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Research Paper

THE STRONGLY ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE RING WITH RESPECT TO AN IDEAL

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Abstract. For a commutative ring R with identity, $\mathrm{SAG}(R)$ be the graph whose vertices are the nonzero annihilating ideals of R and with two distinct nonzero annihilating ideals I and J joined by an edge when $I \cap \mathrm{Ann}(J) \neq (0)$ and $J \cap \mathrm{Ann}(I) \neq (0)$. Also, strongly Annihilating-ideal graph with respect to an ideal (I), that it is shown by $\mathrm{SAG}_I(R)$, is the graph whose vertices are all ideals of R such that $K \nsubseteq I$ and for some ideal J that $J \nsubseteq I$, $KJ \subseteq I$, and distinct vertices K and J are adjacent if and only if $J \cap \mathrm{Ann}_I(K) \nsubseteq I$ and $K \cap \mathrm{Ann}_I(J) \nsubseteq I$. In this paper, we study the notion of $\mathrm{SAG}_I(R)$. Also, among other results, we give some results about the relationships between $\mathrm{SAG}_I(R)$ and $\mathrm{SAG}(R/I)$.

1. Introduction

Throughout this paper, all rings are assumes to be commutative with identity. First we state some definitions and notions used throughout the paper. The *girth* of a graph G, denoted by

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gr(G), is the length of a shortest cycle in G. If G has no cycles, we define the girth of G to be infinite. A graph is called complete if all its vertices are connected. We denote the set of zero-divisors of R by $Z(R)$, and we write $I \trianglelefteq R$ to denote I is an ideal of R. A *uniserial* ring is a ring whose ideals are totally ordered by inclusion. For ideals I and J, $\mathrm{Ann}_I(J) = \{r \in R|\ rj \in I,\ \text{for every } j \in J\}$ and if $I = (0)$, then we write $\mathrm{Ann}(J)$ instead of $\mathrm{Ann}_I(J)$. Also $A(R) = \{J \trianglelefteq R|\ \mathrm{Ann}(J) \neq 0\}$ and $N_I(R) = \{J \trianglelefteq R|\ J \nsubseteq I\}$.

A nonzero ideal I of R is called an *annihilating-ideal* if there exists a nonzero ideal J of R such that $IJ = 0$. In [2] Anderson and Livingston defined the zero-divisor graph of R, $\Gamma(R)$, with vertices $Z(R) \setminus \{0\}$, and for distinct $x, y \in Z(R) \setminus \{0\}$, the vertices x and y are adjacent if and only if $xy = 0$. As an extension of the zero-divisor graph of a commutative ring R, Redmond defined in [6], the ideal-based *zero-divisor graph* of a commutative ring R, denoted by $\Gamma_I(R)$, where for an ideal I of R, the vertices of $\Gamma_I(R)$ are $\{x \in R - I \big|\ xy \in I$ for some $y \in R\}$, and distinct vertices x and y are adjacent if and only if $xy \in I$. He found some relationships between $\Gamma\left(\frac{R}{I}\right)$ and $\Gamma_I(R)$. Later in [3], Behboodi and Rakeei introduced the *annihilating-ideal graph* of R, AG(R), with the vertex set $A(R)^* = A(R) \setminus \{0\}$ and two distinct vertices joined by an edge when the product of the vertices is the zero ideal. Also Aliniaeifard and Behboodi in [1], defined the *annihilating-ideal graph with respect to an ideal I* of R, $\mathrm{AG}_I(R)$, whose vertices are $A_I(R) = \{K \in N_I(R) \big|\ KJ \subseteq I$ for some $J \in N_I(R)\}$ and distinct vertices K and J are adjacent if and only if $KJ \subseteq I$. Also they get some relationships between AG(R) and $\mathrm{AG}_I(R)$ (see also [4]). Tohidi, Nikmehr and Nikandish in [8] defined the *strongly annihilating-ideal graph* of R, SAG(R), with the vertex set $A(R)^*$ and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}(J) \neq (0)$ and $J \cap \mathrm{Ann}(I) \neq (0)$.

In this paper, we extend the notion of the strongly annihilating-ideal graph of a ring R to the *strongly annihilating-ideal graph with respect to an ideal I* of R, denoted $\mathrm{SAG}_I(R)$, whose the set of vertices is $A_I(R)$ and two distinct vertices K and J are adjacent if $K \cap \mathrm{Ann}_I(J) \nsubseteq I$ and $J \cap \mathrm{Ann}_I(K) \nsubseteq I$. Thus, $\mathrm{AG}_I(R)$ is a subgraph of $\mathrm{SAG}_I(R)$. Also for $I = (0)$, $\mathrm{SAG}_I(R) =$ SAG(R). In Section 2, we prove some basic properties of $\mathrm{SAG}_I(R)$. In particular, it is proved that $\mathrm{SAG}_I(R)$ is connected with diameter at most 2 and $\mathrm{gr}\left(\mathrm{SAG}_I(R)\right) \leq 4$, if it contains a cycle. In Section 3, we prove some relationships between $\mathrm{SAG}_I(R)$ and $\mathrm{SAG}\left(\frac{R}{I}\right)$, especially among other results, we get the condition that $\mathrm{SAG}_I(R)$ and $\mathrm{SAG}\left(\frac{R}{I}\right)$ are isomorphic. At last in Section 4, we prove some results about connectivity of $\mathrm{SAG}_I(R)$.

