Algebraic Structures and Their Applications Vol. 13 No. 1 (2026) pp 1-15.

Research Paper

NON-NILPOTENT ELEMENT GRAPH OF A MODULE OVER A COMMUTATIVE RING

PARTHA PROTIM GOGOI AND JITUPARNA GOSWAMI*

Abstract. Let R be a commutative ring with non-zero unity and M be a unitary R-module. Let $Nil(M)$ be the set of all the nilpotent elements of M and $Non(M) = M - Nil(M)$ be the set of all non-nilpotent elements of M. The non-nilpotent element graph of M over R is an undirected simple graph $G_{NN}(M)$ with $Non(M)$ as vertex set and any two distinct vertices x and y are adjacent if and only if $x + y \in Nil(M)$. In this paper, we study the basic properties of the graph $G_{NN}(M)$. We also study the diameter and girth of $G_{NN}(M)$. Further, we determine the domination number and the bondage number of $G_{NN}(M)$. We establish a relation between the diameter and domination number of $G_{NN}(M)$. We also establish a relation between the girth and bondage number of $G_{NN}(M)$.

DOI: 10.22034/as.2025.20681.1679

MSC(2010): Primary: 05C25, 16D10, 05C75.

Keywords: Bondage number, Diameter, Domination number, Girth, Nilpotent elements.

Received: 26 September 2023, Accepted: 9 May 2025.

*Corresponding author

1. Introduction

In the past two decades, research on graphs connected to diverse algebraic structures has grown to be fascinating. This field has recently experienced remarkable growth, which has produced numerous intriguing findings and questions. In 1998 I. Beck [7] first introduced the 'zero - divisor graph' of commutative ring. He was interested in the coloring of such graphs and then this investigation was continued by Anderson and Naseer in [5]. In [4], Anderson and Livingston studied zero-divisor graph whose vertices are non zero divisors. After that, many fundamental papers assigning graphs to rings and modules have been published, for instance see [23] and [1]. In 2008, Anderson and Badawi [3] introduced the total graph of a commutative ring and later this notion was generalized to many algebraic structures, in particular to module over a commutative ring (see [12], [15] and [10]). Atani and Habibi [6] generalized the total graph by introducing the total torsion element graph of a module over a commutative ring. They studied various properties like connectedness, completeness, diameter, girth of this total graph and it's induced subgraphs.

In graph theory, the ideas of dominating sets and domination numbers are crucial. Many books on graph theory concentrate on domination properties, for example, see [17] and [16]. Although, there has not been much investigation on the domination properties of graphs related to algebraic structures like groups, rings, and modules. However, recently some works have appeared on the domination of graphs related to rings and modules, for instance see [22], [20], [21], [11], [13], [14] and [9].

Recently, the second author along with the co-researchers [18] have introduced and studied the non-nilpotent graph of commutative rings. In this paper, we introduce the non-nilpotent element graph $G_{NN}(M)$ of a module M over a commutative ring R. We study the basic properties of the graph $G_{NN}(M)$ and determine the diameter and girth of $G_{NN}(M)$. Further, we investigate the domination number and the bondage number of $G_{NN}(M)$. We establish a relationship between the diameter and domination number of $G_{NN}(M)$ and also a relationship between the girth and bondage number of $G_{NN}(M)$.

2. Preliminaries

In this section, we recall all the basic definitions, concepts and results which are needed in the later sections. Throughout this paper, R is a commutative ring with non-zero unity and M is an unitary R-module, unless otherwise specified.

Let N be a submodule of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$. The annihilator of M denoted by $ann_R(M)$ is $(0 :_R M)$. An R-module is called faithful if $ann_R(M) = 0$. An R-module is called a multiplication module if every submodule N of M has the form $N = IM$ for some ideal I of R. Note that since $I \subseteq (N :_R M)$, then $N = IM \subseteq (N :_R M)M \subseteq N$.

Therefore, we have $N = (N :_R M)M$. If K is a multiplication submodule of M, then for all submodules N of M, $N \cap K = ((N \cap K) : K)K = (N : K)K$. If M is a finitely generated faithful multiplication R-module, then $(IN : M) = I(N : M)$. A proper submodule N of M is prime whenever $rm \in N$ for $r \in R$, $m \in M$ implies $m \in N$ or $r \in (N :_R M)$. Here $P = (N :_R M)$ is a prime ideal of R and N is called a P-prime submodule of M.

An ideal I of R is called nilpotent if $I^k = 0$ for some positive integer k and an element r of R is nilpotent if $r^k = 0$ for some $k \in \mathbb{N}$. We denote all the nilpotent elements of R by $Nil(R)$. If $Nil(R) = 0$ for a ring R, then we call it a reduced ring. A submodule N of M is called nilpotent if $(N :_R M)^k N = 0$ for some positive integer k. We say $m \in M$ is nilpotent element if Rm is a nilpotent submodule of M [2]. The set of all nilpotent elements of M is denoted by $Nil(M)$ and $Non(M) = M - Nil(M)$. In general, $Nil(M)$ is not necessarily a submodule of M, but if M is faithful, then $Nil(M)$ is a submodule of M ([2], Theorem 6). If I is a nilpotent ideal of R, or N is a nilpotent submodule of M, then IN is a nilpotent submodule of M. Hence, if $r \in Nil(R)$ or $m \in Nil(M)$, then $rm \in Nil(M)$. If M is a faithful multiplication module, then $Nil(M) = Nil(R)M$. For any undefined terminology in rings and modules we refer to [19].

By a graph G, we mean a simple undirected graph without loops. For a graph G, we denote the set of vertices by $V(G)$ and edges by $E(G)$. We call a graph finite if both $V(G)$ and $E(G)$ are finite sets, and we use the symbol $|V(G)|$ to denote the number of vertices in the graph G. We say that G is a null graph if $E(G) = \phi$.

A graph G is called connected if for any two distinct vertices x, y there is a path from x to y, otherwise G is disconnected. A graph G is complete if any two distinct vertices of the graph are adjacent. We denote a complete graph on n vertices by K_n. A subset S of $V(G)$ is called independent set if no two vertices of S are adjacent.

