Algebraic Structures and Their Applications Vol. 12 No. 4 (2025) pp 333-352.

Research Paper

IMPLICATIVE PRE-HILBERT ALGEBRAS AND THEIR CONNECTIONS WITH OTHER ALGEBRAS OF LOGIC

ANDRZEJ WALENDZIAK*

Abstract. In the paper, implicative pre-Hilbert algebras are introduced and studied, their characterizations and connections with some algebras of logic are presented. Some important results and examples are given. In particular, it is proven that an implicative pre-Hilbert algebra is equivalent to an implicative pre-BCC algebra. It is shown that for any Hilbert algebra, the implicative property is equivalent to the commutative property. Moreover, several old or new characterizations of Tarski algebras are established. We prove that Tarski algebras coincide with commutative pre-Hilbert algebras and with implicative generalized exchange algebras satisfying the property of antisymmetry. Finally, the hierarchies existing between all classes of implicative algebras considered here are shown.

DOI: 10.22034/as.2025.22162.1743

MSC(2010): Primary: 03G25; Secondary: 06A06, 06F35.

Keywords: BCC algebra, BCK algebra, Hilbert algebra, Implicativity, Pre-Hilbert algebra.

Received: 19 September 2024, Accepted: 02 April 2025.

*Corresponding author

1. Introduction

L. Henkin [6] introduced the notion of "implicative model", as a model of positive implicative propositional calculus. In 1960, A. Monteiro [14] has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, K. Iséki [8] introduced a new notion called a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [13]. To solve some problems on BCK algebras, Y. Komori [12] introduced BCC algebras. These algebras (also called BIK^+-algebras) are an algebraic model of BIK^+-logic. In [11], as a generalization of BCK algebras, H. S. Kim and Y. H. Kim defined BE algebras. Later on, in 2010, D. Buşneag and S. Rudeanu [4] introduced the notion of pre-BCK algebra. A BCK algebra is just a pre-BCK with the antisymmetry. In 2016, A. Iorgulescu [7] introduced new generalizations of BCK and Hilbert algebras (RML, pre-BCC, aBE, pi-BE algebras and many others).

In 2021, R. K. Bandaru et al. [1] introduced the concepts of GE algebra (generalized exchange algebra) and transitive GE algebra (tGE algebra for short). These algebras have many connections with other algebras of logic. Recently, A. Walendziak [20] introduced pre-Hilbert algebras as a natural generalization of Hilbert algebras.

The concepts of commutativity and implicativity in the theory of BCK algebras were introduced by K. Iséki and S. Tanaka ([15], [9]). In [16], the property of implicativity for various generalizations of BCK algebras was studied. Implicative BE algebras were presented in [18].

Here we consider RML, BE, GE, tGE, pre-BCC, pre-BCK and pre-Hilbert algebras and present the connections between these algebras. We give some important results and examples. We introduce implicative pre-Hilbert algebras and investigate their properties. We show that the class of implicative pre-Hilbert algebras coincides with the class of implicative pre-BCC algebras. We prove that for any Hilbert algebra the implicative property is equivalent to the commutative property. Moreover, we establish several old or new characterizations of Tarski algebras. In particular, we show that Tarski algebras coincide with commutative pre-Hilbert algebras and with implicative generalized exchange algebras satisfying the antisymmetry property. Finally, we draw the hierarchies existing between all classes of implicative algebras considered here.

The motivation of this study consists of algebraic and logical arguments. Namely, pre-Hilbert algebras are related to Henkin's Positive Implicative Logic, they belong to a wide class of algebras of logic. An additional motivation is the fact that the present paper is a continuation of the author's earlier publications [20] and [21] on pre-Hilbert algebras. Moreover, the results of the paper may have applications for future studies of some generalizations of Hilbert algebras.

2. Preliminaries

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2,0)$. We define the binary relation $\leq$ by: for all $x, y \in A$,

$$x \leq y \Longleftrightarrow x \rightarrow y = 1.$$

We consider the following list of properties ([7]) that can be satisfied by $\mathcal{A}$:

(An) (Antisymmetry) $(x \leq y$ and $y \leq x) \Longrightarrow x = y$,

(B) $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(C) $x \rightarrow (y \rightarrow z) \leq y \rightarrow (x \rightarrow z)$,

(D) $y \leq (y \rightarrow x) \rightarrow x$,

(Ex) (Exchange) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(K) $x \leq y \rightarrow x$,

(L) (Last element) $x \leq 1$,

(M) $1 \rightarrow x = x$,

(Re) (Reflexivity) $x \leq x$,

(Tr) (Transitivity) $(x \leq y$ and $y \leq z) \Longrightarrow x \leq z$,

(*) $y \leq z \Longrightarrow x \rightarrow y \leq x \rightarrow z$,

(**) $y \leq z \Longrightarrow z \rightarrow x \leq y \rightarrow x$.