2. Basic properties of $\mathrm{SAG}_I(R)$

Let R be a commutative ring with identity.

Lemma 2.1. *Let I be an ideal of ring R and $T, J \in \nu\big(\mathrm{SAG}_I(R)\big)$. Then the following statements hold*

(1) *If two vertices T and J are not adjacent, then* $\mathrm{Ann}_I(TJ) = \mathrm{Ann}_I(T)$ *or* $\mathrm{Ann}_I(TJ) = \mathrm{Ann}_I(J)$*. If* $\sqrt{I} = I$*, then the reverse is true.*

(2) *If two vertices T and J are adjacent in* $\mathrm{AG}_I(R)$*, then these two vertices are also adjacent in* $\mathrm{SAG}_I(R)$*. Therefore,* $\mathrm{AG}_I(R)$ *is a subgraph of* $\mathrm{SAG}_I(R)$*.*

(3) *If* $\mathrm{Ann}_I(J) \nsubseteq \mathrm{Ann}_I(T)$ *and* $\mathrm{Ann}_I(J) \nsubseteq \mathrm{Ann}_I(T)$*, then two vertices T and J are connected. In addition, if* $\sqrt{I} = I$*, then the reverse is true.*

(4) *If* $d_{\mathrm{AG}_I(R)}(T, J) = 3$*, then T and J are connected in* $\mathrm{SAG}_I(R)$*.*

(5) *Let for a positive integer $n \geq 1$, $R = R_1 \times \cdots \times R_n$, where for every $1 \leq i \leq n$, R_i is a ring, $I = I_1 \times \cdots \times I_n$ an ideal of R, and $J = J_1 \times \cdots \times J_n$ and $K = K_1 \times \cdots \times K_n$ are two vertices of* $\mathrm{SAG}_I(R)$*. If $J_s \cap \mathrm{Ann}_{I_s}(K_s) \nsubseteq I_s$ and $K_t \cap \mathrm{Ann}_{I_t}(J_t) \nsubseteq I_t$, for some $1 \leq s, t \leq n$, then J is adjacent to K in* $\mathrm{SAG}_I(R)$*. In particular, if J_s is adjacent to K_s in $\mathrm{SAG}_{I_s}(R_s)$ or $J_s = K_s$, $J_s \cap \mathrm{Ann}(J_s) \nsubseteq I_s$, for some $1 \leq s \leq n$, then J and K are adjacent in* $\mathrm{SAG}_I(R)$*.*

(6) *If T and J are not connected in* $\mathrm{SAG}_I(R)$*, then* $d_{\mathrm{AG}_I(R)}(T, J) = 2$*.*

Proof. (1) Let T and J are not connected. Evidently $\mathrm{Ann}_I(T) \subseteq \mathrm{Ann}_I(TJ)$ and $\mathrm{Ann}_I(J) \subseteq \mathrm{Ann}_I(TJ)$. In contrary, let $\mathrm{Ann}_I(TJ) \nsubseteq \mathrm{Ann}_I(T)$ and $\mathrm{Ann}_I(TJ) \nsubseteq \mathrm{Ann}_I(J)$. Then there exist $r_1, r_2 \in R$ such that

$$r_1TJ \subseteq I,\ r_1T \nsubseteq I \text{ and } r_2TJ \subseteq I,\ r_2J \nsubseteq I. \tag{I}$$

We consider the following two cases:

Case 1) $r_1 = r_2$. Then $r_1t \notin I$ and $r_1j \notin I$, for some $j \in J$ and $t \in T$. Therefore $r_1j \in J \cap \mathrm{Ann}_I(T)$ and $r_1t \in T \cap \mathrm{Ann}_I(J)$, so, T and J are connected, a contradiction.

Case 2) $r_1 \neq r_2$. If $r_1J \nsubseteq I$, then for some $j \in J$, $r_1j \notin I$. So by (I), $r_1j \in J \cap \mathrm{Ann}_I(T)$. Also by (I), $I \nsupseteq r_1T \subseteq T \cap \mathrm{Ann}_I(J)$. Then T and J are connected, a contradiction. So $r_1J \subseteq I$. Similarly, it can be proved $r_2T \subseteq I$. Therefore

$$r_1J \subseteq I \text{ and } r_2T \subseteq I. \tag{II}$$

Then by (I) and (II), $(r_1 - r_2)J \nsubseteq I$ and $(r_1 - r_2)T \nsubseteq I$, so $(r_1 - r_2)TJ \subseteq I$. Therefore $J \cap \mathrm{Ann}_I(T) \nsubseteq I$ and $T \cap \mathrm{Ann}_I(J) \nsubseteq I$. Hence T and J are connected, a contradiction.