A graph G is bipartite if $V(G)$ is the union of two disjoint independent sets, say, X and Y called the partite sets of G. If G contains every line joining X and Y, then G is said to be a complete bipartite graph. If $|X| = m$ and $|Y| = n$, then we denote the complete bipartite graph by $K_{m,n}$. Graphs of the form $K_{1,n}$ are called star graph. A graph G is called r-partite if $V(G)$ can be expressed as the union of r independent sets, say, $V_1, V_2, ..., V_r$ called the partite sets of G. If G contains every line joining V_i and $V_j (i \neq j, 1 \leq i, j \leq r)$, then G is said to be a complete r-partite graph.

If the vertices x and y are connected in G, the distance $d(x, y)$ is defined as the length of the shortest path between x and y. If they are not connected, $d(x, y) = \infty$. The diameter of the graph G is defined as

$$diam(G) = sup\{d(x, y) | x, y \in G\}.$$

A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle with n vertices is denoted by C_n. The girth of the graph G, denoted by $gr(G)$ is the length of the shortest cycle in G and $gr(G) = \infty$ if G has no cycles.

A subset S of V is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is defined to be the minimum cardinality of a dominating set in G and such a dominating set is called γ-set of G. If G is a trivial graph, then $\gamma(G) = 0$. The bondage number $b(G)$ is the minimum number of edges whose removal increases the domination number. For basic definitions and results in domination we refer to [16] and for any undefined graph-theoretic terminology we refer to [25, 8].

The following Lemmas are useful in the later sections.

Lemma 2.1. [8]*:*

(i) If G is a graph of order n, then $1 \leq \gamma(G) \leq n$. A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n-1$; while $\gamma(G) = n$ if and only if $G \cong \overline{K_n}$.

(ii) $\gamma(K_n) = 1$ for a complete graph K_n, but the converse is not true in general and $\gamma(\overline{K_n}) = n$ for a null graph $\overline{K_n}$.

(iii) Let G be a complete r-partite graph $(r \geq 2)$ with partite sets $V_1, V_2, ..., V_r$. If $|V_i| \geq 2$ for $1 \leq i \leq r$, then $\gamma(G) = 2$; because one vertex of V_1 and one vertex of V_2 dominate G. If $|V_i| = 1$ for some i, then $\gamma(G) = 1$.

(iv) $\gamma(K_{1,n}) = 1$ for a star graph $K_{1,n}$.

(v) If G is a union of disjoint subgraphs $G_1, G_2, ..., G_k$, then $\gamma(G) = \gamma(G_1) + \gamma(G_2) + ... + \gamma(G_k)$.

Lemma 2.2. [16]*:*

(i) If G is a simple graph of order n, then $1 \leq b(G) \leq n-1$.

(ii) $b(K_n) = n-1$ for a complete graph K_n, but the converse is not true in general and $b(\overline{K_n}) = 0$ for a null graph $\overline{K_n}$.

(iii) Let G be a complete r-partite graph with partite sets $V_1, V_2, ..., V_r$. Then $b(G) = min\{|V_1|, |V_2|, ..., |V_r|\}$. In particular, $b(K_{m,n}) = min\{m, n\}$.

(iv) If G is a union of disjoint subgraphs $G_1, G_2, ..., G_k$, then $b(G) = min\{b(G_1), b(G_2), ..., b(G_k)\}$.

(v) Let C_n and P_n be an n-cycle and a path with n vertices, respectively. Then $b(P_n) = 1$ and $b(C_n) = 2$.

3. Non-nilpotent element graph $G_{NN}(M)$

In this section we define the non-nilpotent graph $G_{NN}(M)$ of a module M. We investigate the basic properties of the graph $G_{NN}(M)$. We also obtain several interesting results by considering the cases such as $Nil(M)$ is a submodule of M and $Nil(M)$ is a prime submodule of M. We begin with the following definition.

Definition 3.1. Let R be a commutative ring and M be an R-module. The non-nilpotent element graph $G_{NN}(M)$ of M is an undirected simple graph defined by taking $Non(M)$ as the vertex set and two distinct vertices x and y are adjacent if and only if $x+y \in Nil(M)$.

Now, we discuss the following example.

Example 3.2. Let us now consider the $\mathbb{Z}$-module $\mathbb{Z}_5$. In this module for any $\overline{x} \neq \overline{0}$, the submodule generated by $\overline{x}$ i.e. $<\overline{x}>$ is equal to $\mathbb{Z}_5$. Hence $(<\overline{x}>:_{\mathbb{Z}} \mathbb{Z}_5) = \mathbb{Z}$. Also, $(<\overline{x}>:_{\mathbb{Z}} \mathbb{Z}_5) <\overline{x}> \neq \overline{0}$, for any $\overline{x} \neq \overline{0}$. Therefore, we have $Nil(\mathbb{Z}_5) = \{\overline{0}\}$ and so $Non(\mathbb{Z}_5) = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$. Thus, the graph $G_{NN}(\mathbb{Z}_5)$ is union of two disjoint complete bipartite graph $K_{1,1}$.

Proposition 3.3. *Let M be a module over a commutative ring R such that $Nil(M)$ is a submodule of M. If two distinct vertices of $G_{NN}(M)$ are connected, then there exists a path of length 2 or 1 between them. In particular, if $G_{NN}(M)$ is connected, then diam($G_{NN}(M)$) ≤ 2.*

Proof. Let x, y be two distinct vertices of $G_{NN}(M)$ which are connected. If x, y are adjacent in $G_{NN}(M)$, then obviously we get a path of length 1 between them.

If x, y are not adjacent in $G_{NN}(M)$, then there exists a path $x \sim x_1 \sim x_2 \sim ... \sim x_{n-1} \sim y$ of length $n(>1)$ in $G_{NN}(M)$.
Now if n is odd, then we have
$x + y = (x + x_1) - (x_1 + x_2) + ... - (x_{n-2} + x_{n-1}) + (x_{n-1} + y) \in Nil(M)$, a contradiction.
Therefore, n must be even and so we have
$x - y = (x + x_1) - (x_1 + x_2) + ... + (x_{n-2} + x_{n-1}) - (x_{n-1} + y) \in Nil(M)$.