Remark 2.1. *The properties in the list are the most important properties satisfied by a BCK algebra.*

Lemma 2.2. ([7]) *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2,0)$. Then the following hold*

(i) *(M) + (B) $\Longrightarrow$ (*), (**);*

(ii) *(M) + (*) $\Longrightarrow$ (Tr);*

(iii) *(M) + (**) $\Longrightarrow$ (Tr);*

(iv) *(C) + (An) $\Longrightarrow$ (Ex);*

(v) *(M) + (L) + (**) $\Longrightarrow$ (K).*

Following Iorgulescu [7], we say that $(A, \rightarrow, 1)$ is an *RML algebra* if it verifies the axioms (Re), (M), (L). We recall now the following definition.

Definition 2.3. *([7]) Let $\mathcal{A} = (A, \rightarrow, 1)$ be an RML algebra. The algebra $\mathcal{A}$ is said to be*

1. an *aRML algebra* if it verifies (An),

2. a *pre-BCC algebra* if it verifies (B),

3. a *BCC algebra* if it verifies (B), (An), that is, it is a pre-BCC algebra with (An),

4. a *BE algebra* if it verifies (Ex),

5. an *aBE algebra* if it verifies (Ex), (An), that is, it is a BE algebra with (An),

6. a *pre-BCK algebra* if it verifies (B), (Ex), that is, it is a pre-BCC algebra with (Ex) or, equivalently, it is a BE algebra with (B),

7. a *BCK algebra* if it is a pre-BCK algebra verifying (An).

Denote by **RML**, **aRML**, **pre-BCC**, **BCC**, **BE**, **aBE**, **pre-BCK**, **BCK** the classes of RML, aRML, pre-BCC, BCC, BE, aBE, pre-BCK, BCK algebras respectively.

By definitions, we have

pre-BCC = **RML** + (B), **BE** = **RML** + (Ex),

pre-BCK = **pre-BCC** + (Ex) = **BE** + (B),

aRML = **RML** + (An), **BCC** = **pre-BCC** + (An) = **aRML** + (B),

aBE = **BE** + (An) = **aRML** + (Ex),

BCK = **pre-BCK** + (An) = **BCC** + (Ex) = **aBE** + (B).

The interrelationships between the classes of algebras mentioned above are visualized in Figure 1.

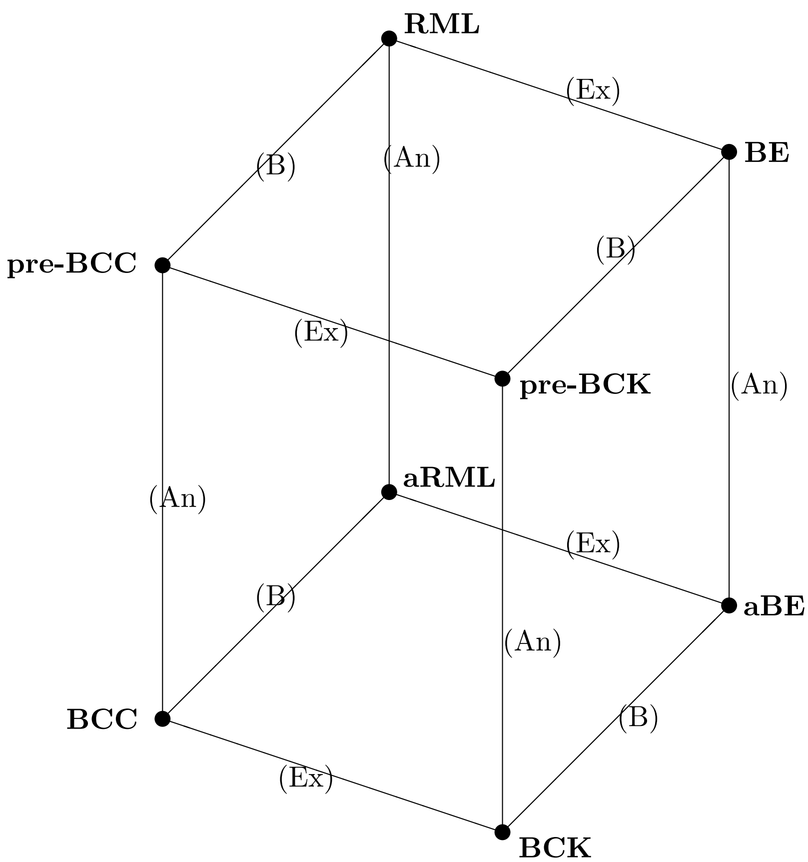

FIGURE 1

It is known that $\leq$ is an order relation in BCC and BCK algebras. By definition, in RML and BE algebras, $\leq$ is a reflexive relation; in aRML and aBE algebras, $\leq$ is reflexive and antisymmetric. By Lemma 2.2 (i)–(ii), in pre-BCC and pre-BCK algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation).

3. On pre-Hilbert and GE algebras

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Now, we consider the following properties (in fact, the properties satisfied by Hilbert algebras):

(pi) $x \rightarrow (x \rightarrow y) = x \rightarrow y$,

(p-1) $x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(p-2) $(x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z)$,

(pimpl) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(GE) $x \rightarrow (y \rightarrow z) = x \rightarrow (y \rightarrow (x \rightarrow z))$.

