	ψ	ω	π	e	ξ
o	o	o	o	o	o	o	o	o	o	o
υ	o	υ	o	υ	υ	o	υ	o	υ	υ
ϕ	o	o	ϕ	ϕ	ϕ	o	o	ϕ	ϕ	ϕ
θ	o	υ	ϕ	θ	θ	o	υ	ϕ	θ	θ
χ	o	υ	ϕ	θ	χ	o	υ	ϕ	θ	χ
ψ	o	o	o	o	o	ψ	ψ	ψ	ψ	ψ
ω	o	υ	o	υ	υ	ψ	ω	ψ	ω	ω
π	o	o	ϕ	ϕ	ϕ	ψ	ψ	π	π	π
e	o	υ	ϕ	θ	θ	ψ	ω	π	e	e
ξ	o	υ	ϕ	θ	χ	ψ	ω	π	e	ξ

$\vee$	o	υ	ϕ	θ	χ	ψ	ω	π	e	ξ
o	o	υ	ϕ	θ	χ	ψ	ω	π	e	ξ
υ	υ	υ	θ	θ	χ	ω	ω	e	e	ξ
ϕ	ϕ	θ	ϕ	θ	χ	π	e	π	e	ξ
θ	θ	θ	θ	θ	χ	e	e	e	e	ξ
χ	χ	χ	χ	χ	χ	ξ	ξ	ξ	ξ	ξ
ψ	ψ	ω	π	e	ξ	ψ	ω	π	e	ξ
ω	ω	ω	e	e	ξ	ω	ω	e	e	ξ
π	π	e	π	e	ξ	π	e	π	e	ξ
e	e	e	e	e	ξ	e	e	e	e	ξ
ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ

Consider a filter $\mathcal{G} = \{\phi, \theta, \chi, \pi, e, \xi\}$ and the dense set $\mathcal{D} = \{e, \xi\}$. Clearly, we have that $(\mathcal{G}, \mathcal{D}) = \mathcal{D}$ and hence $\mathcal{G}$ is a condensed filter of $\mathcal{L}$. But $\mathcal{G}$ is not a regula filter because $((\mathcal{G}, \mathcal{D}), \mathcal{D}) = \mathcal{L} \neq \mathcal{G}$.

It is evident that the collection of all condensed filters in an ADL $\mathcal{L}$ constitutes a sublattice within the lattice of all filters of $\mathcal{L}$. Generally, a proper condensed filter is not necessarily a regular filter. However, several equivalent conditions have been established for a $\mathcal{D}$-filter of $\mathcal{L}$ to qualify as a regular filter.

Theorem 3.10. *If each proper filter is non-condensed, then the conditions listed below are equivalent:*

(1) *each member in* $\mathfrak{F}^{\mathcal{D}}(\mathcal{L})$ *is a member of* $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$*;*

(2) *each member in* $Spec_F^{\mathcal{D}}(\mathcal{L})$ *is a member of* $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$*;*

(3) *each member in* $Spec_F^{\mathcal{D}}(\mathcal{L})$ *is minimal;*

(4) *each member in* $Spec_F^{\mathcal{D}}(\mathcal{L})$ *is maximal.*

Proof. (1) $\Rightarrow$ (2): It is evident.

(2) $\Rightarrow$ (3): Assume (2). Let $\mathcal{Q} \in Spec_F^{\mathcal{D}}(\mathcal{L})$. Then, $(\mathcal{Q},\mathcal{D}),\mathcal{D}) = \mathcal{Q}$. Now, assume $\mathcal{Q} \notin Min_F^{\mathcal{D}}(\mathcal{L})$. This means there is $\mathcal{P} \in Spec_F^{\mathcal{D}}(\mathcal{L})$ such that $\mathcal{P} \subset \mathcal{Q}$. Choose an element $\mu \in \mathcal{Q}\setminus\mathcal{P}$. Let $\theta \in (\mathcal{Q},\mathcal{D})$. Since $\mu \in \mathcal{Q}$, we have $\theta \vee \mu \in \mathcal{D} \subseteq \mathcal{P}$. As $\mathcal{P}$ is prime and $\mu \notin \mathcal{P}$, it follows that $\theta \in \mathcal{P} \subset \mathcal{Q}$. Therefore, $(\mathcal{Q},\mathcal{D}) \subseteq \mathcal{Q} \subseteq ((\mathcal{Q},\mathcal{D}),\mathcal{D})$. Thus, $(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D}) \cap ((\mathcal{Q},\mathcal{D}),\mathcal{D}) = \mathcal{D}$. This leads to the contradiction $\mathcal{Q} = (\mathcal{Q},\mathcal{D}),\mathcal{D}) = \mathcal{L}$. Therefore, $\mathcal{Q} \in Min_F^{\mathcal{D}}(\mathcal{L})$.

(3) $\Rightarrow$ (4): It is obvious.

(4) $\Rightarrow$ (1): Assume (4). Let $\mathcal{G}$ be a non-dense filter. Evidently, $\mathcal{G} \subseteq ((\mathcal{G},\mathcal{D}),\mathcal{D})$. Let $\mu \in ((\mathcal{G},\mathcal{D}),\mathcal{D})$. This implies $(\mathcal{G},\mathcal{D}) \subseteq (\mu,\mathcal{D})$. Suppose $\mu \notin \mathcal{G}$. Then, there is $\mathcal{Q} \in Spec_F^{\mathcal{D}}(\mathcal{L})$ such that $\mathcal{G} \subseteq \mathcal{Q}$, $\mu \notin \mathcal{Q}$. By (4), $\mathcal{Q}$ is maximal. As $\mu \notin \mathcal{Q}$, we obtain $\mathcal{Q} \vee [\mu) = \mathcal{L}$. Thus, $(\mathcal{Q},\mathcal{D}) \cap (\mu,\mathcal{D}) = (\mathcal{Q} \vee [\mu),\mathcal{D}) = (\mathcal{L},\mathcal{D}) = \mathcal{D}$. Therefore, $(\mathcal{Q},\mathcal{D}) = (\mathcal{Q},\mathcal{D}) \cap (\mathcal{G},\mathcal{D}) = \mathcal{D}$, which leads to a contradiction. Hence, $\mu \in \mathcal{G}$, and so, $((\mathcal{G},\mathcal{D}),\mathcal{D}) \subseteq \mathcal{G}$. Consequently, $\mathcal{G}$ is a regular filter of $\mathcal{L}$. □

Given a filter $\mathcal{G}$ of an ADL $\mathcal{L}$, define $Hom_{\mathcal{L}}(\mathcal{G})$ as the set of all homomorphisms on $\mathcal{G}$. It is clear that $Hom_{\mathcal{L}}(\mathcal{G})$ is an ADL when endowed with the pointwise operations.

The following statement can be easily verified.

Proposition 3.11. *For any* $\mathcal{G} \in \mathfrak{F}(\mathcal{L})$, $\tau \in \mathcal{L}$, *define the function* $\phi_\tau : \mathcal{G} \to \mathcal{G}$ *by* $\phi_\tau(\mu) = \mu \vee \tau$ *for all* $\mu \in \mathcal{G}$. *The following statements are valid:*

(1) ϕ_τ *is homomorphism;*

(2) $\phi_{\tau\wedge\omega} = \phi_\tau \wedge \phi_\omega$ *for* $\tau,\omega \in \mathcal{L}$*;*

(3) $\phi_{\tau\vee\omega} = \phi_\tau \vee \phi_\omega$ *for* $\tau,\omega \in \mathcal{L}$.

Definition 3.12. Let $\mathcal{G} \in \mathfrak{F}^{\mathcal{D}}(\mathcal{L})$. A homomorphism $\upsilon : \mathcal{G} \to \mathcal{G}$ is referred as dense-valued if, for each $\mu \in \mathcal{G}$, $\upsilon(\mu) \in \mathcal{D}$.

from the example 3.2, for a filter $\mathcal{G}_2$, define $\upsilon : \mathcal{G}_2 \to \mathcal{G}_2$ as $\upsilon(\theta) = \theta, \upsilon(\vartheta) = \vartheta, \upsilon(e) = \vartheta, \upsilon(\varphi) = \varphi, \upsilon(\rho) = \rho$. Clearly, υ is dense-valued homomorphism.