Hence, there exists a path $x \sim (-y) \sim y$ of length 2 between x and y . Similarly we also get a path $x \sim (-x) \sim y$ of length 2 between x and y. Thus, if $G_{NN}(M)$ is connected, then diam($G_{NN}(M)$) ≤ 2. □

Proposition 3.4. *Let M be a module over a commutative ring R such that $Nil(M)$ is a submodule of M. Then the following conditions are equivalent :*

(1) *$G_{NN}(M)$ is connected.*
(2) *Either $x + y \in Nil(M)$ or $x - y \in Nil(M)$ for all $x, y \in Non(M)$.*
(3) *Either $x + y \in Nil(M)$ or $x + 2y \in Nil(M)$ for all $x, y \in Non(M)$. In particular, either $2x \in Nil(M)$ or $3x \in Nil(M)$ (not both) for all $x \in Non(M)$.*

Proof. (1) $\Rightarrow$ (2) Let $x, y \in Non(M)$ such that $x + y \notin Nil(M)$. If $x = y$, then $x - y = x + (-y) = 0 \in Nil(M)$. Suppose that $x \neq y$. So $x \sim (-y) \sim y$ is a path from x to y by Proposition 3.3 . Hence $x - y = x + (-y) \in Nil(M)$.

(2) $\Rightarrow$ (3) Let $x, y \in Non(M)$ such that $x + y \notin Nil(M)$. Since $(x + y) - y = x \notin Nil(M)$. So by our assumption $(x + y) + y = x + 2y \in Nil(M)$. In particular, if $x \in Non(M)$, then either $2x = x + x \in Nil(M)$ or $3x = x + 2x \in Nil(M)$.

Now, $3x - 2x = x \notin Nil(M)$, so $2x$ and $3x$ both can not be in $Nil(M)$ since $Nil(M)$ is a submodule of M.

(3) $\Rightarrow$ (1) Let x and y be two distinct elements of $Non(M)$ such that they are not adjacent in $G_{NN}(M)$. So $x + y \in Non(M)$. Then by assumption, $x + 2y \in Nil(M)$. If $2y \in Nil(M)$, then $x = (x + 2y) - 2y \in Nil(M)$, since $Nil(M)$ is a submodule of M. It is a contradiction. So $2y \notin Nil(M)$. Thus $3y \in Nil(M)$, by assumption. If $x = 2y$, then $x + y = 2y + y = 3y \in Nil(M)$, a contradiction. Therefore $x \neq 2y$ and so $x \sim 2y \sim y$ is a path from x to y. Therefore, $G_{NN}(M)$ is connected. □

Example 3.5. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{25}$. The submodule generated by $\overline{x}$ is denoted by $<\overline{x}>$. It is clear that $<\overline{x}> = \mathbb{Z}_{25}$ and hence $(<\overline{x}>:_{\mathbb{Z}} \mathbb{Z}_{25}) = \mathbb{Z}$ for any $x \neq \overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}$. If $x = 5k$, for some integer k, $<\overline{x}> = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}\}$ and so $(<\overline{x}>:_{\mathbb{Z}} \mathbb{Z}_{25}) = 5\mathbb{Z}$. The ideal $5\mathbb{Z}$ is not nilpotent, but $(5\mathbb{Z})^2 = 5^2\mathbb{Z}$ and $5^2\mathbb{Z} <\overline{x}> = \overline{0}$ for $x = \overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}$. This implies that $Nil_{\mathbb{Z}}(\mathbb{Z}_{25}) = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}\}$. Clearly $Nil_{\mathbb{Z}}(\mathbb{Z}_{25})$ is a proper submodule of $\mathbb{Z}_{25}$.

We now partition the set $Non(\mathbb{Z}_{25})$ in two sets $X = \{\overline{1}, \overline{4}, \overline{6}, \overline{9}, \overline{11}, \overline{14}, \overline{16}, \overline{19}, \overline{21}, \overline{24}\}$ and $Y = \{\overline{2}, \overline{3}, \overline{7}, \overline{8}, \overline{12}, \overline{13}, \overline{17}, \overline{18}, \overline{22}, \overline{23}\}$. The induced subgraphs of $G_{NN}(\mathbb{Z}_{25})$ by the sets X and Y are disjoint. Therefore, the graph $G_{NN}(\mathbb{Z}_{25})$ is disconnected.

Again, for $\overline{1}, \overline{7} \in Non(\mathbb{Z}_{25})$, we have $\overline{1} + \overline{7}, \overline{1} - \overline{7} \notin Nil_{\mathbb{Z}}(\mathbb{Z}_{25})$ which shows that $G_{NN}(\mathbb{Z}_{25})$ is disconnected as mentioned in Proposition 3.4 .

Lemma 3.6. *([2], Proposition 4(4)) Let R be a ring and M be an R-module. Let I be an ideal of R and N be a submodule of M. If I is a nilpotent ideal of R or N is nilpotent in M, then IN is nilpotent in M.*

Proposition 3.7. *Let M be a module over a commutative ring R with unity such that $Nil(M)$ is a submodule of M, $|Nil(M)| = \lambda$ and $|\frac{M}{Nil(M)}| = \mu$. If $2 = 1_R + 1_R \in Nil(R)$, then $G_{NN}(M)$ is the union of $\mu - 1$ disjoint K_λ's.*

Proof. For every $x \in Non(M)$ and $y \in Nil(M)$, $x+y \in Non(M)$, since $Nil(M)$ is a submodule of M. So $x + Nil(M) \subseteq Non(M)$ for every $x \in Non(M)$.

If $2 \in Nil(R)$, then $2x \in Nil(M)$ for any $x \in Non(M)$ by Lemma 3.6 . Therefore, for $x \in Non(M)$ and x_1, $x_2 \in Nil(M)$, we have $(x + x_1) + (x + x_2) = 2x + x_1 + x_2 \in Nil(M)$, since $Nil(M)$ is a submodule of M. Thus, the coset $x + Nil(M)$ induces a complete graph with λ elements, i.e. K_λ.