Lemma 3.1. *Let $(A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following are true:*

(i) *(Re) + (M) + (pimpl) $\Longrightarrow$ (pi), (p-1), (p-2), (K), (B);*

(ii) *(p-1) + (p-2) + (An) $\Longrightarrow$ (pimpl);*

(iii) *(Ex) + (pi) $\Longrightarrow$ (GE).*

Proof. (i) By Propositions 6.4 and 6.9 of [7].

(ii) Obvious.

(iii) By Proposition 2.7 (ii) of [19]. □

Remark 3.2. *From Lemma 3.1 (i) it follows that in RML algebras, (pimpl) implies (pi). For BCK algebras, (pimpl) and (pi) are equivalent (cf. Theorem 8 of* [9]*).*

Recall the following definitions.

Definition 3.3. *An algebra $(A, \rightarrow, 1)$ is a*

1. *Hilbert algebra* [5] if it verifies the axioms (An), (K), (p-1);

2. *pre-Hilbert algebra* [20] if it verifies the axioms (M), (K), (p-1);

3. *generalized exchange algebra* [1] (briefly, GE algebra) if it verifies the axioms (Re), (M), (GE);

4. *pi-RML algebra* ([7]) if it is an RML algebra verifying (pi).

Let us denote by **GE**, **pre-H** and **H** the classes of GE, pre-Hilbert and Hilbert algebras, respectively. Moreover, let **pi-RML** denote the class of pi-RML algebras; similarly for the subclasses of **RML**.

Remark 3.4. *(1) In* [5]*, A. Diego proved that Hilbert algebras satisfy (Re), (M), (L), (pi), (p-2), (pimpl). Moreover, he showed that the class of all Hilbert algebras is a variety. Since (An) + (K) + (p-1) imply (M) (see* [5]*), a Hilbert algebra is in fact a pre-Hilbert algebra verifying (An), that is,* **H** *=* **pre-H** *+ (An).*

(2) Note that the definition of pre-Hilbert algebra is inspired by Henkin's Positive Implicative

Logic [6] *(see also Remark 3.7 of* [20]*).*
(3) For examples of GE algebras and pre-Hilbert algebras we refer the reader to [1] *and* [20]*, respectively.*
(4) Remark that the connections between subclasses of the class **pi-RML** *have been presented in* [7].

Remark 3.5. *(1) By definitions,*
pi-RML = **RML** + *(pi),* **pi-BE** = **BE** + *(pi),* **pi-pre-H** = **pre-H** + *(pi),*
pi-pre-BCC = **pre-BCC** + *(pi)* = **pi-RML** + *(B),*
pi-pre-BCK = **pi-pre-BCC** + *(Ex)* = **pi-BE** + *(B)* = **pre-BCK** + *(pi),*
pi-BCC = **pi-pre-BCC** + *(An),* **pi-aBE** = **pi-BE** + *(An).*
(2) Note that from Remark 6.19 of [7] *we have* **H** = **pi-BCK**. *Hence*
H = **pi-pre-BCK** + *(An)* = **pi-BCC** + *(Ex)* = **pi-aBE** + *(B).*

Proposition 3.6. ([1], Theorem 3.3) *GE algebras satisfy* (Re), (M), (L), (pi), (K), (C), (D).

Corollary 3.7. *Any GE algebra is a pi-RML algebra.*

Since GE algebras and Hilbert algebras satisfy (pi), we have **pi-GE** = **GE** and **pi-H** = **H**. By Lemma 3.1 (iii), we get

Corollary 3.8. *Any pi-BE algebra is a GE algebra.*

Example 3.9. *(*[19]*, Example 2.5) Consider the set* $A = \{a, b, c, d, e, 1\}$ *and the operation* $\to$ *given by the following table*

$\to$	a	b	c	d	e	1
a	1	1	c	c	1	1
b	a	1	d	d	1	1
c	a	1	1	1	1	1
d	a	1	1	1	1	1
e	a	1	1	1	1	1
1	a	b	c	d	e	1

We can observe that the properties (Re), (M), (L), (GE) (hence (pi)) are satisfied. Therefore, $(A, \to, 1)$ *is a GE algebra. It does not satisfy (An) for* $(x, y) = (c, d)$*; (Ex) for* $(x, y, z) = (a, b, c)$*; (B) and (p-1) for* $(x, y, z) = (a, e, c)$*. Then,* $\mathcal{A}$ *is not a BE algebra, not a pre-BCC algebra and not a pre-Hilbert algebra.*

Theorem 3.10. ([20], Theorem 3.9) *Pre-Hilbert algebras satisfy* (Re), (M), (L), (K), (B), (C), (D), (Tr), (*), (**), (p-1), (p-2).

Corollary 3.11. *Any pre-Hilbert algebra is a pre-BCC algebra.*

Example 3.12 below shows that the converse statement is not true; that is, there are pre-BCC algebras which are not pre-Hilbert algebras.

Example 3.12. *([7], Example 9.24) Let $A = \{a, b, c, d, 1\}$ and $\rightarrow$ be given by the following table*

$\rightarrow$	a	b	c	d	1
a	1	a	c	c	1
b	1	1	d	c	1
c	a	b	1	1	1
d	a	b	1	1	1
1	a	b	c	d	1

Then $(A, \rightarrow, 1)$ verifies (Re), (M), (L), (B). It does not verify (An) for $x = c$, $y = d$; (Ex) for $x = a, y = b, z = c$; (GE) for $x = a, y = 1, z = b$; (pi) for $x = a, y = b$; (p-1) and (pimpl) for $x = y = a, z = b$. Therefore, $(A, \rightarrow, 1)$ is a pre-BCC algebra not verifying (An), (Ex), (GE), (pi), (p-1) and (pimpl).