Now, let $\sqrt{I} = I$ and without lose of generality, let $\mathrm{Ann}_I(TJ) = \mathrm{Ann}_I(J)$. If $j \in J \cap \mathrm{Ann}_I(J)$, then $j^2 \in I$, so $j \in I$, thus $J \cap \mathrm{Ann}_I(J) \subseteq I$. Therefore:

$$J \cap \mathrm{Ann}_I(T) \subseteq J \cap \mathrm{Ann}_I(TJ) = J \cap \mathrm{Ann}_I(J) \subseteq I.$$

Then T and J are not adjacent.

(2) It is obtained by the definition of the edges in $\mathrm{SAG}_I(R)$.

(3) In contrary, suppose that T and J are not connected. Without lose of generality, we can get by (1), $\mathrm{Ann}_I(TJ) \subseteq \mathrm{Ann}_I(T)$. Since $\mathrm{Ann}_I(J) \subseteq \mathrm{Ann}_I(TJ)$, then $\mathrm{Ann}_I(J) \subseteq \mathrm{Ann}_I(T)$, a contradiction.

Now, suppose that T and J are connected and $\sqrt{I} = I$. In contrary, suppose that $\mathrm{Ann}_I(J) \subseteq \mathrm{Ann}_I(T)$. Then $T \cap \mathrm{Ann}_I(T) \subseteq I$. As a result, $T \cap \mathrm{Ann}_I(J) \subseteq I$. This conclusion contradicts the definition of an edge in $\mathrm{SAG}_I(R)$.

(4) Let $d_{\mathrm{AG}_I(R)}(T, J) = 3$. Then there is a path like T —— K —— L —— J, so $LJ \subseteq I$ and $TL \nsubseteq I$. Thus $tl \notin I$, for some $t \in T$ and $l \in L$. On the other hand, since $tl \in L$ and $LJ \subseteq I$, $tl \in \mathrm{Ann}_I(J)$. Therefore $tl \in \big(T \cap \mathrm{Ann}_I(J)\big) \setminus I$, so $T \cap \mathrm{Ann}_I(J) \nsubseteq I$.

In the same way, it can be proved $J \cap \mathrm{Ann}_I(T) \nsubseteq I$. Then the vertices T and J are adjacent in $\mathrm{SAG}_I(R)$.

(5) Since $J_s \cap \mathrm{Ann}_{I_s}(K_s) \nsubseteq I_s$, there exists an element $a_s \in J_s \setminus I_s$ such that $a_s K_s \subseteq I_s$ and thus $(0 \ldots 0, a_s, 0, ..., 0) \in J \cap \mathrm{Ann}_I(K)$. Similarly, $K_t \cap \mathrm{Ann}_{I_t}(J_t) \nsubseteq I_t$ implies that $K \cap \mathrm{Ann}_I(J) \nsubseteq I$. Hence J and K are connected in $\mathrm{SAG}_I(R)$. The "in particular" statement is now clear.

(6) Suppose that T and J are not connected in $\mathrm{SAG}_I(R)$. By (2), T and J are not connected in $\mathrm{AG}_I(R)$, so by [1, Theorem 3.3], $d_{\mathrm{AG}_I(R)}(T, J) = 2$ or 3. If $d_{\mathrm{AG}_I(R)}(T, J) = 3$, then T and J are connected in $\mathrm{SAG}_I(R)$ by (4), a contradiction. Therefore, $d_{\mathrm{AG}_I(R)}(T, J) = 2$.

□

By Lemma 2.1(2), $\mathrm{AG}_I(R)$ is a subgraph of $\mathrm{SAG}_I(R)$, but the next example shows that these two graphs are not identical.

Example 2.2. The ideals of $\mathbb{Z}_8$ are $I_1 = \{0\}$, $I_2 = \{0, 4\}$, $I_3 = \{0, 2, 4, 6\}$ and $I_4 = \mathbb{Z}_8$. Also, the ideals of $\mathbb{Z}_2$ are $J_1 = \{0\}$ and $J_2 = \mathbb{Z}_2$. So, the ideals of $\mathbb{Z}_8 \times \mathbb{Z}_2$ are as follows

$$I_1 \times J_1 = \{(0,0)\}$$

$$I_1 \times J_2 = \{(0,0), (0,1)\}$$

$$I_2 \times J_1 = \{(0,0), (4,0)\}$$

$$I_2 \times J_2 = \{(0,0), (0,1), (4,0), (4,1)\}$$

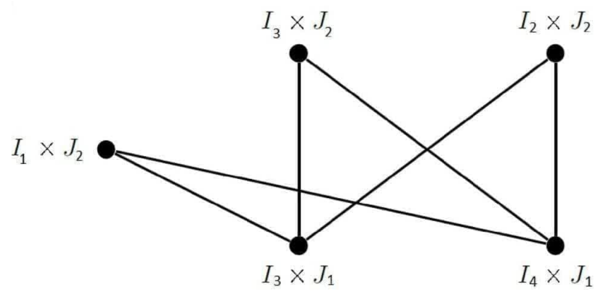
$$I_3 \times J_1 = \{(0,0), (2,0), (4,0), (6,0)\}$$