Assume the collection of all $\upsilon \in Hom_{\mathcal{L}}(\mathcal{G})$ where υ represents a dense-valued homomorphism, denoted as $\mathcal{D}(\mathcal{G})$. It's evident that the identity element of $Hom_{\mathcal{L}}(\mathcal{G})$ belongs to $\mathcal{D}(\mathcal{G})$. Specifically, the mapping $\mathbf{1} : \mathcal{G} \to \mathcal{G}$ defined by $\mathbf{1}(\mu) = \mu$ for all $\mu \in \mathcal{G}$, constitutes a dense-valued homomorphism. Thus, $\mathbf{1}$ belongs to $\mathcal{D}(\mathcal{G})$. Furthermore, It's easy to see that, $\mathcal{D}(\mathcal{G})$ forms a filter on $Hom_{\mathcal{L}}(\mathcal{G})$. Additionally, for every $e \in \mathcal{D}$, $\Phi_e \in \mathcal{D}(\mathcal{G})$.

Definition 3.13. Let $\mathcal{G} \in \mathfrak{F}^{\mathcal{D}}(\mathcal{L})$ with $\rho : \mathcal{L} \to Hom_{\mathcal{L}}(\mathcal{G})$ is homomorphism. The dense kernel of ρ, denoted as $Ker^{\mathcal{D}}(\rho)$, is defined by $Ker^{\mathcal{D}}(\rho) = \{\mu \in \mathcal{L} \mid \rho(\mu) \in \mathcal{D}(\mathcal{G})\}$. Additionally, let $\Phi_{\mathcal{G}} : \mathcal{L} \to Hom_{\mathcal{L}}(\mathcal{G})$ be a map such that $\Phi_{\mathcal{G}}(\tau) = \Phi_\tau$ for all $\tau \in \mathcal{L}$. It is noted that $Ker^{\mathcal{D}}(\rho)$ forms a filter in $\mathcal{L}$.

From the example 3.2, for a filter $\mathcal{G}_2$, define $\rho : \mathcal{L} \to Hom\mathcal{L}(\mathcal{G}_2)$ as $\rho(i) = \phi_i$, for all $i \in \mathcal{G}_2$. Claerly, we have that $Ker^{\mathcal{D}}(\rho) = \{\theta, \vartheta, \sigma, \varphi, \rho\}$

Theorem 3.14. *For each $\mathcal{G} \in \mathfrak{F}^{\mathcal{D}}(\mathcal{L})$, we have $(\mathcal{G}, \mathcal{D}) = Ker^{\mathcal{D}}(\Phi_{\mathcal{G}})$. So, the pair $(\mathcal{G}, \mathcal{D})$ can be regarded as the dense kernel of a homomorphism.*

Proof. Suppose $\tau \in Ker^{\mathcal{D}}(\Phi_{\mathcal{G}})$. By definition, this means $\Phi_\tau \in \mathcal{D}(\mathcal{G})$, which implies that $\mu \vee \tau = \Phi_\tau(\mu)$ is a dense element for all $\mu \in \mathcal{G}$. Thus, we conclude that $\tau \in (\mathcal{G}, \mathcal{D})$. Conversely, assume $\tau \in (\mathcal{G}, \mathcal{D})$. This indicates that $\mu \vee \tau \in \mathcal{D}$ for every $\mu \in \mathcal{G}$. Therefore, Φ_τ maps every element of $\mathcal{G}$ to a dense element. As a result, $\Phi_{\mathcal{G}}(\tau) = \Phi_\tau \in \mathcal{D}(\mathcal{G})$. Hence, $\tau \in Ker^{\mathcal{D}}(\Phi_{\mathcal{G}})$. □

Theorem 3.15. *If each member in $\mathfrak{F}^{\mathcal{D}}(\mathcal{L})$ is a member of $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$, any two prime $\mathcal{D}$-filters are incomparable.*

Proof. Suppose each member in $\mathfrak{F}^{\mathcal{D}}(\mathcal{L})$ is a member of $\mathfrak{F}^{\mathcal{R}}(\mathcal{L})$. If there exist $\mathcal{Q},\ \mathcal{P} \in Spec_F^{\mathcal{D}}(\mathcal{L})$ with $\mathcal{Q} \neq \mathcal{P},\ \mathcal{Q} \subset \mathcal{P}$. Select an element $\eta \in \mathcal{P} \setminus \mathcal{Q}$. For any $\mu \in (\mathcal{P}, \mathcal{D})$, it follows that $\mu \vee \eta \in \mathcal{D} \subseteq \mathcal{Q}$. As $\mathcal{Q}$ is prime and $\eta \notin \mathcal{Q}$, we conclude that $\mu \in \mathcal{Q}$. Thus, $(\mathcal{P}, \mathcal{D}) \subseteq \mathcal{Q} \subseteq \mathcal{P}$. This implies $(\mathcal{P}, \mathcal{D}) = \mathcal{P} \cap (\mathcal{P}, \mathcal{D}) = \mathcal{D}$. Since every $\mathcal{D}$-filter is assumed to be regular, $\mathcal{P}$ is also regular. Therefore, $\mathcal{P} = ((\mathcal{P}, \mathcal{D}), \mathcal{D}) = (\mathcal{D}, \mathcal{D}) = \mathcal{L}$, it gives a contradiction. □

The following result establishes a sufficient condition, expressed by using regular filters, for an ADL to be relatively complemented.

Theorem 3.16. *Suppose each principal filter is a $\mathcal{D}$-filter. Then every $\mathcal{D}$-filter is regular if and only if $\mathcal{L}$ is relatively complemented.*

Proof. Let $\mathcal{L}$ be an ADL where every principal filter is a $\mathcal{D}$-filter, and every $\mathcal{D}$-filter is a regular filter. Assume, for contradiction, that $\mathcal{L}$ is not relatively complemented. Then there exist elements $\theta, \vartheta, \sigma \in \mathcal{L}$ such that $\vartheta < \sigma < \theta$, and σ lacks a complement within the interval $[\vartheta, \theta]$. Define the set $\mathcal{I} = \{\mu \in \mathcal{L} \mid \sigma \wedge \mu \leq \vartheta\}$. It is straightforward to verify that $\mathcal{I}$ is an ideal in $\mathcal{L}$. Now, construct the ideal $\mathcal{C} = \mathcal{I} \vee (\sigma]$. Assume $\theta \in \mathcal{C}$. Then θ can be expressed as $\theta = \sigma \vee i$ for some $i \in \mathcal{I}$. Therefore, $\theta = \theta \vee \vartheta = (\sigma \vee i) \vee \vartheta = \sigma \vee (i \vee \vartheta)$, and $(i \vee \vartheta) \wedge \sigma = (i \wedge \sigma) \vee (\vartheta \wedge \sigma) = (\sigma \wedge i) \vee \vartheta = \vartheta$, since $i \in \mathcal{I}$. This implies that $i \vee \vartheta$ is a relative complement of σ in the interval $[\vartheta, \theta]$, which contradicts the assumption that σ