Theorem 3.13. *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. The following are equivalent:*

(i) *$\mathcal{A}$ is a pre-Hilbert algebra;*
(iii) *$\mathcal{A}$ satisfies* (M), (L), (B) *and* (p-1).

Proof. (i) $\Longrightarrow$ (ii). Follows from Theorem 3.10.

(ii) $\Longrightarrow$ (i). By Lemma 2.2 (i), (M) + (B) imply (**). By Lemma 2.2 (v), (M) + (L) + (**) imply (K). Then $\mathcal{A}$ satisfies (M), (K) and (p-1). Thus $\mathcal{A}$ is a pre-Hilbert algebra. □

Remark 3.14. *Pre-Hilbert algebras do not have to satisfy (An), (Ex), (GE), (pi), (pimpl); see example below.*

Example 3.15. *([20], Example 3.13) Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\rightarrow$ given by the following table*

$\rightarrow$	a	b	c	d	1
a	1	c	b	d	1
b	a	1	1	d	1
c	a	1	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

We can observe that the properties (M), (K), (p-1) are verified. Then, $(A, \to, 1)$ *is a pre-Hilbert algebra. It does not verify (An) for* $(x, y) = (b, c)$*; (Ex) and (pimpl) for* $(x, y, z) = (a, d, b)$*; (pi) for* $(x, y) = (a, b)$*; (GE) for* $(x, y, z) = (a, 1, b)$.

Now we give a characterization of pi-pre-Hilbert algebras.

Theorem 3.16. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$*. The following are equivalent:*

(i) $\mathcal{A}$ *is a pi-pre-Hilbert algebra;*

(ii) $\mathcal{A}$ *is a pi-pre-BCC algebra with (C).*

Proof. (i) $\Longrightarrow$ (ii). Follows from Theorem 3.10.

(ii) $\Longrightarrow$ (i). Let $\mathcal{A}$ be a pi-pre-BCC algebra verifying (C). Then $\mathcal{A}$ verifies (Re), (M), (L), (B) (hence, by Lemma 2.2, (*), (**), (Tr), (K)), (pi), (C). To prove (p-1), let $x, y, z \in A$. From (B) we conclude that $y \to z \leq (x \to y) \to (x \to z)$. Using (*), we get

$$x \to (y \to z) \leq x \to ((x \to y) \to (x \to z)). \tag{1}$$

By (C),

$$x \to ((x \to y) \to (x \to z)) \leq (x \to y) \to (x \to (x \to z)). \tag{2}$$

Applying (Tr) and (pi), from (1) and (2) we obtain $x \to (y \to z) \leq (x \to y) \to (x \to (x \to z)) = (x \to y) \to (x \to z)$. Consequently, $\mathcal{A}$ is a pi-pre-Hilbert algebra. □

Example 3.17. *Let* $A = \{a, b, c, d, e, 1\}$ *and* $\to$ *be defined as follows*

$\to$	a	b	c	d	e	1
a	1	1	e	d	e	1
b	1	1	d	d	e	1
c	a	b	1	1	1	1
d	a	b	1	1	1	1
e	a	b	1	1	1	1
1	a	b	c	d	e	1

It is easy to see that the algebra $(A, \to, 1)$ *verifies properties (Re), (M), (L), (B) and (pi). From Therem 3.16 we conclude that* $(A, \to, 1)$ *is a pi-pre-Hilbert algebra. It does not verify (An) for* $x = a$*,* $y = b$*; (Ex), (GE) and (pimpl) for* $x = a, y = b, z = c$.

Example 3.18. *([7], Example 10.3) Let* $A = \{a, b, c, d, 1\}$ *and* $\to$ *be given by the following table*

$\to$	a	b	c	d	1
a	1	b	b	b	1
b	a	1	c	c	1
c	a	1	1	1	1
d	a	1	1	1	1
1	a	b	c	d	1

Then $(A, \to, 1)$ is a pi-pre-BCC algebra. It does not verify (An) for $x = c$, $y = d$; (Ex) and (GE) for $x = a, y = b, z = c$; (p-1) for $x = b$, $y = a$, $z = c$.

Remark 3.19. *(1) From Theorem 3.13 we see that* **pre-H** = **pre-BCC** *+ (p-1). By Theorem 3.16,* **pi-pre-H** = **pi-pre-BCC** *+ (C). Hence*
pi-pre-H *+ (Ex) =* **pi-pre-BCC** *+ (C) + (Ex) =* **pi-pre-BCC** *+ (Ex) =* **pi-pre-BCK**, *since (Re) + (Ex) imply (C).*
(2) By definition and Corollary 3.7, **GE** = **RML** *+ (GE) =* **RML** *+ (pi) + (GE) =* **pi-RML** *+ (GE). Hence, by Lemma 3.1 (iii),*
GE *+ (Ex) =* **pi-RML** *+ (GE) + (Ex) =* **pi-RML** *+ (Ex) =* **pi-BE**.