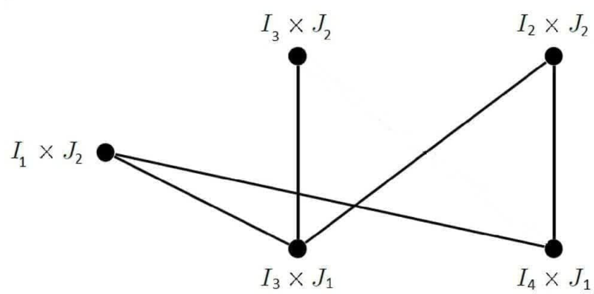
$$I_3 \times J_2 = \{(0,0), (2,0), (4,0), (6,0), (0,1), (2,1), (4,1), (6,1)\}$$

$$I_4 \times J_1 = \{(0,0), (1,0), (2,0), (3,0), (4,0), (5,0), (6,0), (7,0)\}$$

$$I_4 \times J_2 = \{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),$$
$$(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1),(7,1)\}.$$

Hence, $\mathrm{SAG}_{I_2\times J_1}(\mathbb{Z}_8\times\mathbb{Z}_2)$ and $\mathrm{AG}_{I_2\times J_1}(\mathbb{Z}_8\times\mathbb{Z}_2)$ are as follows, respectively





Evidently

$$d_{\mathrm{AG}_{I_2\times J_1}(\mathbb{Z}_8\times\mathbb{Z}_2)}(I_3\times J_2, I_4\times J_1)=3.$$

However

$$d_{\mathrm{SAG}_{I_2\times J_1}(\mathbb{Z}_8\times\mathbb{Z}_2)}(I_3\times J_2, J_4\times J_1)=1.$$

Theorem 2.3. *Let R be a ring. Then*

(1) $\mathrm{SAG}_I(R)$ *is a connected graph.*
(2) $\mathrm{diam}\,(\mathrm{SAG}_I(R)) \le 2$.
(3) *If* $\mathrm{SAG}_I(R)$ *contains at least one cycle, then* $\mathrm{gr}\,(\mathrm{SAG}_I(R)) \le 4$.

Proof. (1) By Lemma 2.1(2), $\mathrm{AG}_I(R)$ is a subgraph of $\mathrm{SAG}_I(R)$. According to [1, Theorem 3.3], $\mathrm{AG}_I(R)$ is connected, hence $\mathrm{SAG}_I(R)$ is connected.

(2) Let T and J be two vertices of $\mathrm{SAG}_I(R)$.

a) If T and J are adjacent, then the distance between T and J in $\mathrm{SAG}_I(R)$ is equal to 1.

b) If T and J are not adjacent, then the distance between T and J in $\mathrm{AG}_I(R)$ is equal to 2, by Lemma 2.1(6).

According to the previous lemma, $\mathrm{AG}_I(R)$ is a subgraph of $\mathrm{SAG}_I(R)$, hence $d_{\mathrm{SAG}_I(R)}(T,J)=2$. Therefore, by (a) and (b), $\mathrm{diam}\,(\mathrm{SAG}_I(R)) \le 2$.

(3) In contrary, let $\operatorname{gr}\big(\mathrm{SAG}_I(R)\big) > 4$. Hence, there is a path like the following sequence, which is one of the shortest sequences in $\mathrm{SAG}_I(R)$:

$$(1) \quad I_1 \text{ —— } I_2 \text{ —— } I_3 \text{ —— } I_4 \text{ —— } I_5 \text{ —— } I_6 \text{ —— } \cdots \text{ —— } I_{n-1} \text{ —— } I_n \text{ —— } I_1.$$

Since I_1 and I_3 are not connected, by Lemma 2.1(6), $d_{\mathrm{AG}_I(R)}(I_1, I_3) = 2$. Therefore, $L_1 \in V\big(\mathrm{AG}_I(R)\big)$, where I_1 —— L_1 —— I_3 is a path from I_1 to I_3 in $\mathrm{AG}_I(R)$. If $L_1 = I_{i_0}$, where $1 \leq i_0 \leq n$ and $i_0 \neq 2, n$, then I_1 and I_{i_0} are adjacent in $\mathrm{AG}_I(R)$. Therefore, according to Lemma 2.1(2), I_1 and I_{i_0} are adjacent in $\mathrm{SAG}_I(R)$. This conclusion contradicts (1). Hence, for each $1 \leq i \leq n$, where $i \neq 2, n$, $L_1 \neq I_i$. Now, we prove that $L_1 \neq I_n$. If in contrary, $L_1 = I_n$, then

$$L_1 \text{ —— } I_3 \text{ —— } I_4 \text{ —— } \cdots \text{ —— } I_{n-1} \text{ —— } L_1 \ .$$

It will be a cycle of the length of $n-2$ in $\mathrm{SAG}_I(R)$. This conclusion contradictions Relation (1). Therefore, $L_1 \neq I_n$. Since I_3 and I_5 are not connected in $\mathrm{SAG}_I(R)$, then according to Lemma 2.1(6), $d_{\mathrm{AG}_I(R)}(I_3, I_5) = 2$, hence, $L_2 \in V\big(\mathrm{AG}_I(R)\big)$ where

$$I_1 \text{ —— } L_1 \text{ —— } I_3 \text{ —— } L_2 \text{ —— } I_5 \ .$$

is a path from I_1 to I_5 in $\mathrm{AG}_I(R)$.