has no complement. Thus, $\theta \notin \mathcal{C}$, leading to $[\theta) \cap \mathcal{C} = \emptyset$. Since $[\theta)$ is a $\mathcal{D}$-filter, there exists a prime $\mathcal{D}$-filter $\mathcal{Q}$ in $\mathcal{L}$ such that $[\theta) \subseteq \mathcal{Q}$ and $\mathcal{Q} \cap \mathcal{C} = \emptyset$. Consequently, $\mathcal{Q} \cap \mathcal{I} = \emptyset$ and $\mathcal{Q} \cap (\sigma] = \emptyset$. Now define $\mathcal{G} = [\sigma) \vee \mathcal{Q}$. Clearly, $\mathcal{G}$ is a $\mathcal{D}$-filter in $\mathcal{L}$. Suppose $\vartheta \in \mathcal{G}$. Then $\vartheta \in [\sigma) \vee \mathcal{Q}$, which means $\vartheta = \sigma \vee \delta$ for some $\delta \in \mathcal{Q}$. This implies $\delta \in \mathcal{I}$, contradicting the fact that $\delta \in \mathcal{Q} \cap \mathcal{I} = \emptyset$. Hence, $\vartheta \notin \mathcal{G}$, and $\mathcal{G} \cap (\vartheta] = \emptyset$. Then there exists a prime $\mathcal{D}$-filter $\mathcal{P}$ such that $\mathcal{G} \subseteq \mathcal{P}$ and $(\vartheta] \cap \mathcal{P} = \emptyset$. Thus, $\mathcal{Q} \subset \mathcal{G} \subseteq \mathcal{P}$. This implies that $\mathcal{Q}$ and $\mathcal{P}$ are distinct prime $\mathcal{D}$-filters with $\mathcal{Q} \subset \mathcal{P}$. This shows that two prime $\mathcal{D}$-filters are comparable, which contradicts the earlier result. Therefore, $\mathcal{L}$ must be relatively complemented. Conversely assume that $\mathcal{L}$ is relatively complemented. Let $\mathcal{G}$ be a $\mathcal{D}$-filter of $\mathcal{L}$. Clearly we have that $\mathcal{G} \subseteq ((\mathcal{G}, \mathcal{D}), \mathcal{D})$. Let $\mu \in ((\mathcal{G}, \mathcal{D}), \mathcal{D})$. Then $[\mu) \subseteq ((\mathcal{G}, \mathcal{D}), \mathcal{D})$ and hence $(\mathcal{G}, \mathcal{D}) = ((\mathcal{G}, \mathcal{D}), \mathcal{D}) = ([\mu), \mathcal{D})$. If $\mu \notin \mathcal{G}$ then there exists a prime $\mathcal{D}$-filter $\mathcal{Q}$ such that $\mathcal{G} \subseteq \mathcal{Q}$ and $\mu \notin \mathcal{Q}$. There fore $(\mathcal{Q}, \mathcal{D}) \subseteq (\mathcal{G}, \mathcal{D})$. We prove that $\mathcal{Q}$ is maximal. Suppose $\mathcal{P}$ is a prime $\mathcal{D}$-filter such that $\mathcal{Q} \subsetneq \mathcal{P}$. Let $\theta \in \mathcal{L}$. Choose $\theta \in \mathcal{P} \setminus \mathcal{Q}$ and $\nu \in \mathcal{Q}$. Then $0 < \theta < \theta \vee \vartheta \vee \nu$. Since $\mathcal{L}$ relatively complemented, there exists a relative complement $\psi \in [0,\ \theta \vee \vartheta \vee \nu]$ such that $\theta \wedge \psi = 0$ and $\theta \vee \psi = \theta \vee \vartheta \vee \nu$. Since $\mathcal{Q}$ is a filter and $\nu \in \mathcal{Q}$, we have $\theta \vee \vartheta \vee \nu \in \mathcal{Q}$ and hence $\theta \vee \psi \in \mathcal{Q}$. Since $\mathcal{Q}$ is prime, we get $\theta \in \mathcal{Q}$ or $\psi \in \mathcal{Q}$. Since $\theta \notin \mathcal{Q}$, we get $\psi \in \mathcal{Q}$ and hence $\psi \in \mathcal{P}$. Since $\theta, \psi \in \mathcal{P}$, we have $\theta \wedge \psi \in \mathcal{P}$ and hence $0 \in \mathcal{P}$. Therefore $\mathcal{P} = \mathcal{L}$, we get a contradiction. Thus $\mathcal{Q}$ is maximal. Since $\mu \notin \mathcal{Q}$, we get $\mathcal{Q} \vee [\mu) = \mathcal{L}$. Therefore $(\mathcal{Q}, \mathcal{D}) \cap ([\mu), \mathcal{D}) = (\mathcal{Q} \vee [\mu), \mathcal{D}) = (\mathcal{L}, \mathcal{D}) = \mathcal{D}$. Now $(\mathcal{Q}, \mathcal{D}) = (\mathcal{Q}, \mathcal{D}) \cap (\mathcal{G}, \mathcal{D}) \subseteq (\mathcal{Q}, \mathcal{D}) \cap ([\mu), \mathcal{D}) = (\mathcal{L}, \mathcal{D}) = \mathcal{D}$, we get a contradiction. Therefore $\mu \in \mathcal{G}$ and hence $((\mathcal{G}, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{G}$. Thus $\mathcal{G}$ is a regular filter of $\mathcal{L}$. □

4. π-filters of ADLs

This section explores the notion of π-filters in an ADL. It provides a characterization of these filters through regular filters and congruences. Additionally, a series of equivalent criteria are established for an ADL can be transformed into a Boolean algebra.

Definition 4.1. A filter $\mathcal{G}$ in $\mathcal{L}$ is called a π-*filter* if, for every $\mu \in \mathcal{G}$, the condition $((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{G}$ holds.

From the example 3.2, clearly we have that $\mathcal{G}_3$ is a π-filter of an ADL $\mathcal{L}$.

Lemma 4.2. *In an ADL, the subsequent properties are true:*

(1) *$\mathcal{D}$ is the smallest π-filter;*

(2) *Every regular filter is a π-filter.*

Proposition 4.3. *Each minimal prime $\mathcal{D}$-filter in $\mathcal{L}$ is a π-filter.*

Proof. Let $\mathcal{Q} \in Min_F^{\mathcal{D}}(\mathcal{L})$, and assume $\mu \in \mathcal{Q}$. Then there exists an element $\pi \notin \mathcal{Q}$ such that $\mu \vee \pi \in \mathcal{D}$. Consider $\nu \in ((\mu, \mathcal{D}), \mathcal{D})$. By the definition of the filter, $(\mu, \mathcal{D}) \subseteq (\nu, \mathcal{D})$. As a result, $\pi \in (\nu, \mathcal{D})$. This leads to the conclusion that $\nu \in ((\nu, \mathcal{D}), \mathcal{D}) \subseteq (\pi, \mathcal{D}) \subseteq \mathcal{Q}$ since π is not in $\mathcal{Q}$. Hence, we have $((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. This shows that $\mathcal{Q}$ is a π-filter of $\mathcal{L}$. □

Definition 4.4. For each $\mathcal{G} \in \mathfrak{F}(\mathcal{L})$, define the set $\mathcal{G}^{\varphi} = \{\mu \in \mathcal{L} \mid (\theta, \mathcal{D}) \subseteq (\mu, \mathcal{D})$ for some $\theta \in \mathcal{G}\}$. This set $\mathcal{G}^{\varphi}$ is referred to as an extension of $\mathcal{G}$.

This result is a straightforward consequence of the previous definition

Lemma 4.5. *For any $\mathcal{G}$, $\mathcal{U} \in \mathfrak{F}(\mathcal{L})$, the properties listed below hold:*

(1) $\mathcal{D} \subseteq \mathcal{G}^{\varphi}$ *and* $\mathcal{D}^{\varphi} = \mathcal{D}$;

(2) $\mathcal{G} \subseteq \mathcal{U}$ *implies* $\mathcal{G}^{\varphi} \subseteq \mathcal{U}^{\varphi}$;

(3) $(\mathcal{G} \cap \mathcal{U})^{\varphi} = \mathcal{G}^{\varphi} \cap \mathcal{U}^{\varphi}$;

(4) $(\mathcal{G}^{\varphi})^{\varphi} = \mathcal{G}^{\varphi}$.