Now if $x + Nil(M)$ and $y + Nil(M)$ are distinct cosets for some $x,\ y \in Non(M)$ and $x+x_1,\ y+y_1$ are adjacent for some $x_1,\ y_1 \in Nil(M)$, then $x+y = (x+x_1)+(y+y_1)-(x_1+y_1) \in Nil(M)$ and hence $x - y = x + y - 2y \in Nil(M)$ as $2y \in Nil(M)$ and $Nil(M)$ is a submodule of M. Thus, $x + Nil(M) = y + Nil(M)$, which is a contradiction.

Hence, $G_{NN}(M)$ is the union of $\mu - 1$ disjoint (induced) subgraphs $x + Nil(M)$ and so $G_{NN}(M)$ is the union of $\mu - 1$ disjoint K_λ's. □

Example 3.8. Let $R = \mathbb{Z}_8$. Then $M = R$ is a module over itself. Here $Nil(M) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$ and clearly, $Nil(M)$ is a submodule of M. Also $2 \in Nil(R)$. Now, $Non(M) = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$, $|Nil(M)| = 4$ and $|\dfrac{M}{Nil(M)}| = 2$. We can observe that the graph $G_{NN}(M)$ is the complete graph K_4.

Lemma 3.9. *([2], Theorem 6 (2)) Let R be a ring and M be a faithful multiplication $R-$module, then $Nil(M) = Nil(R)M$.*

Lemma 3.10. *([24], Theorem 9) Let A and B be ideals of a ring R and M be a finitely generated multiplication $R-$module. Then $AM \subseteq BM$ if and only if $A \subseteq B + ann(M)$.*

Lemma 3.11. *Let M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M and $m \in Non(M)$. Then $2m \in Nil(M)$ if and only if $2 \in Nil(R)$.*

Proof. If $2 \in Nil(R)$, then $2m \in Nil(M)$ by Lemma 3.6 .
Now, if $2m \in Nil(M)$, then $2 \in (Nil(M) :_R M)$, since $Nil(M)$ is a prime submodule of M and $m \in Non(M)$. Again $Nil(M) = Nil(R)M$ by Lemma 3.9 . Also $(Nil(M) :_R M) = Nil(R)(M :_R M)$ by Lemma 3.10 .

Hence, $2 \in (Nil(M) :_R M) = Nil(R)(M :_R M) = Nil(R)$. □

Proposition 3.12. *Let R be a commutative ring with unity and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \lambda$ and $|\dfrac{M}{Nil(M)}| = \mu$. If $2 = 1_R + 1_R \notin Nil(R)$, then $G_{NN}(M)$ is the union of $\dfrac{\mu - 1}{2}$ disjoint $K_{\lambda,\lambda}$'s.*

Proof. Let $x \in Non(M)$ and $x_1, x_2 \in Nil(M)$, then $(x + x_1) + (x + x_2) = 2x + (x_1 + x_2) \in Nil(M)$. So $2x \in Nil(M)$ and $2 \in Nil(R)$ by Lemma 3.11, which is a contradiction. Hence no two distinct elements of $x+Nil(M)$ are adjacent. Now, the cosets $x+Nil(M)$ and $-x+Nil(M)$ are disjoint as $2x \notin Nil(M)$. Also, each element of $x + Nil(M)$ is adjacent to each element of $-x+Nil(M)$, since $Nil(M)$ is a submodule of M . Thus, $(x+Nil(M))\cup(-x+Nil(M))$ forms a complete bipartite subgraph $K_{\lambda,\lambda}$ of $G_{NN}(M)$.

On the other hand, if $x+x_1$ is adjacent to $x'+x_2$ for some $x,x'\in Non(M)$ and $x_1,x_2\in Nil(M)$, then $(x+x_1)+(x'+x_2)\in Nil(M)$. So $x+x'\in Nil(M)$, since $Nil(M)$ is a submodule of M . Therefore, $x+Nil(M)=-x'+Nil(M)$. Thus, $G_{NN}(M)$ is the union of $\dfrac{\mu-1}{2}$ disjoint complete bipartite(induced) subgraphs $K_{\lambda,\lambda}$. □

Example 3.13. Let $R=\mathbb{Z}_5$. Then $R=M$ is a module over itself. Since $ann_R(M)=0$, so M is a finitely generated faithful multiplication module. Also, $Nil(M)=\{\overline{0}\}$ and so $Non(M)=\{\overline{1},\overline{2},\overline{3},\overline{4}\}$. Clearly, $Nil(M)$ is a prime submodule of M and $2=1_R+1_R\notin Nil(R)$. Thus, we have $|Nil(M)|=1$ and $|\dfrac{M}{Nil(M)}|=5$. We can see that the graph $G_{NN}(M)$ is the union of two disjoint complete bipartite graphs $K_{1,1}$.

Proposition 3.14. *Let R be a commutative ring with unity and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. Then*

(1) *$G_{NN}(M)$ is complete if and only if either $|\dfrac{M}{Nil(M)}|=2$ or $|\dfrac{M}{Nil(M)}|=|M|=3$.*

(2) *$G_{NN}(M)$ is connected if and only if either $|\dfrac{M}{Nil(M)}|=2$ or $|\dfrac{M}{Nil(M)}|=3$.*

(3) *$G_{NN}(M)$ is totally disconnected if and only if $Nil(M)=\{0\}$ and $2\in Nil(R)$.*

Proof. (1) If $G_{NN}(M)$ is complete, then $G_{NN}(M)$ is a complete graph K_λ, where $\lambda=|Nil(M)|$ or the complete bipartite graph $K_{1,1}$ by Proposition 3.7 and Proposition 3.12.

If $2\in Nil(R)$, $\mu-1=1$, so $\mu=|\dfrac{M}{Nil(M)}|=2$.

If $2\notin Nil(R)$, $\lambda=1$, $\dfrac{\mu-1}{2}=1$, so $Nil(M)=0$ and $\mu=|\dfrac{M}{Nil(M)}|=|M|=3$.

Conversely, we assume either $|\dfrac{M}{Nil(M)}|=2$ or $|\dfrac{M}{Nil(M)}|=|M|=3$.