Remark 3.20. *(1) Examples 3.12 and 3.15 show that the inclusions below are proper.*
pre-BCC $\supset$ **pre-H** $\supset$ **pi-pre-H**.
(2) By Examples 3.9 and 3.18, **GE** $\supset$ **pi-BE** *and* **pi-pre-BCC** $\supset$ **pi-pre-H**, *respectively.*

By Remarks 3.5, 3.19 and 3.20, we can draw now the hierarchy between **RML** and **H**, in the next Figure 2.

Following [1], we say that a GE algebra is *transitive* if it satisfies (B). For examples of transitive GE algebras we refer to [1, 19]. From Corollary 3.8 of [19] it follows that any transitive GE algebra verifies (p-1). Since any GE algebra verifies properties (M), (K) and (pi), we have

Proposition 3.21. *Any transitive GE algebra is a pi-pre-Hilbert algebra.*

Proposition 3.22. *We have* **tGE** = **GE** $\cap$ **pi-pre-H** = **GE** $\cap$ **pre-H**, *where* **tGE** *denotes the class of all transitive GE algebras.*

Proof. If $\mathcal{A}$ belongs to **GE** and **pre-H**, then $\mathcal{A}$ is a tGE algebra, since a pre-Hilbert algebra satisfies (B). Hence **GE** $\cap$ **pi-pre-H** $\subseteq$ **GE** $\cap$ **pre-H** $\subseteq$ **tGE**. The converse inclusions follow from Proposition 3.21. □

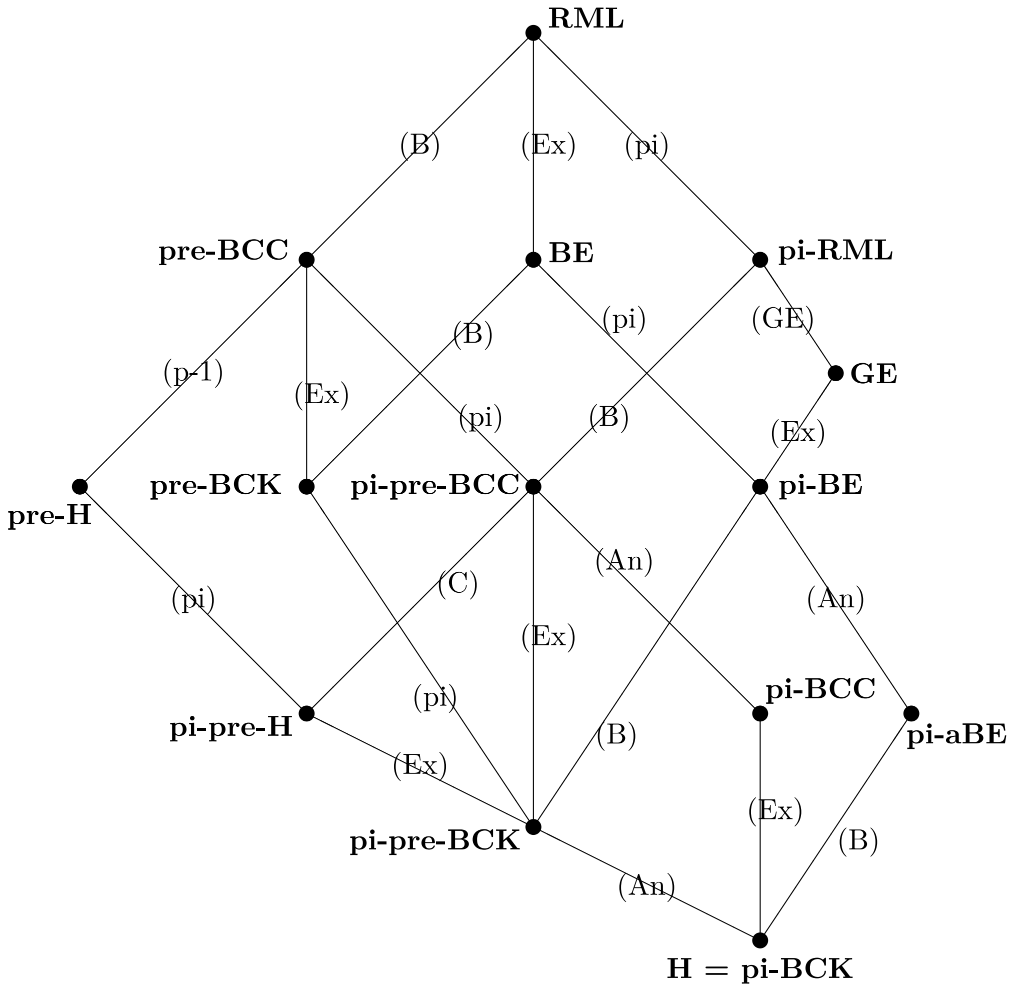

FIGURE 2.

Following [3], we say that a GE algebra is *antisymmetric* (*aGE algebra* for short) if it verifies (An). Denote by **aGE** the class of antisymmetric GE algebras. From Remark 3.4 of [19] we see that **aGE** = **pi-aBE**.