Proposition 4.6. *For any $\mathcal{G} \in \mathfrak{F}(\mathcal{L})$, $\mathcal{G}^{\varphi}$ is the smallest π-filter such that $\mathcal{G} \subseteq \mathcal{G}^{\varphi}$.*

Proof. It is evident that $\mathcal{D} \subseteq \mathcal{G}^{\varphi}$. Let $\mu, \pi \in \mathcal{G}^{\varphi}$. Then, there exist $\theta, \vartheta \in \mathcal{G}$ such that $(\theta, \mathcal{D}) \subseteq (\mu, \mathcal{D})$ and $(\vartheta, \mathcal{D}) \subseteq (\pi, \mathcal{D})$. Therefore, $(\theta \wedge \vartheta, \mathcal{D}) = (\theta, \mathcal{D}) \cap (\vartheta, \mathcal{D}) \subseteq (\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = (\mu \wedge \pi, \mathcal{D})$, implying $\mu \wedge \pi \in \mathcal{G}^{\varphi}$. Next, let $\mu \in \mathcal{G}^{\varphi}$ and $\mu \leq \pi$. Then, for some $\theta \in \mathcal{G}$, we have $(\theta, \mathcal{D}) \subseteq (\mu, \mathcal{D}) \subseteq (\pi, \mathcal{D})$. This shows that $\mathcal{G}^{\varphi}$ is a filter of $\mathcal{L}$. Furthermore, it is clear that $\mathcal{G} \subseteq \mathcal{G}^{\varphi}$. Now, let $\mu \in \mathcal{G}^{\varphi}$ and $\nu \in ((\mu, \mathcal{D}), \mathcal{D})$. Then, there exists $\theta \in \mathcal{G}$ such that $(\theta, \mathcal{D}) \subseteq (\mu, \mathcal{D}) \subseteq (\nu, \mathcal{D})$. Hence, $((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{G}^{\varphi}$, meaning $\mathcal{G}^{\varphi}$ is a π-filter of $\mathcal{L}$. Finally, let $\mathcal{U}$ be a π-filter of $\mathcal{L}$ such that $\mathcal{G} \subseteq \mathcal{U}$. Let $\mu \in \mathcal{G}^{\varphi}$. Then, there exists $\theta \in \mathcal{G} \subseteq \mathcal{U}$ such that $(\theta, \mathcal{D}) \subseteq (\mu, \mathcal{D})$. Since $\mathcal{U}$ is a π-filter, we conclude that $\mu \in ((\mu, \mathcal{D}), \mathcal{D}) \subseteq ((\theta, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{U}$. Therefore, $\mathcal{G}^{\varphi} \subseteq \mathcal{U}$, showing that $\mathcal{G}^{\varphi}$ is the smallest π-filter of $\mathcal{L}$ such that $\mathcal{G} \subseteq \mathcal{G}^{\varphi}$. □

From the previous results, it follows that a filter $\mathcal{G}$ is a π-filter if and only if $\mathcal{G} = \mathcal{G}^{\varphi}$, establishing that $\mathcal{D}$ is the minimal π-filter in $\mathcal{L}$. Additionally, combining these observations, we conclude that the set of all π-filters of an ADL $\mathcal{L}$, denoted $\mathfrak{F}^{\pi}(\mathcal{L})$, forms a complete distributive lattice. In this lattice, the meet operation is given by $\mathcal{G} \wedge \mathcal{U} = \mathcal{G} \cap \mathcal{U}$, and the join operation is defined by $\mathcal{G} \underline{\vee} \mathcal{U} = (\mathcal{G} \vee \mathcal{U})^{\varphi}$, where the least element is $\mathcal{D}$.

Theorem 4.7. *For any $\mathcal{G} \in \mathfrak{F}(\mathcal{L})$ and $\mu, \pi \in \mathcal{L}$, define a binary relation $\Theta(\mathcal{G})$ on $\mathcal{L}$ as follows:*

$$(\mu, \pi) \in \Theta(\mathcal{G}) if and only if \{\mathcal{D} \vee [\mu)\} \cap (\theta, \mathcal{D}) = \{\mathcal{D} \vee [\pi)\} \cap (\theta, \mathcal{D}),$$

for some $\theta \in \mathcal{G}$. Then $\Theta(\mathcal{G})$ is congruence on $\mathcal{L}$.

Proof. It is evident that $\Theta(\mathcal{G})$ defines an equivalence relation on $\mathcal{L}$. Let $(\mu, \pi) \in \Theta(\mathcal{G})$. Then $\{\mathcal{D} \vee [\mu)\} \cap (\theta, \mathcal{D}) = \{\mathcal{D} \vee [\pi)\} \cap (\theta, \mathcal{D})$ for some $\theta \in \mathcal{G}$. For any $\sigma \in \mathcal{L}$, we have $\{\mathcal{D} \vee [\mu \vee \sigma)\} \cap (\theta, \mathcal{D}) = \{\mathcal{D} \vee [\mu)\} \cap \{\mathcal{D} \vee [\sigma)\} \cap (\theta, \mathcal{D}) = \{\mathcal{D} \vee [\pi)\} \cap \{\mathcal{D} \vee [\sigma)\} \cap (\theta, \mathcal{D})$

$= \{\mathcal{D} \vee [\pi \vee \sigma)\} \cap (\theta, \mathcal{D})$. Therefore $(\mu \vee \sigma, \pi \vee \sigma) \in \Theta(\mathcal{G})$. Again, $\{\mathcal{D} \vee [\mu \wedge \sigma)\} \cap (\theta, \mathcal{D}) = \{\mathcal{D} \vee [\mu) \vee [\sigma)\} \cap (\theta, \mathcal{D}) = \{\{\mathcal{D} \vee [\mu)\} \cap (\theta, \mathcal{D})\} \vee \{[\sigma) \cap (\theta, \mathcal{D})\} = \{\{\mathcal{D} \vee [\pi)\} \cap (\theta, \mathcal{D})\} \vee \{[\sigma) \cap (\theta, \mathcal{D})\} = \{\mathcal{D} \vee [\pi \wedge \sigma)\} \cap (\theta, \mathcal{D})$. Hence $(\mu \wedge \sigma, \pi \wedge \sigma) \in \Theta(\mathcal{G})$. Therefore $\Theta(\mathcal{G})$ is a congruence on $\mathcal{L}$. □

Lemma 4.8. *Let $\mathcal{L}$ be an ADL. For any $\mu \in \mathcal{L}$, the following properties hold:*

(1) $\{\mathcal{D} \vee (([\mu), \mathcal{D}), \mathcal{D})\} = ((\mu, \mathcal{D}), \mathcal{D})$;

(2) $\{\mathcal{D} \vee [\mu)\} \cap (\mu, \mathcal{D}) = \mathcal{D}$.

Proposition 4.9. *For each $\mathcal{G} \in \mathfrak{F}(\mathcal{L})$, define the dense-kernel $Ker^{\mathcal{D}}\Theta(\mathcal{G})$ of $\Theta(\mathcal{G})$ as follows:*

$$Ker^{\mathcal{D}}\Theta(\mathcal{G}) = \{\mu \in \mathcal{L} \mid \{\mathcal{D} \vee [\mu)\} \cap (\theta, \mathcal{D}) = \mathcal{D} \text{ for some } \theta \in \mathcal{G}\}.$$

Then $Ker^{\mathcal{D}}\Theta(\mathcal{G}) \in \mathfrak{F}(\mathcal{L})$ containing $\mathcal{G}$.

Proof. It is evident that $\mathcal{D} \subseteq Ker^{\mathcal{D}}\Theta(\mathcal{G})$. Let $\mu, \pi \in Ker^{\mathcal{D}}\Theta(\mathcal{G})$. Then $\{\mathcal{D} \vee [\mu)\} \cap (\theta, \mathcal{D}) = \{\mathcal{D} \vee [\pi)\} \cap (\vartheta, \mathcal{D}) = \mathcal{D}$ for some $\theta, \vartheta \in \mathcal{G}$. Now $\{\mathcal{D} \vee [\mu \wedge \pi)\} \cap ((\theta \wedge \vartheta), \mathcal{D}) = \{\mathcal{D} \vee [\mu) \vee \mathcal{D} \vee [\pi)\} \cap (\theta, \mathcal{D}) \cap (\vartheta, \mathcal{D}) = \{(\mathcal{D} \vee [\mu)) \cap (\theta, \mathcal{D}) \cap (\vartheta, \mathcal{D})\} \vee \{(\mathcal{D} \vee [\pi)) \cap (\theta, \mathcal{D}) \cap (\vartheta, \mathcal{D})\} = \{\mathcal{D} \cap (\vartheta, \mathcal{D})\} \vee \{\mathcal{D} \cap (\theta, \mathcal{D})\} = \mathcal{D}$. Hence $\mu \wedge \pi \in Ker^{\mathcal{D}}\Theta(\mathcal{G})$. Let $\mu \in Ker^{\mathcal{D}}\Theta(\mathcal{G})$ and $\mu \leq \pi$. Then there is $\theta \in \mathcal{G}$ such that $\{\mathcal{D} \vee [\pi)\} \cap (\theta, \mathcal{D}) \subseteq \{\mathcal{D} \vee [\mu)\} \cap (\theta, \mathcal{D}) = \mathcal{D}$. Which gives $\pi \in Ker^{\mathcal{D}}\Theta(\mathcal{G})$. Hence $Ker^{\mathcal{D}}\Theta(\mathcal{G}) \in \mathfrak{F}(\mathcal{L})$. Let $\mu \in \mathcal{G}$. From the above result, we get $\mu \in Ker^{\mathcal{D}}\Theta(\mathcal{G})$. Therefore $\mathcal{G} \subseteq Ker^{\mathcal{D}}\Theta(\mathcal{G})$. □