If $|\dfrac{M}{Nil(M)}|=2$, then $\dfrac{M}{Nil(M)}=\{Nil(M),x+Nil(M)\}$, where $x\in Non(M)$. Therefore, $x+Nil(M)=-x+Nil(M)$ and so $2x\in Nil(M)$. This gives $2\in Nil(R)$ by Lemma 3.11. Thus, $G_{NN}(M)$ contains one K_λ and hence $G_{NN}(M)$ is complete by Proposition 3.7.

If $|\dfrac{M}{Nil(M)}|=|M|=3$, $Nil(M)=\{0\}$. Now, if $2\in Nil(R)$, then $2x\in Nil(M)$ for all $x\in M$ by Lemma 3.11 . This yields $2x=0$ for all $x\in M$, a contradiction as M is a cyclic group of order 3. Therefore, $2\notin Nil(R)$ and hence by Proposition 3.12, we have $G_{NN}(M)$ is the complete graph $K_{1,1}$.

(2) Let us suppose that $G_{NN}(M)$ is connected.

If $2\in Nil(R)$ then the graph $G_{NN}(M)$ is a K_λ by Proposition 3.7 and hence $\mu-1=1$, where $\mu=|\dfrac{M}{Nil(M)}|$ and $\lambda=|Nil(M)|$. This gives $\mu=|\dfrac{M}{Nil(M)}|=2$.

If $2 \notin Nil(R)$, then the graph $G_{NN}(M)$ is a $K_{\lambda,\lambda}$ by Proposition 3.12 and hence $\dfrac{\mu-1}{2} = 1$. Therefore, $\mu = |\dfrac{M}{Nil(M)}| = 3$. Thus, either $|\dfrac{M}{Nil(M)}| = 2$ or $|\dfrac{M}{Nil(M)}| = 3$.

Conversely, let us assume that either $|\dfrac{M}{Nil(M)}| = 2$ or $|\dfrac{M}{Nil(M)}| = 3$.

If $|\dfrac{M}{Nil(M)}| = 2$, $G_{NN}(M)$ is complete by Part(1) and thus connected. So we assume that $|\dfrac{M}{Nil(M)}| = 3$.

If $2 \in Nil(R)$, then $2x \in Nil(M)$ for all x by Lemma 3.11. Now, we have $\dfrac{M}{Nil(M)} = \{Nil(M), x + Nil(M), y + Nil(M)\}$ where $x,\ y \in Non(M)$. So, $x + y + Nil(M) = Nil(M)$ as $\dfrac{M}{Nil(M)}$ is a cyclic group of order 3. This implies $x + y \in Nil(M)$ and $2y \in Nil(M)$. Now, $x+y-2y = x-y \in Nil(M)$. Which gives $x+Nil(M) = y+Nil(M)$, a contradiction. Therefore, $2 \notin Nil(R)$. Thus, $G_{NN}(M)$ is a complete bipartite graph $K_{\lambda,\lambda}$, where $\lambda = |Nil(M)|$. Hence $\dfrac{M}{Nil(M)} = \{Nil(M), x + Nil(M), 2x + Nil(M)\}$, where $x \in Non(M)$ and $3x \in Nil(M)$, as $2x \notin Nil(M)$, by Lemma 3.11.

Now, we show that $G_{NN}(M)$ is connected. Let x_1 and x_2 be two distinct elements of $G_{NN}(M)$ such that they are not adjacent. Therefore, $x_1 + x_2 \notin Nil(M)$. Without loss of generality, we may assume that $x_1 + Nil(M) \neq x + Nil(M)$. So, $x_1 + Nil(M) = 2x + Nil(M)$.

If $x_2 + Nil(M) = x + Nil(M)$, then

$$x_1 + x_2 + Nil(M) = (2x + Nil(M)) + (x + Nil(M)) = 3x + Nil(M) = Nil(M).$$

This gives $x_1 + x_2 \in Nil(M)$, which is a contradiction. Hence, we take $x_2 + Nil(M) = 2x + Nil(M)$. Now, $x_1 + x_1 + x_2 - 6x = (x_1 - 2x) + (x_1 - 2x) + (x_2 - 2x) \in Nil(M)$ and $x_1 + x_2 - 6x + x_2 = (x_1 - 2x) + (x_2 - 2x) + (x_2 - 2x) \in Nil(M)$, as $Nil(M)$ is a submodule of M. Thus, $x_1 \sim (x_1 + x_2 - 6x) \sim x_2$ is a path from x_1 to x_2 and hence $G_{NN}(M)$ is connected.

(3) $G_{NN}(M)$ is totally disconnected if and only if it is the union of disjoint K_1's. Now, if $G_{NN}(M)$ is the union of disjoint K_1's, then $|Nil(M)| = 1$ which gives $Nil(M) = \{0\}$. Therefore, by Proposition 3.7 and Proposition 3.12, we have $2 \in Nil(R)$. Conversely, if $Nil(M) = \{0\}$ and $2 \in Nil(R)$, then $G_{NN}(M)$ is the disjoint union of K_λ's and $\lambda = |Nil(M)| = 1$. □

4. Diameter and Girth of $G_{NN}(M)$

In this section, we discuss the diameter and the girth of the non-nilpotent graph $G_{NN}(M)$. We begin with the following Proposition.

Proposition 4.1. *Let R be a ring and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. Then*

(1) *$diam(G_{NN}(M)) = 0$ if and only if $Nil(M) = \{0\}$ and $|M| = 2$.*
(2) *$diam(G_{NN}(M)) = 1$ if and only if $Nil(M) \neq \{0\}$ and $|\frac{M}{Nil(M)}| = 2$ or $Nil(M) = \{0\}$ and $|M| = 3$.*
(3) *$diam(G_{NN}(M)) = 2$ if and only if $Nil(M) \neq \{0\}$ and $|\frac{M}{Nil(M)}| = 3$.*
(4) *Otherwise, $diam(G_{NN}(M)) = \infty$.*

Proof. (1) If $Nil(M) = 0$ and $|M| = 2$, then $|Non(M)| = 1$, so $diam(G_{NN}(M)) = 0$.