If $L_2 = I_{i_0}$ when $1 \leq i_0 \leq n$ and $i_0 \neq 4$, then I_3 and I_{i_0} are adjacent in $\mathrm{AG}_I(R)$. Therefore, according to Lemma 2.1(2), these two vertices are adjacent in $\mathrm{SAG}_I(R)$. This conclusion contradicts Relation (1). Thus, for each $1 \leq i \leq n$ when $i \neq 4$, there will be $L_2 \neq I_i$. We should now prove that $L_2 \neq L_1$. Let $L_2 = L_1$, then

$$I_1 \text{ —— } L_1 \text{ —— } I_5 \text{ —— } I_6 \text{ —— } \cdots \text{ —— } I_{n-1} \text{ —— } I_n \text{ —— } I_1 \ .$$

This is a cycle of length $n-2$ in $\mathrm{SAG}_I(R)$. This conclusion contradicts Relation (1). Therefore, $L_2 \neq L_1$.

Similarly, the procedure is kept on. Now if I_1 and I_n are adjacent in $\mathrm{AG}_I(R)$, then

$$I_1 \text{ —— } L_1 \text{ —— } I_3 \text{ —— } L_2 \text{ —— } I_5 \text{ —— } L_3 \text{ —— } I_7 \text{ —— } \cdots \text{ —— } L_s \text{ —— } I_n \text{ —— } I_1 \ .$$

This is a cycle of length $n > 4$ in $\mathrm{AG}_I(R)$. According to [1, Theorem 3.3], $\operatorname{gr}\big(\mathrm{AG}_I(R)\big) \leq 4$. At the same time, $\mathrm{AG}_I(R)$ is the subgraph of $\mathrm{SAG}_I(R)$; hence, $\operatorname{gr}\big(\mathrm{SAG}_I(R)\big) \leq 4$. This conclusion contradicts Relation (1). Therefore, the contradiction is rejected, and the theorem is proven. If I_1 and I_n are not connected in $\mathrm{AG}_I(R)$, hence

$$I_1 \text{ —— } L_1 \text{ —— } I_3 \text{ —— } L_2 \text{ —— } I_5 \text{ —— } \cdots \text{ —— } L_s \text{ —— } I_n \text{ —— } L_{s+1} \text{ —— } I_1 \ .$$

This is a cycle of $n+1$ in $\mathrm{AG}_I(R)$. According to [1, Theorem 3.3], $\operatorname{gr}\big(\mathrm{AG}_I(R)\big) \leq 4$. Then by Lemma 2.1(2), $\operatorname{gr}\big(\mathrm{SAG}_I(R)\big) \leq 4$. This conclusion contradicts Relation (1), hence, the contradiction is rejected and the proof is proved. □

Corollary 2.4. *If* $\mathrm{AG}_I(R)$ *contains a cycle of the length* n*, then this cycle also exists in* $\mathrm{SAG}_I(R)$*. If* $\mathrm{SAG}_I(R)$ *includes the following* n*-long cycle:*

$$I_1 \text{ --- } I_2 \text{ --- } I_3 \text{ --- } I_4 \text{ --- } I_5 \text{ --- } \cdots \text{ --- } I_{n-1} \text{ --- } I_n \text{ --- } I_1 \ .$$

Then $\mathrm{AG}_I(R)$ *includes one of the following cycles of the length either* n *or* $n+1$*:*

$$I_1 \text{ --- } L_1 \text{ --- } I_2 \text{ --- } L_2 \text{ --- } I_3 \text{ --- } \cdots \text{ --- } L_s \text{ --- } I_n \text{ --- } I_1 \ .$$

or

$$I_1 \text{ --- } L_1 \text{ --- } I_2 \text{ --- } L_2 \text{ --- } I_3 \text{ --- } \cdots \text{ --- } L_s \text{ --- } I_n \text{ --- } L_{s+1} \text{ --- } I_1 \ .$$

where, $L_1, \ldots, L_{s+1} \in V\big(\mathrm{AG}_I(R)\big)$*.*

Lemma 2.5. *Let* I *and* L *be the ideals of* R *such that for every* $n \in \mathbb{N}$*,* $L^n \subseteq I$*. Then for every ideal* J *of* R *such that* $J \nsubseteq I$*,* $\mathrm{Ann}_I(L) \cap J \nsubseteq I$*.*

Proof. Let J be an ideal of R such that $J \nsubseteq I$ and $J \cap \mathrm{Ann}_I(L) \subseteq I$. Then $JL \nsubseteq I$. Now, let k be the smallest natural number shch that $JL^{k-1} \nsubseteq I$. Since $JL^{k-1} \subseteq \mathrm{Ann}_I(L)$ and $JL^{k-1} \subseteq J$, $J \cap \mathrm{Ann}_I(L) \nsubseteq I$. This conclusion contradicts the assumption, and the proof is complete. □

3. Relationships between $\mathrm{SAG}_I(R)$ and $\mathrm{SAG}(\frac{R}{I})$

Let I be an ideal of R and $\phi : R \to \frac{R}{I}$, where for every $r \in R$, $\phi(r) = r + I$. For every $r \in R$ and ideal J of R, we denote $\phi(r)$ by $\bar{r}$, and $\phi(J)$ by $\bar{J}$.