4. Implicative pre-Hilbert algebras

The well-known implicative and commutative BCK algebras were introduced by K. Iséki and S. Tanaka ([9], [15]).

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We first consider the following properties: for all $x, y \in A$,

(im) (implicative) $(x \to y) \to x = x$,

(Com) (commutative) $(x \to y) \to y = (y \to x) \to x$.

Lemma 4.1. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$. *Then*

(i) *(Re) + (im)* $\Longrightarrow$ *(M),*

(ii) *(M) + (im)* $\Longrightarrow$ *(L),*

(iii) *(im)* $\Longrightarrow$ *(pi),*

(iv) *(M) + (K) + (*) + (**) + (im)* $\Longrightarrow$ *(C),*

(v) *(M) + (Com)* $\Longrightarrow$ *(An),*

(vi) *(M) + (K) + (pimpl) + (Com)* $\Longrightarrow$ *(im),*

(vii) *(Re) + (M) + (pimpl) + (Com)* $\Longrightarrow$ *(im).*

Proof. (i)–(iii) follow from Proposition 3.5 of [16].

(iv) By Lemma 2.2 (ii), $\mathcal{A}$ satisfies (Tr). To prove (C), let $x, y, z \in A$. From (K) it follows that $z \leq x \to z$. Applying (*) twice, we get

$$x \to (y \to z) \leq x \to (y \to (x \to z)). \tag{3}$$

Since $x \to z \leq y \to (x \to z)$, by (**) and (im), $(y \to (x \to z)) \to x \leq (x \to z) \to x = x$, that is, $(y \to (x \to z)) \to x \leq x$. Similarly, using (**) and (im), we have $x \to (y \to (x \to z)) \leq [(y \to (x \to z)) \to x] \to (y \to (x \to z)) = y \to (x \to z)$. Therefore,

$$x \to (y \to (x \to z)) \leq y \to (x \to z). \tag{4}$$

From (3) and (4) we see that (C) holds in $\mathcal{A}$.

(v) follows from Proposition 3.3 (i) of [17].

(vi) follows from Proposition 3.2 (vii) of [18].

(vii) Let $\mathcal{A}$ satisfy (Re), (M), (pimpl) and (Com). By Lemma 3.1 (i), $\mathcal{A}$ also satisfies (K). Hence, from above (vi) we obtain (vii). □

Recall the definition of Tarski algebras. A *Tarski algebra* is an algebra $\mathcal{A} = (A, \to, 1)$ of type $(2, 0)$ satisfying the following axioms ([10]): (Re), (M), (pimpl) and (Com). By definition and Lemma 4.1 (vii), Tarski algebras satisfy (Com) and (im). Note that Hilbert algebras do not have to satisfy these properties.

As in the case of BCK algebras, we now define:

Definition 4.2. *An RML algebra $\mathcal{A}$ is called*

- *implicative* if it satisfies (im),
- *commutative* if it satisfies (Com).

Remark that implicative RML algebras and commutative RML algebras were investigated in [16] and [17], respectively. From Lemma 4.1 (i) and (ii) we have

Proposition 4.3. *The class of implicative RML algebras is characterized by the axioms:* (Re) *and* (im).

Remark 4.4. *In 2017, A. Borumand Saeid, H. S. Kim and A. Rezaei introduced BI-algebras ([2]). They defined a BI-algebra as an algebra $(A, \to, 1)$ of type $(2,0)$ satisfying the following axioms:*

(B1) $x \to x = 1$,

(B2) $(x \to y) \to x = x$.

Note that (B1) is (Re) and (B2) is (im). Thus, BI-algebras are implicative RML algebras.

Proposition 4.5. ([1], Theorem 3.9) *If $\mathcal{A}$ is a commutative GE algebra, then it is a Hilbert algebra.*

Denote by **T**, **im-RML** and **com-RML** the classes of Tarski algebras, implicative RML algebras and commutative RML algebras, respectivelly. Similarly for the subclasses of **RML**.

Now we give several characterizations of implicative pre-Hilbert algebras.

Theorem 4.6. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2,0)$. The following are equivalent:*

(i) *$\mathcal{A}$ is an implicative pre-Hilbert algebra;*

(ii) *$\mathcal{A}$ is an implicative pre-BCC algebra;*

(iii) *$\mathcal{A}$ satisfies (Re), (B), (im);*

(iv) *$\mathcal{A}$ satisfies (Re), (im), (*), (**);*

(v) *$\mathcal{A}$ is an implicative RML algebra satisfying (*) and (**).*

Proof. (i) $\Longrightarrow$ (ii). Follows from Theorem 3.10.

(ii) $\Longrightarrow$ (iii). Obvious.

(iii) $\Longrightarrow$ (iv). By Lemma 4.1 (i), (Re) + (im) imply (M). By Lemma 2.2 (i), (M) + (B) imply (*) and (**).

(iv) $\Longrightarrow$ (v). Follows from Lemma 4.1 (i) and (ii).