Theorem 4.10. *Let $\mathcal{G}$ be a filter in an ADL $\mathcal{L}$. Then the following are equivalent:*

(1) *$\mathcal{G}$ is a π-filter;*

(2) *$\mathcal{G} = Ker^{\mathcal{D}}\Theta(\mathcal{G})$;*

(3) *for $\mu, \pi \in \mathcal{L}, (\mu, \mathcal{D}) = (\pi, \mathcal{D})$ and $\mu \in \mathcal{G} \Rightarrow \pi \in \mathcal{G}$;*

(4) *$\mu \in \mathcal{G} \Leftrightarrow \mu \in ((\theta, \mathcal{D}), \mathcal{D})$ for some $\theta \in \mathcal{G}$.*

Proof. (1) $\Rightarrow$ (2): Assume (1). Clearly $\mathcal{G} \subseteq Ker^{\mathcal{D}}\Theta(\mathcal{G})$. Let $\mu \in Ker^{\mathcal{D}}\Theta(\mathcal{G})$. Then $\{\mathcal{D} \vee [\mu)\} \cap (\theta, \mathcal{D}) = \mathcal{D}$ for some $\theta \in \mathcal{G}$. Since $\mathcal{G}$ is a π-filter, $\mu \in \mathcal{D} \vee [\mu) \subseteq ((\theta, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{G}$. Therefore $Ker^{\mathcal{D}}\Theta(\mathcal{G}) \subseteq \mathcal{G}$. Hence $\mathcal{G} = Ker^{\mathcal{D}}\Theta(\mathcal{G})$.

(2) $\Rightarrow$ (3): Assume that $\mathcal{G} = Ker^{\mathcal{D}}\Theta(\mathcal{G})$. Let $\theta, \vartheta \in \mathcal{L}$ such that $(\theta, \mathcal{D}) = (\vartheta, \mathcal{D})$. Suppose $\theta \in \mathcal{G}$. Then $\{\mathcal{D} \vee [\theta)\} \cap (\nu, \mathcal{D}) = \mathcal{D}$ for some $\nu \in \mathcal{G}$. Then we get $\{\mathcal{D} \vee [\theta)\} \cap (\nu, \mathcal{D}) = \mathcal{D} \Rightarrow ((\{\mathcal{D} \vee [\theta)\}, \mathcal{D}), \mathcal{D}) \cap (\nu, \mathcal{D}) = ((\mathcal{D}, \mathcal{D}), \mathcal{D}) = \mathcal{D} \Rightarrow ((\theta, \mathcal{D}), \mathcal{D}) \cap (\nu, \mathcal{D}) = \mathcal{D} \Rightarrow ((\vartheta, \mathcal{D}), \mathcal{D}) \cap (\nu, \mathcal{D}) = \mathcal{D} \Rightarrow \{\mathcal{D} \vee [\vartheta)\} \cap (\nu, \mathcal{D}) \subseteq ((\{\mathcal{D} \vee [\vartheta)\}, \mathcal{D}), \mathcal{D}) \cap (\nu, \mathcal{D}) = \mathcal{D} \Rightarrow \vartheta \in Ker^{\mathcal{D}}\Theta(\mathcal{G}) = \mathcal{G}$.

(3) $\Rightarrow$ (4): Assume (3). Let $\mu \in \mathcal{G}$. Then clearly $\mu \in ((\mu, \mathcal{D}), \mathcal{D})$. Again, let $\mu \in ((\theta, \mathcal{D}), \mathcal{D})$ for some $\theta \in \mathcal{G}$. Hence $((\mu, \mathcal{D}), \mathcal{D}) \subseteq ((\theta, \mathcal{D}), \mathcal{D})$, which yields $((\mu, \mathcal{D}), \mathcal{D}) = ((\mu, \mathcal{D}), \mathcal{D}) \cap ((\theta, \mathcal{D}), \mathcal{D}) = (((\mu \vee \theta), \mathcal{D}), \mathcal{D})$. Thus $(\mu, \mathcal{D}) = ((\mu \vee \theta), \mathcal{D})$ and $\mu \vee \theta \in \mathcal{G}$. By (3), we have $\mu \in \mathcal{G}$.

(4) $\Rightarrow$ (1): Assume (4). Let $\mu \in \mathcal{G}$. Hence $\mu \in ((\theta, \mathcal{D}), \mathcal{D})$ for some $\theta \in \mathcal{G}$. Let $\nu \in ((\mu, \mathcal{D}), \mathcal{D})$. Then for this $\theta \in \mathcal{G}$, we get that $\nu \in ((\mu, \mathcal{D}), \mathcal{D}) \subseteq ((\theta, \mathcal{D}), \mathcal{D})$. Hence by (4), we have $\nu \in \mathcal{G}$. Therefore $((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{G}$. Thus F is a π-filter of $\mathcal{L}$. $\square$

It is a well known fact that an ADL is a Boolean algebra if and only if it has a unique dense element. Hence the following result is a direct consequence.

Theorem 4.11. *The following assertions are equivalent in an ADL $\mathcal{L}$.*

(1) *$\mathcal{L}$ is a Boolean algebra;*

(2) *every filter is a $\mathcal{D}$-filter;*

(3) *every filter is a π-filter;*

(4) *every prime filter is a π-filter.*

It has been observed that every minimal prime $\mathcal{D}$-filter is also a prime π-filter. However, the reverse is not necessarily true. Nevertheless, a condition is provided that is sufficient for a prime π-filter to become a minimal prime $\mathcal{D}$-filter. let's denote the set of all prime π-filters of $\mathcal{L}$ as $Spec^{\pi}_{\mathcal{G}}(\mathcal{L})$.

Proposition 4.12. *If every principal filter of the form $(\mu, \mathcal{D})$ for $\mu \in \mathcal{L}$ is a principal filter, then every prime π-filter is a minimal prime $\mathcal{D}$-filter.*

Proof. Let $\mathcal{Q} \in Spec^{\pi}F(\mathcal{L})$ and $\mu \in \mathcal{Q}$. Then $(\mu, \mathcal{D}) = [\pi)$ for some $\pi \in \mathcal{L}$. Therefore $\mu \vee \pi \in \mathcal{D}$. Now $((\mu \wedge \pi), \mathcal{D}) = (\mu, \mathcal{D}) \cap (\pi, \mathcal{D}) = (\mu, \mathcal{D}) \cap ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{D}$. Hence $\mu \wedge \pi \notin \mathcal{Q}$, which implies that $\pi \notin \mathcal{Q}$. Therefore $\mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter. $\square$

Theorem 4.13. *In an ADL, the following are equivalent:*

(1) *any π-filter is a principal filter;*

(2) *any $(\mu, \mathcal{D})$ is a principal filter and every minimal prime $\mathcal{D}$-filter is non-condensed;*

(3) *any prime π-filter is a principal filter.*

Proof. (1) $\Rightarrow$ (2): Assume each $(\mu, \mathcal{D})$ is a π-filter. To prove that every minimal prime $\mathcal{D}$-filter is non-condensed, consider a minimal prime $\mathcal{D}$-filter $\mathcal{Q}$. From Proposition 4.3, we know that $\mathcal{Q}$ is a π-filter, which implies that $\mathcal{Q} = [\theta)$ for some $\theta \in \mathcal{L}$. Suppose, for the sake of contradiction, that $(\mathcal{Q}, \mathcal{D}) = \mathcal{D}$. In this case, we would have $(\theta, \mathcal{D}) = \mathcal{D}$, which implies $\mathcal{L} = ((\theta, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$, leading to a contradiction. Thus, $(\mathcal{Q}, \mathcal{D}) \neq \mathcal{D}$.