Lemma 3.1. *Let* R *be a ring and let* I_1, I_2, I, J *and* K *be some ideals of* R *such that* $I_1, I_2 \subseteq I$*. Then the following statements are equivalent:*

(1) $(J + I_1) \cap \mathrm{Ann}_I(K + I_2) \nsubseteq I$.
(2) $(J + I_1) \cap \mathrm{Ann}_I(K) \nsubseteq I$.
(3) $J \cap \mathrm{Ann}_I(K + I_2) \nsubseteq I$.
(4) $J \cap \mathrm{Ann}_I(K) \nsubseteq I$.
(5) $\bar{J} \cap \mathrm{Ann}(\bar{K}) \neq (0)$.

Proof. Since for every ideals A, B, I and I_0 of R, such that $I_0 \subseteq I$, $\mathrm{Ann}_I(A + I_0) = \mathrm{Ann}_I(A)$, (1) and (2) (similarly, (3) and (4)) are equivalent. Now, evidently

$$(A + I_0) \cap \mathrm{Ann}_I(B) \supseteq (A \cap \mathrm{Ann}_I(B)) + (I_0 \cap \mathrm{Ann}_I(B)) = (A \cap \mathrm{Ann}_I(B)) + I_0$$

so, if $x \in ((A + I_0) \cap \mathrm{Ann}_I(B)) \setminus I$, then there exists $x = a + i_0 \in A + I_0 \setminus I$, for some $a \in A$ and $i_0 \in I_0$, such that $xB \subseteq I$, then $a \in (A \cap \mathrm{Ann}_I(B)) \setminus I$. Thus, $(A + I_0) \cap \mathrm{Ann}_I(B) \nsubseteq I$ if and only if $A \cap \mathrm{Ann}_I(B) \nsubseteq I$. Therefore (1) and (3) (similarly, (2) and (4)) are equivalent.

Also, it is easy to see that (4) and (5) are equivalent. □

Theorem 3.2. *Let I be an ideal of a ring R, and let $J, K \in N_I(R)$. Then the following statements are true:*

(1) *If $\bar{J} \neq \bar{K}$, then J is adjacent to K in $\mathrm{SAG}_I(R)$ if and only if $\bar{J}$ is adjacent to $\bar{K}$ in $\mathrm{SAG}(\frac{R}{I})$.*
(2) *If $\bar{J} = \bar{K}$, then J is adjacent to K in $\mathrm{SAG}_I(R)$ if and only if $J \cap \mathrm{Ann}_I(J) \nsubseteq I$.*

Proof. It is an easy consequence of Lemma 3.1. □

Corollary 3.3. *Let J and K be (distinct) adjacent vertices in $\mathrm{SAG}_I(R)$. Then all (distinct) ideals of the form $J + I_1$ is adjacent to all ideals of the form $K + I_2$ in $\mathrm{SAG}_I(R)$, where I_1 and I_2 are ideals of R and $I_1, I_2 \subseteq I$. If $J \cap \mathrm{Ann}_I(J) \nsubseteq I$, then distinct ideals of the forms $J + I_1$, for every ideals I_1 of R and $I_1 \subseteq I$, are adjacent in $\mathrm{SAG}_I(R)$.*

Proof. It is clear that $\bar{J} = \bar{K}$ if and only if $J + I = K + I$. Now the result follows from Lemma 3.1 and Theorem 3.2. □

Corollary 3.4. *Let I be an ideal of a ring R. Then $\mathrm{SAG}_I(R)$ contain a copy of $\mathrm{SAG}(\frac{R}{I})$.*

Proof. Since there is a one-to-one correspondence between the ideals of R that contain I, and the ideals of $\frac{R}{I}$, the result is an immediate consequence of Theorem 3.2(1). □

Now, let's check when the graphs $\mathrm{SAG}_I(R)$ and $\mathrm{SAG}(\frac{R}{I})$ are isomorphic. First, we introduce some symbols.

Notation 1. *Let R be a ring and I be an ideal of R. For simplicity, let $V = V\big(\mathrm{SAG}_I(R)\big)$ and $V' = V\big(\mathrm{SAG}(\frac{R}{I})\big)$. For any ideal $I \subsetneq K$ of R, set*

$$C(K) = \{T \mid I \nsubseteq T \text{ is an ideal of } R, \text{ and } T + I = K\}.$$

For distinct ideals K and J of R, where $I \subsetneq J, K$, evidently, $C(K) \cap C(J) = \emptyset$. If for an $I \subset K$, $K \in V$, we call the set $C(K)$ the column of K in $\mathrm{SAG}_I(R)$. Also, set:

$$\mathcal{M} = \{K \mid K \in V, C(K) \neq \emptyset\}.$$

By Theorem 3.2 and the above notation, if $K \in V$, then the following two cases can be considered

1) If $I \subset K$, then $\frac{K}{I} \in V'$.
2) If $I \nsubseteq K$, then $K + I \in V$, therefore $\frac{K+I}{I} \in V'$.