(v) $\Longrightarrow$ (i). By Lemma 2.2 (v), (M) + (L) + (**) imply (K). From Lemma 4.1 (iii) and (iv) we see that $\mathcal{A}$ satisfies (pi) and (C). Now observe that $\mathcal{A}$ also satisfies (D). By (Re) and (C),

$$1 = (y \to x) \to (y \to x) \le y \to ((y \to x) \to x).$$

Then $1 = 1 \to (y \to ((y \to x) \to x)) = y \to ((y \to x) \to x)$ by (M). Thus (D) holds in $\mathcal{A}$. To prove (p-1), let $x, y, z \in A$. From (D) it follows that $y \le (y \to z) \to z$. Hence, applying (*) and (C), we have

$$x \to y \le x \to [(y \to z) \to z] \le (y \to z) \to (x \to z),$$

that is, $x \to y \le (y \to z) \to (x \to z)$. Then, by (D) and (**),

$$y \to z \le [(y \to z) \to (x \to z)] \to (x \to z) \le (x \to y) \to (x \to z).$$

Thus $y \to z \leq (x \to y) \to (x \to z)$, and hence, using (*), we obtain

$$x \to (y \to z) \leq x \to ((x \to y) \to (x \to z)). \tag{5}$$

We have $x \to ((x \to y) \to (x \to z)) \overset{(C)}{\leq} (x \to y) \to [x \to (x \to z)] \overset{(pi)}{=} (x \to y) \to (x \to z)$. Thus

$$x \to ((x \to y) \to (x \to z)) \leq (x \to y) \to (x \to z). \tag{6}$$

From (5) and (6) we get $x \to (y \to z) \leq (x \to y) \to (x \to z)$, that is, (p-1) holds in $\mathcal{A}$. Consequently, $\mathcal{A}$ is an implicative pre-Hilbert algebra. □

Example 4.7. *Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table*

$\to$	a	b	c	d	1
a	1	b	b	d	1
b	a	1	a	a	1
c	1	1	1	1	1
d	a	1	1	1	1
1	a	b	c	d	1

The algebra $(A, \to, 1)$ is an implicative RML algebra. It does not verify (An) and (Com) for $(x, y) = (c, d)$; (Ex) and (GE) for $(x, y, z) = (b, a, d)$; (B) and (p-1) for $(x, y, z) = (d, c, a)$.

Example 4.8. *Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table*

$\to$	a	b	c	d	1
a	1	1	c	c	1
b	d	1	1	d	1
c	a	a	1	a	1
d	b	b	b	1	1
1	a	b	c	d	1

It is easy to see that the properties (im), (An), (Re), (M) and (L) are satisfied. Hence, $\mathcal{A} = (A, \to, 1)$ is an implicative aRML algebra. It does not satisfy (Ex) and (GE) for $(x, y, z) = (a, b, a)$, (B) and (p-1) for $(x, y, z) = (a, b, c)$.

Example 4.9. *(*[19]*, Example 4.12) Let $A = \{a, b, c, d, e, 1\}$ and $\to$ be defined as follows*

$\to$	a	b	c	d	e	1
a	1	1	e	d	e	1
b	1	1	c	d	d	1
c	b	b	1	1	1	1
d	a	b	1	1	1	1
e	a	a	1	1	1	1
1	a	b	c	d	e	1

The properties (Re), (M), (L), (K), (p-1) and (im) are satisfied. Consequently, $(A, \to, 1)$ *is an implicative pre-Hilbert algebra. It does not satisfy (An) and (Com) for* $(x, y) = (a, b)$*; (Ex) and (GE) for* $(x, y, z) = (a, b, c)$.

Example 4.10. *Let* $A = \{a, b, c, d, 1\}$ *and* $\to$ *be given by the following table*

$\to$	a	b	c	d	1
a	1	b	b	1	1
b	a	1	1	1	1
c	1	1	1	d	1
d	1	c	c	1	1
1	a	b	c	d	1

We can observe that the properties (Re), (M), (L), (GE) and (im) are verified. Then $(A, \to, 1)$ *is an implicative GE algebra. It does not verify (An) and (Com) for* $x = b$, $y = c$*; (Ex) for* $x = a$, $y = d$, $z = b$*; (B) for* $x = c$, $y = b$, $z = d$.

Remark 4.11. *(1) By definitions,*
im-RML = **RML** + *(im),* **im-GE** = **im-RML** + *(GE),*
im-tGE = **im-GE** + *(B),* **im-aRML** = **im-RML** + *(An).*
(2) By Lemma 3.1 (iii), (pi) + (Ex) imply (GE). Hence (im) + (Ex) imply (GE), because (im) implies (pi). Consequently,
im-BE = **im-RML** + *(Ex)* = **im-RML** + *(GE)* + *(Ex)* = **im-GE** + *(Ex).*
(3) By Theorem 4.6, **im-pre-H** = **im-pre-BCC** = **im-RML** + *(B). Hence*
im-pre-H + *(GE)* = **im-RML** + *(B)* + *(GE)* = **im-GE** + *(B)* = **im-tGE**.
Then, **im-tGE** + *(Ex)* = **im-GE** + *(Ex)* + *(B)* = **im-BE** + *(B)* = **im-pre-BCK**.