(2) $\Rightarrow$ (3): Assume condition (2) holds. Let $\mathcal{Q}$ be a prime π-filter of $\mathcal{L}$. Since every $(\mu, \mathcal{D})$ is a principal filter, by the previous result, we deduce that $\mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter and that $(\mathcal{Q}, \mathcal{D}) \neq \mathcal{D}$. Therefore, there exists an element $\mu \neq \mathcal{D}$ such that $\mu \in (\mathcal{Q}, \mathcal{D})$. This leads to the conclusion that $\mathcal{Q} \subseteq ((\mathcal{Q}, \mathcal{D}), \mathcal{D}) \subseteq (\mu, \mathcal{D})$. On the other hand, let $\nu \in (\mu, \mathcal{D})$. Then

$\nu \vee \mu \in \mathcal{D} \subseteq \mathcal{Q}$. Since $\mathcal{Q}$ is prime and $\mu \notin (\mu, \mathcal{D}) = \mathcal{Q}$, it follows that $\nu \in \mathcal{Q}$. Therefore, $\mathcal{Q} = (\mu, \mathcal{D})$, which shows that, under condition (2), $\mathcal{Q} = (\mu, \mathcal{D})$ is indeed a principal filter.
(3) $\Rightarrow$ (1): Suppose every prime π-filter in $\mathcal{L}$ is principal. Consider a π-filter $\mathcal{G}$ in $\mathcal{L}$, and assume that $\mathcal{G}$ is not principal.

$$\mathfrak{G} = \{\mathcal{U} | \mathcal{U} is a \pi - filter which is not a principal filter\}.$$

Note that $\mathcal{G} \in \mathfrak{G}$, and consequently, there exists an index $i \in \Delta$ such that $\theta \in \mathcal{U}_i$. This implies that $[\theta) \subseteq \mathcal{U}_i$ for some $i \in \Delta$. Since $\mathcal{U}_i$ is a filter, we also know that $\mathcal{U}_i \subseteq \bigcup_{i \in \Delta} \mathcal{U}_i = [\theta)$, so we must have $\mathcal{U}_i = [\theta)$ for some $i \in \Delta$. This leads to a contradiction, since $\mathcal{U}_i$ cannot equal $[\theta)$ by assumption. Now, $\bigcup_{i \in \Delta} \mathcal{U}_i$ serves as an upper bound for the set $\{\mathcal{U}_i\}_{i \in \Delta}$ within $\mathfrak{G}$. By Zorn's Lemma, there exists a maximal filter $\mathcal{N}$ in $\mathfrak{G}$ containing $\mathcal{G}$. Next, let $\mu \notin \mathcal{N}$ and $\pi \notin \mathcal{N}$. Since $\mathcal{N}$ is a filter, we have $\mathcal{N} \subseteq \{\mathcal{N} \vee [\mu)\}^{\varphi}$ and $\mathcal{N} \subseteq \{\mathcal{N} \vee [\pi)\}^{\varphi}$. This implies that $\{\mathcal{N} \vee [\mu)\}^{\varphi} = [\vartheta)$ and $\{\mathcal{N} \vee [\pi)\}^{\varphi} = [\sigma)$ for some $\vartheta, \sigma \in \mathcal{L}$. Therefore, we can conclude that $\{\mathcal{N} \vee [\mu \vee \pi)\}^{\varphi} = \{\mathcal{N} \vee [\mu)\}^{\varphi} \cap \{\mathcal{N} \vee [\pi)\}^{\varphi} = [\vartheta) \cap [\sigma) = [\vartheta \vee \sigma)$. If $\mu \vee \pi \in \mathcal{N}$, then $\mathcal{N} = \mathcal{N}^{\varphi} = [\vartheta \vee \sigma)$, which contradicts the assumption in condition (3). Therefore, it follows that $\mathcal{G}$ must be a principal filter. □

5. The space of prime π-filters

This section explores the topological properties of the set of all prime π-filters within an ADL. It provides several equivalent conditions under which the prime π-filter space of an ADL becomes a Hausdorff space.

Theorem 5.1. *Given an ideal $\mathcal{I}$, let $\mathcal{G}$ be a π-filter of $\mathcal{L}$ with $\mathcal{G} \cap \mathcal{I} = \emptyset$. Then there is $\mathcal{Q} \in Spec_F^{\pi}(\mathcal{L})$ such that $\mathcal{G} \subseteq \mathcal{Q}$ and $\mathcal{Q} \cap \mathcal{I} = \emptyset$.*

Proof. Consider

$$\mathfrak{G} = \{\mathcal{U} \mid \mathcal{U} \text{ is a } \pi\text{-filter, } \mathcal{G} \subseteq \mathcal{U} \text{ and } \mathcal{U} \cap \mathcal{I} = \emptyset\}.$$

It is clear that $\mathcal{G} \in \mathfrak{G}$ and $\mathfrak{G}$ satisfies the hypothesis of Zorn's Lemma. Hence choose a maximal element $\mathcal{N}$ in $\mathfrak{G}$. Let $\mu, \pi \in \mathcal{L}$ be such that $\mu \notin \mathcal{N}$ and $\pi \notin \mathcal{N}$. Then $\mathcal{N} \subset \mathcal{N} \vee [\mu) \subseteq \{\mathcal{N} \vee [\mu)\}^{\varphi}$ and $\mathcal{N} \subset \mathcal{N} \vee [\pi) \subseteq \{\mathcal{N} \vee [\pi)\}^{\varphi}$. As $\mathcal{N}$ is maximal, $\{\mathcal{N} \vee [\mu)\}^{\varphi} \cap \mathcal{I} \neq \emptyset$ and $\{\mathcal{N} \vee [\pi)\}^{\varphi} \cap \mathcal{I} \neq \emptyset$. Choose $\theta \in \{\mathcal{N} \vee [\mu)\}^{\varphi} \cap \mathcal{I}$ and $\vartheta \in \{\mathcal{N} \vee [\pi)\}^{\varphi} \cap \mathcal{I}$. Therefore $\theta \vee \vartheta \in \mathcal{I}$ and $\theta \vee \vartheta \in \{\mathcal{N} \vee [\mu)\}^{\varphi} \cap \{\mathcal{N} \vee [\pi)\}^{\varphi} = \{(\mathcal{N} \vee [\mu)) \cap (\mathcal{N} \vee [\pi))\}^{\varphi} = \{\mathcal{N} \vee [\mu \vee \pi)\}^{\varphi}$. If $\mu \vee \pi \in \mathcal{N}$. Then $\theta \vee \vartheta \in \mathcal{N}^{\varphi} = \mathcal{N}$. Hence $\theta \vee \vartheta \in \mathcal{N} \cap \mathcal{I}$, which is a contradiction. Thus $\mathcal{N} \in Spec_F^{\pi}(\mathcal{L})$. □

Corollary 5.2. *Let $\mathcal{G}$ be a π-filter of $\mathcal{L}$ and $\mu \notin \mathcal{G}$. Then there is $\mathcal{Q} \in Spec_F^{\pi}(\mathcal{L})$ such that $\mu \notin \mathcal{Q}$ and $\mathcal{G} \subseteq \mathcal{Q}$.*

Corollary 5.3. *For any π-filter $\mathcal{G}$ of $\mathcal{L}$, we obtain*

$$\mathcal{G} = \bigcap\{\mathcal{Q} \mid \mathcal{Q} \in Spec^{\pi}_{F}(\mathcal{L}),\ \mathcal{G} \subseteq \mathcal{Q}\}.$$

Corollary 5.4. *$\mathcal{D}$ is equal to the intersection of all members of $Spec^{\pi}_{F}(\mathcal{L})$.*

For every $\mathcal{S} \subseteq \mathcal{L}$, $\mathcal{J}'(\mathcal{S}) = \{\mathcal{Q} \in Spec^{\pi}_{F}(\mathcal{L}) | \mathcal{S} \nsubseteq \mathcal{Q}\}$. In specific for $\mu \in \mathcal{L},\ \mathcal{J}'(\mu) = \mathcal{J}'(\mu)$.