Conversely, if $diam(G_{NN}(M)) = 0$, $G_{NN}(M)$ contains one K_1 by Proposition 3.7. So, $\lambda = 1$ and $\mu = 2$. Thus $Nil(M) = \{0\}$ and $|M| = 2$.

(2) $diam(G_{NN}(M)) = 1$ if and only if $G_{NN}(M)$ is a complete graph which is a $K_2 = K_{1,1}$ by Proposition 3.7 and Proposition 3.12. The proof follows from Proposition 3.14(1).

(3) If $diam(G_{NN}(M)) = 2$, $G_{NN}(M)$ is a $K_{2,2}$ or K_4. So, $\lambda = |Nil(M)| = 2$ and $\mu = |\frac{M}{Nil(M)}| = 3$ or $\lambda = |Nil(M)| = 4$ and $\mu = |\frac{M}{Nil(M)}| = 2$. In each case, $Nil(M) \neq \{0\}$. Also, if $|\frac{M}{Nil(M)}| = 2$, then $diam(G_{NN}(M)) = 1$ by Part 2 above. Therefore, $|\frac{M}{Nil(M)}| = 3$.

Conversely, if $Nil(M) \neq \{0\}$ and $|\frac{M}{Nil(M)}| = 3$, $G_{NN}(M)$ is connected by Proposition 3.14(2) and $diam(G_{NN}(M)) \leq 2$ by Proposition 3.3. So by Part 1 and Part 2 above, we have $diam(G_{NN}(M)) = 2$.

(4) First, we assume that $Nil(M) = \{0\}$.

If $2 \in Nil(R)$, then $G_{NN}(M)$ is totally disconnected graph by Proposition 3.14(3) and so $diam(G_{NN}(M)) = \infty$.

If $2 \notin Nil(R)$, then $G_{NN}(M)$ is the union of $\frac{\mu - 1}{2}$ disjoint $K_{\lambda,\lambda}$ by Proposition 3.12 where $\lambda = |Nil(M)| = 1$. So, $diam(G_{NN}(M)) = 1$ or ∞.

Now, let us suppose that $Nil(M) \neq \{0\}$. Since $diam(G_{NN}(M)) \neq 0, 1$ or 2, so $|\frac{M}{Nil(M)}| \neq 2$ or 3, by Part 2 and Part 3 above. Hence, $G_{NN}(M)$ is disconnected by Proposition 3.14(2). Therefore, $diam(G_{NN}(M)) = \infty$. □

Proposition 4.2. *Let R be a ring and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. Then the following holds:*

(1) *$gr(G_{NN}(M)) = 3$ if and only if $2 \in Nil(R)$ and $|Nil(M)| \geq 3$.*
(2) *$gr(G_{NN}(M)) = 4$ if and only if $2 \notin Nil(R)$ and $|Nil(M)| \geq 2$.*
(3) *Otherwise, $gr(G_{NN}(M)) = \infty$.*

Proof. (1) If $2 \in Nil(R)$ and $|Nil(M)| \geq 3$, then $G_{NN}(M)$ is disjoint union of K_λ, $\lambda \geq 3$ by Proposition 3.7. Therefore, K_λ contains a 3-cycle which implies $gr(G_{NN}(M)) = 3$.

Conversely, if $gr(G_{NN}(M)) = 3$, then $G_{NN}(M)$ can not be union of complete bipartite graphs and so $2 \in Nil(R)$ by Proposition 3.12. Therefore, $G_{NN}(M)$ is the union of complete graphs K_λ's by Proposition 3.7 and hence $\lambda = |Nil(M)| \geq 3$.

(2) If $2 \notin Nil(R)$ and $|Nil(M)| \geq 2$, then $G_{NN}(M)$ is union of complete bipartite graph $K_{\lambda,\lambda}$, $\lambda \geq 2$ by Proposition 3.12. So $gr(G_{NN}(M)) = 4$.

Conversely, if $gr(G_{NN}(M)) = 4$, then $G_{NN}(M)$ is the union of complete bipartite graphs $K_{\lambda,\lambda}$ with $\lambda = |Nil(M)| \geq 2$ and so $2 \notin Nil(R)$ by Proposition 3.12.

(3) If $G_{NN}(M)$ contains a cycle, then $G_{NN}(M)$ contains either complete graphs or complete bipartite graphs. So if $G_{NN}(M)$ contains a cycle, then $gr(G_{NN}(M))$ is either 3 or 4. Hence, in all the other cases, $gr(G_{NN}(M)) = \infty$. □

5. Some Domination parameters of $G_{NN}(M)$

In this section we study some domination parameters such as domination number and bondage number of the the non-nilpotent graph $G_{NN}(M)$. We establish a relationship between the diameter and domination number of $G_{NN}(M)$. We also establish a relationship between the girth and bondage number of $G_{NN}(M)$. We begin with the following proposition.

Proposition 5.1. *Let R be a ring and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \lambda \geq 2$ and $|\frac{M}{Nil(M)}| = \mu$, then $\gamma(G_{NN}(M)) = \mu - 1$.*

Proof. Let us consider the following two cases for $Nil(R)$.

Case 1: Suppose that $2 = 1_R + 1_R \in Nil(R)$. Then we have from Proposition 3.7 that the graph $G_{NN}(M)$ is the union $\mu - 1$ disjoint K_λ's and we know that $\gamma(K_\lambda) = 1$. Thus $\gamma(G_{NN}(M)) = (\mu - 1) \times 1 = \mu - 1$.

Case 2: Suppose that $2 = 1_R + 1_R \notin Nil(R)$. Then again we have from Proposition 3.12 that the graph $G_{NN}(M)$ is the union of $\frac{\mu - 1}{2}$ disjoint $K_{\lambda,\lambda}$'s and we know that $\gamma(K_{\lambda,\lambda}) = 2$. Thus $\gamma(G_{NN}(M)) = (\frac{\mu - 1}{2}) \times 2 = \mu - 1$. Hence, $\gamma(G_{NN}(M)) = \mu - 1$. □

Proposition 5.2. *Let R be a reduced ring and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M with $|\frac{M}{Nil(M)}| = \mu$, Then $\gamma(G_{NN}(M)) = \frac{\mu - 1}{2}$.*

Proof. Since R is a reduced ring , so we have $Nil(R) = \{0\}$. Again, $Nil(M) = Nil(R)M$ by Lemma 3.9 . So, $Nil(M) = \{0\}$. Therefore, $|\frac{M}{Nil(M)}| = |M| = \mu$.