Thus, if K is a vertex of $\mathrm{SAG}_I(R)$, then either $\frac{K}{I}$ is a vertex of $\mathrm{SAG}(\frac{R}{I})$ or $\frac{J}{I}$ is a vertex of $\mathrm{SAG}(\frac{R}{I})$, for an ideal J that $K \in C(J)$. Hence, all vertices of $\mathrm{SAG}_I(R)$ can be determined in this method and the following theorem explains how to draw the edges of $\mathrm{SAG}_I(R)$.

Theorem 3.5. *With our notations,*

a) *If L and K are adjacent in $\mathrm{SAG}_I(R)$, then L is connected to all members of the column of K.*
b) *If $I \subset K \in V$ and $K \cap \mathrm{Ann}_I(K) \nsubseteq I$, then K is adjacent with all members of its column. Moreover, all members of K column are connected.*
c) *If $I \subset K \in V$ and $K \cap \mathrm{Ann}_I(K) \subseteq I$, then K is not adjacent with any members of its column. In addition, no two members of column K are connected to each other.*

Proof. It is enough to note that for every ideals A, B and I of R, $\mathrm{Ann}_I(A + I) = \mathrm{Ann}_I(A)$. Also,

$$(A + I) \cap \mathrm{Ann}_I(B) \supseteq (A \cap \mathrm{Ann}_I(B)) + (I \cap \mathrm{Ann}_I(B)) = (A \cap \mathrm{Ann}_I(B)) + I$$

then $(A + I) \cap \mathrm{Ann}_I(B) \nsubseteq I$ if and only if $A \cap \mathrm{Ann}_I(B) \nsubseteq I$. □

Corollary 3.6. $\mathrm{SAG}_I(R) \cong \mathrm{SAG}(\frac{R}{I})$ *if and only if* $\mathcal{M} = \emptyset$.

Corollary 3.7. *If R is a uniserial ring, then* $\mathrm{SAG}_I(R) \cong \mathrm{SAG}(\frac{R}{I})$.

Proposition 3.8. *For a ring R, $\mathrm{SAG}_I(R)$ is a complete graph if and only if $\mathrm{SAG}(\frac{R}{I})$ is complete and for every $J \in \mathcal{M}$, $J \cap \mathrm{Ann}_I(J) \nsubseteq I$.*

Proof. First suppose that $\mathrm{SAG}_I(R)$ is complete and $\bar{K}$ and $\bar{J}$ are two vertex of $\mathrm{SAG}(\frac{R}{I})$. Evidently, $J \neq K$, so J and K are adjacent in $\mathrm{SAG}_I(R)$. Then $\bar{K}$ and $\bar{J}$ are adjacent in $\mathrm{SAG}(\frac{R}{I})$, by Theorem 3.2 (1).

Suppose that $J \in \mathcal{M}$ such that $J \cap \mathrm{Ann}_I(J) \subseteq I$, then there exists an $T \in C(J)$. Therefore, J and T are not adjacent by Theorem 3.5 (c), a contradiction. Then $J \cap \mathrm{Ann}_I(J) \nsubseteq I$.

Now, let $\mathrm{SAG}(\frac{R}{I})$ be complete and for every $J \in \mathcal{M}$, $J \cap \mathrm{Ann}_I(J) \nsubseteq I$. Let K and L be two distinct vertices of $\mathrm{SAG}_I(R)$. If $\bar{J} \neq \bar{K}$, then J and K are adjacent, by Theorem 3.2 (1). If $\bar{J} = \bar{K}$, then J and K are adjacent, by Theorem 3.2 (2). Therefore $\mathrm{SAG}_I(R)$ is complete. □

The next example shows $\mathrm{SAG}_I(R)$ and $\mathrm{SAG}(\frac{R}{I})$ are not isomorphic in general.

Example 3.9. Let $R = \mathbb{Z}_8 \times \mathbb{Z}_2$ and $I = I_2 \times J_1$. By Example 2.2, the proper ideals of $\frac{R}{I}$ are as follows

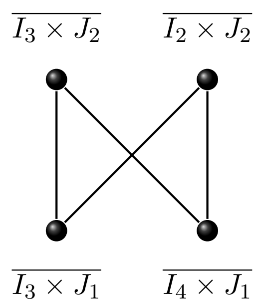
$$\overline{I_2 \times J_2} = \{\overline{(0,0)}, \overline{(0,1)}\} = \overline{I_1 \times J_2}$$

$$\overline{I_3 \times J_1} = \{\overline{(0,0)}, \overline{(2,0)}\}$$

$$\overline{I_3 \times J_2} = \{\overline{(0,0)}, \overline{(2,0)}, \overline{(0,1)}, \overline{(2,1)}\}$$

$$\overline{I_4 \times J_1} = \{\overline{(0,0)}, \overline{(1,0)}, \overline{(2,0)}, \overline{(3,0)}\}$$

Hence, $\mathrm{SAG}_I(\frac{R}{I})$ is as follows



Then, by Example 2.2, $\mathrm{SAG}_I(R) \ncong \mathrm{SAG}(\frac{R}{I})$.

In addition, $I_1 \times J_2 \in C(I_2 \times J_2)$ and $(I_1 \times J_2) \cap \mathrm{Ann}_{J_2 \times I_1}(I_2 \times J_2) = J_2 \times I_1$, then as mentioned in Theorem 3.5(c), $I_2 \times J_2$ is not connected to $I_1 \times J_2$ in $\mathrm{SAG}_I(R)$.