Lemma 5.5. *Every $\mu, \pi \in \mathcal{L}$ gives us*

(1) $\bigcup_{\mu \in \mathcal{L}} \mathcal{J}'(\mu) = Spec^{\pi}_{F}(\mathcal{L})$;

(2) $\mathcal{J}'(\mu) \cap \mathcal{J}'(\pi) = \mathcal{J}'(\mu \vee \pi)$;

(3) $\mathcal{J}'(\mu) \cup \mathcal{J}'(\pi) = \mathcal{J}'(\mu \wedge \pi)$;

(4) $\mathcal{J}'(\mu) = \emptyset$ *if and only if* $\mu \in \mathcal{D}$;

(5) $\mathcal{J}'(0) = Spec^{\pi}_{F}(\mathcal{L})$.

It is simple to see from the above lemma that a topology on $Spec^{\pi}_{F}(\mathcal{L})$ has as its basis $\{\mathcal{J}'(\mu) | \mu \in \mathcal{L}\}$.

Theorem 5.6. *The set of all compact open sets of $Spec^{\pi}_{F}(\mathcal{L})$ is the base $\{\mathcal{J}'(\mu) | \mu \in \mathcal{L}\}$.*

Proof. Let $\mu \in \mathcal{L}$. Let $\mathcal{S} \subseteq \mathcal{L}$ with $\mathcal{J}'(\mu) \subseteq \bigcup_{\pi \in \mathcal{S}} \mathcal{J}'(\pi)$ and $\mathcal{G} = [\mathcal{S})$. If $\mu \notin \mathcal{G}^{\varphi}$. By Corollary 4.2, there is $\mathcal{Q} \in Spec^{\pi}_{F}(\mathcal{L})$ such that $\mathcal{G}^{\varphi} \subseteq \mathcal{Q}$ and $\mu \notin \mathcal{Q}$. Hence $\mathcal{Q} \in \mathcal{J}'(\mu) \subseteq \bigcup_{\pi \in \mathcal{S}} \mathcal{J}'(\pi)$. Therefore $\pi \notin \mathcal{Q}$ for some $\pi \in \mathcal{S}$, which gives a contradiction. Therefore $\mu \in \mathcal{G}^{\varphi}$. Then there is $\theta \in \mathcal{G}$ such that $\mu \in ((\theta, \mathcal{D}), \mathcal{D})$. AS $\mathcal{G} = [\mathcal{S})$, there are $\theta_1, \theta_2, \ldots, \theta_n \in \mathcal{S}$ such that $\theta = \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n$. Hence $\mu \in ((\theta, \mathcal{D}), \mathcal{D}) = ((\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n, \mathcal{D})\mathcal{D})$. It is noted that $\mathcal{J}'(\mu) \subseteq \bigcup_{i=1}^{n} \mathcal{J}'(\theta_i)$. Thus $\mathcal{J}'(\mu)$ is compact in $Spec^{\pi}_{F}(\mathcal{L})$. It suffices to show that every compact open subset of $Spec^{\pi}_{F}(\mathcal{L})$ can be expressed as $\mathcal{J}'(\mu)$ for some $\mu \in \mathcal{L}$. Let $\mathcal{B}$ be a compact open subset of $Spec^{\pi}_{F}(\mathcal{L})$. Since $\mathcal{B}$ is open, so that $\mathcal{B} = \bigcup_{\theta \in \mathcal{S}} \mathcal{J}'(\theta)$ for some $\mathcal{S} \subseteq \mathcal{L}$. Since $\mathcal{B}$ is compact, there are $\theta_1, \theta_2, \ldots, \theta_n \in \mathcal{S}$ such that $\mathcal{B} = \bigcup_{i=1}^{n} \mathcal{J}'(\theta_i) = \mathcal{J}'(\bigwedge_{i=1}^{n} \theta_i)$. Hence $\mathcal{B} = \mathcal{J}'(\mu)$ for some $\mu \in \mathcal{L}$. □

A maximal π-filter is a filter that is maximal within the collection of proper π-filters in an ADL. Since the set of all π-filters forms a distributive lattice, it follows that every maximal π-filter is necessarily a prime π-filter. The following derivation outlines a set of equivalent conditions under which every prime π-filter becomes a minimal prime $\mathcal{D}$-filter.

Theorem 5.7. *In an ADL, the following subsequent statements are equivalent:*

(1) *each member of $Spec^{\pi}_{F}(\mathcal{L})$ is a member of $Min^{\mathcal{D}}_{F}(\mathcal{L})$;*

(2) *$Spec^{\pi}_{F}(\mathcal{L})$ is a T_1-space;*

(3) *each member of* $Spec^{\pi}_{F}(\mathcal{L})$ *is maximal;*

(4) *each member of* $Spec^{\pi}_{F}(\mathcal{L})$ *is minimal;*

(5) *for each* $\mu \in \mathcal{L}, (\mu, \mathcal{D}) \underline{\vee} ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$*;*

(6) $Spec^{\pi}_{F}(\mathcal{L})$ *is a Hausdorff space;*

(7) *for any* $\mu, \pi \in \mathcal{L}$*, there exists* $\psi \in \mathcal{L}$ *such that* $\mu \vee \psi \in \mathcal{D}$ *and*

$$\mathcal{J}'(\pi) \cap \{Spec^{\pi}_{F}(\mathcal{L}) - \mathcal{J}'(\mu)\} = \mathcal{J}'(\pi \vee \psi).$$

Proof. (1) ⇒ (2): Assume (1). Let $\mathcal{Q}, \mathcal{P} \in Spec^{\pi}_{F}(\mathcal{L})$ with $\mathcal{Q} \neq \mathcal{P}$. As $\mathcal{Q}$ and $\mathcal{P}$ are minimal, we have $\mathcal{Q} \nsubseteq \mathcal{P}$ and $\mathcal{P} \nsubseteq \mathcal{Q}$. Select $\mu \in \mathcal{Q} \setminus \mathcal{P}$ and $\pi \in \mathcal{P} \setminus \mathcal{Q}$. As a result, $\mathcal{P}$ lies in the open set $\mathcal{J}'(\mu) \setminus \mathcal{J}'(\pi)$, and $\mathcal{Q}$ lies in the open set $\mathcal{J}'(\pi) \setminus \mathcal{J}'(\mu)$. This shows that $Spec^{\pi}_{F}(\mathcal{L})$ satisfies the conditions of a T_1-space.

(2) ⇒ (3): Assume (2). Let $\mathcal{Q} \in Spec^{\pi}_{F}(\mathcal{L})$.Suppose $\mathcal{P}$ is a maximal π-filter of $\mathcal{L}$ such that $\mathcal{Q} \subset \mathcal{P}$. Since $Spec^{\pi}_{F}(\mathcal{L})$ is a T_1-space, there exist two basic open sets $\mathcal{J}'(\mu)$ and $\mathcal{J}'(\pi)$ such that $\mathcal{P} \in \mathcal{J}'(\mu) \setminus \mathcal{J}'(\pi)$ and $\mathcal{Q} \in \mathcal{J}'(\pi) \setminus \mathcal{J}'(\mu)$. Since $\mu \in \mathcal{Q} \subset \mathcal{P}$, it follows that $\mathcal{P} \notin \mathcal{J}'(\mu)$, leading to a contradiction. Therefore, $\mathcal{Q}$ must be a maximal π-filter.

(3) ⇒ (4): It is straightforward.

(4) ⇒ (5): Assume (4). Then every prime π-filter is also a minimal prime $\mathcal{D}$-filter. Consider the case where $(\mu, \mathcal{D}) \sqcup ((\mu, \mathcal{D}), \mathcal{D}) \neq \mathcal{L}$ for some $\mu \in \mathcal{L}$. In this scenario, there exists a prime π-filter $\mathcal{Q}$ such that $((\mu, \mathcal{D}), \mathcal{D}) \sqcup ((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. This implies $\mu \in ((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. Since $\mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter and $((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$, we obtain a contradiction, as $\mu \notin \mathcal{Q}$. Therefore, we must have $(\mu, \mathcal{D}) \sqcup ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$.