Now, $2 = 1_R + 1_R \notin Nil(R)$ and Proposition 3.12 implies that the graph $G_{NN}(M)$ is the union of $\frac{\mu-1}{2}$ disjoint $K_{1,1}$'s. Thus $\gamma(G_{NN}(M)) = (\frac{\mu-1}{2}) \times 1 = \frac{\mu-1}{2}$. □

Example 5.3. As mentioned in the Example 3.13, let us consider the module $R = M = \mathbb{Z}_5$ over itself. Since $ann_R(M) = 0$, so M is a finitely generated faithful multiplication module. Also, $Nil(M) = \{\overline{0}\}$ and so $Non(M) = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$. Clearly, $Nil(M)$ is a prime submodule of M. Thus, we have $\lambda = |Nil(M)| = 1$ and $\mu = |\frac{M}{Nil(M)}| = 5$. Also, we observe that the graph $G_{NN}(M)$ is the union of two disjoint complete bipartite graphs $K_{1,1}$. Hence,

$$\gamma(G_{NN}(M)) = \gamma(K_{1,1} \cup K_{1,1}) = \gamma(K_{1,1}) + \gamma(K_{1,1}) = 1 + 1 = 2 = \frac{\mu-1}{2}.$$

Proposition 5.4. *Let R be a ring and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M with $M - Nil(M) \neq \phi$. Then $\gamma(G_{NN}(M)) = 1$ if and only if $|\frac{M}{Nil(M)}| = 2$ or $|\frac{M}{Nil(M)}| = |M| = 3$.*

Proof. Let us assume that $\gamma(G_{NN}(M)) = 1$. Then, clearly $G_{NN}(M)$ is connected.

If $2 \in Nil(R)$, Then $\mu - 1 = 1$ by Proposition 3.7, and so $\mu = 2$, where $\mu = |\frac{M}{Nil(M)}|$. Thus $|\frac{M}{Nil(M)}| = 2$.

If $2 \notin Nil(R)$, Then $\frac{\mu-1}{2} = 1$ and so $\mu = |\frac{M}{Nil(M)}| = 3$, by Proposition 3.12 . Also, by our assumption, $\lambda = |Nil(M)| = 1$ and hence $Nil(M) = \{0\}$. Thus $|\frac{M}{Nil(M)}| = |M| = 3$.

Conversely, let us assume that $|\frac{M}{Nil(M)}| = 2$ or $|\frac{M}{Nil(M)}| = |M| = 3$. Then by Proposition 3.14(1), $G_{NN}(M)$ is complete and hence $\gamma(G_{NN}(M)) = 1$. □

Corollary 5.5. *Let R be a ring and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M with $M - Nil(M) \neq \phi$. Then (1) $diam(G_{NN}(M)) = 1$ if and only if $\gamma(G_{NN}(M)) = 1$.*
(2) $diam(G_{NN}(M)) = 2$ if and only if $\gamma(G_{NN}(M)) = 2$.

Proof. (1) It is clear by Proposition 4.1(2) and Proposition 5.4.

(2) If $diam(G_{NN}(M)) = 2$, then $Nil(M) \neq \{0\}$ and $|\frac{M}{Nil(M)}| = 3$, by Proposition 4.1(3). Hence $G_{NN}(M)$ is connected, by Proposition 3.14(2). Therefore $G_{NN}(M)$ is a complete bipartite graph $K_{\lambda,\lambda}$ with $\lambda \geq 2$. So $\gamma(G_{NN}(M)) = 2$.

Conversely, if $\gamma(G_{NN}(M)) = 2$, then $G_{NN}(M)$ is the union of two K_λ's by Proposition 3.7 or is a complete bipartite graph $K_{\lambda,\lambda}$ with $\lambda \geq 2$ by Proposition 3.12. So, $\mu - 1 = 2$ or $\frac{\mu-1}{2} = 1$ by Propositions 3.7 and Proposition 3.12. In either case, $\mu = |\frac{M}{Nil(M)}| = 3$ and $\lambda = |Nil(M)| \geq 2$. Thus $Nil(M) \neq \{0\}$ and $diam(G_{NN}(M)) = 2$, by Proposition 4.1(3). □

Proposition 5.6. *Let R be a ring and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \lambda$ and $|\frac{M}{Nil(M)}| = \mu$. Then*

$$b(G_{NN}(M)) = \begin{cases} \lambda - 1, & \text{if } 2 = 1_R + 1_R \in Nil(R), \\ \lambda, & \text{if } 2 = 1_R + 1_R \notin Nil(R). \end{cases}$$

Proof. Let us assume that $2 = 1_R + 1_R \in Nil(R)$. Then, by Proposition 3.7, the graph $G_{NN}(M)$ is the union of $\mu - 1$ disjoint K_λ's and we know that $b(K_\lambda) = \lambda - 1$. Thus, $b(G_{NN}(M)) = \lambda - 1$.

Again, let us suppose that $2 = 1_R + 1_R \notin Nil(R)$. Then, by Proposition 3.12, $G_{NN}(M)$ is the union of $\frac{\mu - 1}{2}$ disjoint $K_{\lambda,\lambda}$'s and we know that $b(K_{\lambda,\lambda}) = \lambda$. Hence, $b(G_{NN}(M)) = \lambda$. □

Example 5.7. As mentioned in the Example 3.13, let us consider the module $R = M = \mathbb{Z}_5$ over itself. Since $ann_R(M) = 0$, so M is a finitely generated faithful multiplication module. Also, $Nil(M) = \{\overline{0}\}$ and so $Non(M) = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$. Clearly, $Nil(M)$ is a prime submodule of M and $2 = 1_R + 1_R \notin Nil(R)$. Again, we have $\lambda = |Nil(M)| = 1$ and $\mu = |\frac{M}{Nil(M)}| = 5$. Therefore, we observe that the graph $G_{NN}(M)$ is the union of two disjoint complete bipartite graphs $K_{1,1}$. Thus, $b(G_{NN}(M)) = b(K_{1,1} \cup K_{1,1}) = min\{b(K_{1,1}), b(K_{1,1})\} = min\{1, 1\} = 1 = \lambda$.