4. Completeness

Theorem 4.1. *Let I be a non-trivial ideal of a ring R, and $\frac{R}{I} \cong F_1 \times F_2$, where F_1 and F_2 are two fields. Then $\mathrm{SAG}_I(R)$ is not complete.*

Proof. Since F_1 and F_2 are considered fields, the proper (maximal) ideals of $F_1 \times F_2$ are $F_1 \times (0)$ and $(0) \times F_2$. Therefore, according to the assumption, $\frac{R}{I} = \frac{K}{I} + \frac{J}{I}$, in which K and J are proper (maximal) ideals of R that contain I, $K \cap J = I$, $K^2 + I = K$, $J^2 + I = J$, $J \cap \mathrm{Ann}_I(J) \subseteq I$ and $K \cap \mathrm{Ann}_I(K) \subseteq I$.

Since $\mathrm{SAG}(\frac{R}{I})$ is complete, by Proposition 3.8, $K, J \notin \mathcal{M}$. If L is another maximal ideal of R, where $J \neq L \neq K$, then $L + I = R$, so $LJ + I = J$. Since $I \nsubseteq LJ$, $J \in \mathcal{M}$, a contradiction. Therefore K and J are maximal ideals of R.

If $a \in J \setminus I$, then $\frac{<a>+I}{I} = \frac{J}{I}$, so $\langle a \rangle + I = J$. Since $J \notin \mathcal{M}$, then $J = \langle a \rangle$. Similarly, for every $b \in K \setminus I$, $K = \langle b \rangle$. Evidently, $a^2 \in J \setminus I$, so $J = \langle a^2 \rangle$, then $a(1 - ra) = 0$, for some $r \in R$. Since a is not invertible, $1 - ra \neq 0$, thus either $1 - ra \in J$ or $1 - ra \in K$. Since $ra \in J$, $1 - ra \in K \setminus I$ and thus $K = \langle 1 - ra \rangle$. Thus, $JK = 0$. But, since J and K are both maximal ideals of R, they are comaximal and thus $JK = J \cap K$. Therefore, $I = 0$, which contradicts the assumption, thus $\mathrm{SAG}_I(R)$ is not complete. □

Lemma 4.2. $\mathrm{SAG}_I(R)$ *is complete if and only if for every* $T \in V$*,* $\mathrm{Ann}(\frac{T+I}{I}) \in \mathrm{ESS}(\frac{R}{I})$*.*

Proof. Let $\mathrm{SAG}_I(R)$ be complete. According to Lemma 3.8, $\mathrm{SAG}(R/I)$ is complete. Therefore, by [8, Theorem 3.2], R satisfies in one of the following conditions:

a) $\frac{R}{I} = F_1 \times F_2$, in which F_1 and F_2 are fields.
b) For every $T \in V$, $\mathrm{Ann}(\frac{T+I}{I}) \in \mathrm{ESS}(\frac{R}{I})$.

According to Theorem 4.1, only the second case occurs.

Now, suppose that for every $T \in V$, $\mathrm{Ann}(\frac{T+I}{I}) \in \mathrm{ESS}(\frac{R}{I})$. Then by [8, Theorem 3.2], $\mathrm{SAG}(\frac{R}{I})$ is a complete graph.

Now, consider $J \in \mathcal{M}$. Since $\frac{J}{I} \in V'$, then $\mathrm{Ann}(\frac{J}{I}) \in \mathrm{ESS}(\frac{R}{I})$. So by the definition of $\mathrm{ESS}(\frac{R}{I})$, $\mathrm{Ann}(\frac{J}{I}) \cap \frac{J}{I} \neq I$. As a result, $\mathrm{Ann}_I(J) \cap J \nsubseteq I$. Therefore, According to Lemma 3.8, $\mathrm{SAG}_I(R)$ is complete. □

Theorem 4.3. *Let R be a ring and I be an ideal of R. If $Z(\frac{R}{I})$ is a zero ideal of $\frac{R}{I}$ when $Z^2(\frac{R}{I}) = I$, then* $\mathrm{SAG}_I(R)$ *is complete.*

Proof. According to [1, Theorem 6.5], $\mathrm{AG}_I(R)$ is complete. Hence, according to Part (2) of Lemma 2.1, $\mathrm{SAG}_I(R)$ is complete, too. □

Theorem 4.4. *Let I be an ideal of a ring R. If $\sqrt{I} = I$, then* $\mathrm{SAG}_I(R)$ *is not complete.*

Proof. Suppose by contradiction, $\mathrm{SAG}_I(R)$ is complete. According to Lemma 4.2, for every $T \in V'$, $\mathrm{Ann}(\frac{T}{I}) \in \mathrm{ESS}(\frac{R}{I})$, so $\mathrm{Ann}(\frac{T}{I}) \cap \frac{T}{I} \neq 0$. Then, there exists $t_0 \in T \setminus I$, such that $t_0 T \subseteq I$, thus $t_0^2 \in I$. Now, Since $\sqrt{I} = I$, $t_0 \in I$, a contradiction. □

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