(5) ⇒ (6): Assume (5). Let $\mathcal{Q}, \mathcal{P} \in Spec^{\pi}_{F}(\mathcal{L})$ with $\mathcal{Q} \neq \mathcal{P}$. Let $\mu \in \mathcal{Q}$ be such that $\mu \notin \mathcal{P}$. According to the assumption, we have $((\mu, \mathcal{D}), \mathcal{D}) \sqcup ((\mu, \mathcal{D}), \mathcal{D}) = \mathcal{L}$. Consequently, $0 \in (\mu, \mathcal{D}) \sqcup ((\mu, \mathcal{D}), \mathcal{D}) = \{(\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D})\}^{\varphi}$. Thus, there exists some $\theta \in ((\mu, \mathcal{D}) \vee ((\mu, \mathcal{D}), \mathcal{D}))$ such that $(\theta)^{\circ} \subseteq (0, \mathcal{D}) = \mathcal{D}$. Now, $\theta = \tau \wedge \omega$ for some $\tau \in (\mu, \mathcal{D})$ and $\omega \in ((\mu, \mathcal{D}), \mathcal{D})$, which implies $\tau \vee \mu \in \mathcal{D}$. Suppose $\tau \in \mathcal{Q}$. Since $\mathcal{Q}$ is a π-filter, we get $((\tau, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. Then, $(\tau, \mathcal{D}) \cap (\omega, \mathcal{D}) = ((\tau \wedge \omega), \mathcal{D}) = (\theta, \mathcal{D}) = \mathcal{D}$, which means $(\omega, \mathcal{D}) \subseteq ((\tau, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$. Since $\omega \in ((\mu, \mathcal{D}), \mathcal{D})$, we have $((\mu, \mathcal{D}), \mathcal{D}) \subseteq (\omega, \mathcal{D}) \subseteq \mathcal{Q}$, and since $\mu \in \mathcal{Q}$, it follows that $\mathcal{L} = (\mu, \mathcal{D}) \sqcup ((\mu, \mathcal{D}), \mathcal{D}) \subseteq \mathcal{Q}$, which is a contradiction. Therefore, $\tau \notin \mathcal{Q}$. Hence, $\mathcal{Q} \in \mathcal{J}'(\tau)$. Similarly, $\mathcal{P} \in \mathcal{J}'(\mu)$. Since $\mu \vee \tau \in \mathcal{D}$, we have $\mathcal{J}'(\mu) \cap \mathcal{J}'(\tau) = \mathcal{J}'(\mu \vee \tau) = \emptyset$. Therefore, $Spec^{\pi}_{F}(\mathcal{L})$ is Hausdorff.

(6) ⇒ (7): Assume (6). Then, for each $\theta \in \mathcal{L}$, $\mathcal{J}'(\theta)$ is a compact subset of $Spec^{\pi}_{F}(\mathcal{L})$. Consequently, $\mathcal{J}'(\theta)$ is a clopen set in $Spec^{\pi}_{F}(\mathcal{L})$. Now, let $\mu, \pi \in \mathcal{L}$ be distinct elements. The intersection $\mathcal{J}'(\pi) \cap (Spec^{\pi}_{F}(\mathcal{L}) \setminus \mathcal{J}'(\mu))$ is a compact subset of the compact space $\mathcal{J}'(\pi)$. Since $\mathcal{J}'(\pi)$ is open in $Spec^{\pi}_{F}(\mathcal{L})$, this intersection is a compact open subset of $Spec^{\pi}_{F}(\mathcal{L})$. By Theorem 5.6, there exists an element $\psi \in \mathcal{L}$ such that $\mathcal{J}'(\psi) = \mathcal{J}'(\pi) \cap (Spec^{\pi}_{F}(\mathcal{L}) \setminus \mathcal{J}'(\mu))$. This implies that $\mathcal{J}'(\pi) \cap (Spec^{\pi}_{F}(\mathcal{L}) \setminus \mathcal{J}'(\mu)) = \mathcal{J}'(\pi) \cap \mathcal{J}'(\psi) = \mathcal{J}'(\pi \vee \psi)$. Furthermore, we

have $\mathcal{J}'(\mu \vee \psi) = \mathcal{J}'(\mu) \cap \mathcal{J}'(\psi) = \emptyset$, which implies that $\mu \vee \psi \in \mathcal{D}$.
(7) $\Rightarrow$ (1): For each $\mathcal{Q} \in Spec_F^{\pi}(\mathcal{L})$, choose $\mu, \pi \in \mathcal{L}$ such that $\mu \in \mathcal{Q}$ and $\pi \notin \mathcal{Q}$. Then by the condition (7), there is $\psi \in \mathcal{L}$ such that $\mu \vee \psi \in \mathcal{D}$ and

$$\mathcal{J}'(\pi) \cap \{Spec_F^{\pi}(\mathcal{L}) \setminus \mathcal{J}'(\mu)\} = \mathcal{J}'(\pi \vee \psi).$$

It follows that $\mathcal{Q} \in \mathcal{J}'(\pi) \cap (Spec_F^{\pi}(\mathcal{L}) \setminus \mathcal{J}'(\mu)) = \mathcal{J}'(\pi \vee \psi)$. If $\psi \in \mathcal{Q}$, then $\pi \vee \psi \in \mathcal{Q}$, which contradicts the assumption that $\mathcal{Q} \in \mathcal{J}'(\pi \vee \psi)$. Therefore, $\psi \notin \mathcal{Q}$. Thus, for every $\mu \in \mathcal{Q}$, there exists a $\psi \notin \mathcal{Q}$ such that $\mu \vee \psi \in \mathcal{D}$. This implies that $\mathcal{Q}$ is a minimal prime $\mathcal{D}$-filter of $\mathcal{L}$. $\square$

6. Conclusions

This study established conditions for converting $\mathcal{D}$-filters into regular filters and identified a homomorphism with a dense kernel as a regular filter. A necessary condition for ADLs to become relatively complemented was derived, along with equivalent conditions for ADLs to behave as Boolean algebras. Additionally, a topological investigation of prime π-filters provided. Future work may explore the extension of $\mathcal{D}$-filters to fuzzy $\mathcal{D}$-filters in an ADL, investigating their algebraic and topological properties.

7. Acknowledgments

The authors sincerely appreciate the reviewers' helpful comments and suggestions, which have strengthened the paper

References

[1] G. Birkhoff, *Lattice Theory*, American Mathematical Soc., USA, 1967.

[2] G. Grätzer, *General Lattice Theory*, Academic Press, New York, 1978.

[3] N. Rafi, P. Vijaya Saradhi and M. Balaiah, *The space of minimal prime $D-$filters of almost distributive lattices*, Discussiones Mathematicae - General Algebra and Applications, **44** (2024) 343-368.

[4] N. Rafi, P. Vijaya Saradhi and M. Balaiah, *$w-$filters of almost distributive lattices*, J. Algebr. Syst., **13** No. 2 (2025) 37-52.

[5] G. C. Rao, *Almost Distributive Lattices*, Doctoral Thesis, Andhra University, Visakhapatnam, 1980.

[6] G. C. Rao and S. R. Kumar, *Minimal prime ideals in an ADL*, Int. J. Contemp. Sciences, **4** No. 10 (2009) 475-484.

[7] M. S. Rao and A. P. Kumar, *Regular filters of distributive lattices*, J. Algebr. Syst., **12** No. 2 (2025) 283-299.

[8] U. M. Swamy and G. C. Rao, *Almost distributive lattices*, Jour. Aust. Math. Soc. Series A, **31** (1981) 77-91.

Rafi Noorbhasha

Department of Mathematics,

Bapatla Engineering College,

Bapatla, Andhra Pradesh,

India-522 102.

rafimaths@gmail.com

Mukkamala Sambasiva Rao

Department of Mathematics,

MVGR College of Engineering,

Vizianagaram, India-535 005.

mssraomaths35@rediffmail.com