Proposition 5.8. *Let R be a ring and M be a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \lambda$ and $|\frac{M}{Nil(M)}| = \mu$. Then*

(1) *$gr(G_{NN}(M)) = 3$ if and only if $b(G_{NN}(M)) = \lambda - 1$ and $|Nil(M)| \geq 3$.*

(2) *$gr(G_{NN}(M)) = 4$ if and only if $b(G_{NN}(M)) = \lambda$ and $|Nil(M)| \geq 2$.*

Proof. (1) If $gr(G_{NN}(M)) = 3$, then $2 \in Nil(R)$ and $|Nil(M)| \geq 3$, by Proposition 4.2(1). Since $2 \in Nil(R)$, we have $b(G_{NN}(M)) = \lambda - 1$, by Proposition 5.6.

Conversely, let us asume that $b(G_{NN}(M)) = \lambda - 1$ and $|Nil(M)| \geq 3$. If $2 \notin Nil(R)$, then $G_{NN}(M)$ is the union of $\frac{\mu - 1}{2}$ disjoint $K_{\lambda,\lambda}$'s, by Proposition 3.12 and hence $b(G_{NT}(M)) = \lambda$, a contradiction to our assumption. Therefore $2 \in Nil(R)$, and then $gr(G_{NN}(M)) = 3$, by Proposition 4.2(1).

(2) If $gr(G_{NN}(M)) = 4$, then $2 \notin Nil(R)$ and $|Nil(M)| \geq 2$, by Proposition 4.2(2). Since $2 \notin Nil(R)$, we have $b(G_{NN}(M)) = \lambda$, by Proposition 5.6.

Conversely, let us suppose that $b(G_{NN}(M)) = \lambda$ and $|Nil(M)| \geq 2$. If $2 \in Nil(R)$, then $b(G_{NN}(M)) = \lambda - 1$, by Proposition 5.6, a contradiction. Therefore, $2 \notin Nil(R)$ and hence $gr(G_{NN}(M)) = 4$ by Proposition 4.2(2). □

6. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

References

[1] S. Akbari, D. Kiani, F. Mohammadi and S. Moradi, *The total graph and regular graph of a commutative ring*, J. Pure Appl. Algebra, **213** (2009) 2224-2228.

[2] M. M. Ali, *Idempotent and Nilpotent Submodules of Multiplication Modules*, Commun. Algebra, **36** (2008) 4620-4642.

[3] D. F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra, **320** (2008) 2706-2719.

[4] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, **217** (1999) 434-447.

[5] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra, **159** (1993) 500-514.

[6] S. E. Atani and S. Habibi, *The total torsion element graph of a module over a commutative ring*, An. St. Univ. Ovidius Constanta, **19** No. 1 (2011) 23-34.

[7] I. Beck, *Coloring of commutative rings*, J. Algebra, **116** (1988) 208-226.

[8] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, Tata McGraw-Hill, 2006.

[9] T. T. Chelvam and T. Asir, *Domination in the total graph of a commutative ring*, J. Comb. Math. Comb. Comput., **87** (2013) 147-158.

[10] P. P. Gogoi and J. Goswami, *Non-torsion element graph of a module over a commutative ring*, Bol. Soc. Parana. Mat., **42** (2024) 1-11.

[11] J. Goswami, *Some domination properties of the total graph of a module with respect to singular submodule*, Online J. Anal. Comb., **15** (2020).

[12] J. Goswami and H. K. Saikia, *On the Line graph associated to the Total graph of a module*, Mat., **31** No. 1 (2015) 7-13.

[13] J. Goswami and M. Sarmah, *On domination in the total torsion element graph of a module* , Proyecciones, **42** No. 3 (2023) 795-814.

[14] J. Goswami and M. Shabani, *Domination in the entire nilpotent element graph of a module over a commutative ring*, Proyecciones, **40** No. 6 (2021) 1411-1430.

[15] J. Goswami, K. K. Rajkhowa and H. K. Saikia, *Total graph of a module with respect to singular submodule*, Arab j. math. sci., **22** (2016) 242-249.

[16] T. W. Haynes, S. T. Hedetniemi and P. J. Slatar, *Fundamental of Domination in Graphs*, Marcel Dekker. Inc., 1998.

[17] T. W. Haynes, S. T. Hedetniemi and P. J. Slatar, *Domination in Graphs-Advanced Topics*, Marcel Dekker. Inc., 2000.

[18] H. M. I. Hoque, H. K. Saikia, J. Goswami and D. Patwari, *Non-nilpotent graph of Commutative rings*, J. Algebr. Syst., **12** No. 1 (2024) 149-162.

[19] J. Lambeck, *Lectures on Rings and Modules*, Blaisdell Publishing Company, Waltham, 1966.

[20] D. A. Mojdeh and A. M. Rahimi, *Dominating sets of some graphs associated to commutative rings*, Commun. Algebra, **40** (2012) 3389-3396.

[21] D. Patwari, H. K. Saikia and J. Goswami, *Some results on domination in the generalized total graph of a commutative ring*, J. algebr. relat. top., **10** No. 1 (2022) 119-128.

[22] A. Shariatinia and A. Tehranian, *Domination number of total graph of module*, Alg. Struc. Appl., **2** No. 1 (2015) 1-8.

[23] M. H. Shekarriz, M. H. S. Haghighi and H. Sharif, *On the total graph of a finite commutative ring*, Commun. Algebra, **40** (2012) 2798-2807.

[24] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math., **50** (1988) 223-235.

[25] D. B. West, *Introduction to graph theory*, Second edition, Pearson Education Inc, 2002.

Partha Protim Gogoi

Department of Mathematics,

Gauhati University, Guwahati-14,

Assam, India.

partha.gogoi34@gmail.com

Jituparna Goswami

Department of Mathematics,

Gauhati University, Guwahati-14,

Assam, India.

jituparnagoswami18@gmail.com