Remark 4.12. *(1) From Example 4.7 it follows that* **im-pre-H**, **im-GE** *and* **im-aRML** *are proper subclasses of* **im-RML**.
(2) Examples 4.9 and 4.10 show that the following inclusions:
im-tGE $\subset$ **im-pre-H**, **im-tGE** $\subset$ **im-GE** *and* **im-BE** $\subset$ **im-GE**
are proper.

(3) By Example 4.13 below, **im-pre-BCK** $\subset$ **im-tGE**. *It is clear that* **im-pre-BCK** *is also a proper subclass of* **im-BE** *(see* [18]*).*

Example 4.13. *(*[19]*, Example 2.6) Let* $A = \{a, b, c, d, 1\}$ *and* $\rightarrow$ *be defined as follows*

$\rightarrow$	a	b	c	d	1
a	1	1	c	c	1
b	1	1	d	d	1
c	a	a	1	1	1
d	b	b	1	1	1
1	a	b	c	d	1

The algebra $\mathcal{A} = (A, \rightarrow, 1)$ *verifies (Re), (M), (L), (GE), (B), (im). It does not verify (An) for* $x = a, y = b$*; (Ex) for* $x = a, y = b, z = c$*. Thus* $\mathcal{A}$ *is an implicative tGE algebra which is not a pre-BCK algebra.*

Example 4.14. *Consider the set* $A = \{a, b, c, 1\}$ *and the operation* $\rightarrow$ *given by the following table*

$\rightarrow$	a	b	c	1
a	1	b	1	1
b	a	1	c	1
c	1	b	1	1
1	a	b	c	1

The algebra $\mathcal{A} = (A, \rightarrow, 1)$ *satisfies properties (im), (Re), (M), (L), (B) (hence also (*), (**), (Tr)) and (Ex). It does not satisfy (An) for* $(x, y) = (a, c)$*. Hence,* $\mathcal{A}$ *is an implicative pre-BCK algebra which is not a BCK algebra.*

Lemma 4.15. ([22]) *Let* $\mathcal{A}$ *be an implicative BE algebra. Then* $\mathcal{A}$ *satisfies:*

$$x \rightarrow y = (z \rightarrow x) \rightarrow (x \rightarrow y) \tag{7}$$

for $x, y, z \in A$*.*

Lemma 4.16. *Let* $\mathcal{A}$ *be an implicative BE algebra and* $x, y \in A$*. If* $z = ((x \rightarrow y) \rightarrow y) \rightarrow x$*, then*

$$x \rightarrow (z \rightarrow y) = x \rightarrow y, \tag{8}$$

$$[x \rightarrow (z \rightarrow y)] \rightarrow y = (x \rightarrow y) \rightarrow (z \rightarrow y). \tag{9}$$

Proof. Set $t = (x \rightarrow y) \rightarrow y$. Then $z = t \rightarrow x$. We first prove (8). Applying (Ex) and (7), we get $x \rightarrow (z \rightarrow y) = z \rightarrow (x \rightarrow y) = (t \rightarrow x) \rightarrow (x \rightarrow y) = x \rightarrow y$, that is, (8) holds. Now we show (9). We have

$$[x \to (z \to y)] \to y \overset{(8)}{=} (x \to y) \to y = t \overset{(\mathrm{im})}{=} (t \to x) \to t \tag{10}$$

$$= z \to ((x \to y) \to y) \overset{(\mathrm{Ex})}{=} (x \to y) \to (z \to y). \tag{11}$$

Therefore, (9) also holds. □

Theorem 4.17. *Implicative aGE algebras are commutative Hilbert algebras.*

Proof. Let $\mathcal{A}$ be an implicative aGE algebra. By definition and Proposition 3.6, $\mathcal{A}$ satisfies (An), (K), (C) and (Ex), since (An) + (C) imply (Ex). Let $x, y \in A$. Set $t = (x \to y) \to y$ and $z = t \to x$. We have

$$\begin{aligned}(z \to y) \to y &\overset{(7)}{=} (x \to (z \to y)) \to ((z \to y) \to y)\\ &\overset{(\mathrm{Ex})}{=} (z \to y) \to [(x \to (z \to y)) \to y]\\ &\overset{(9)}{=} (z \to y) \to [(x \to y) \to (z \to y)] \overset{(\mathrm{K})}{=} 1.\end{aligned}$$

Hence $z \to y \leq y$. By (K), $y \leq z \to y$. Using (An), we obtain $z \to y = y$. Therefore, $(z \to y) \to z = y \to z$, that is,

$$z = y \to (t \to x). \tag{12}$$

Now we prove

$$[(x \to y) \to y] \leq [(y \to x) \to x]. \tag{13}$$

We have $[(x \to y) \to y] \to [(y \to x) \to x] \overset{(\mathrm{Ex})}{=} (y \to x) \to z \overset{(12)}{=} (y \to x) \to [y \to (t \to x)] \overset{(\mathrm{Ex})}{=}$ $(y \to x) \to [t \to (y \to x)] \overset{(\mathrm{K})}{=} 1$. Thus (13) holds. Since $\mathcal{A}$ satisfies (An), we see that it is commutative. Therefore, by Proposition 4.5, $\mathcal{A}$ is a Hilbert algebra. □

Proposition 4.18. *For Hilbert algebras, the implicative property is equivalent to the commutative property.*

Proof. Let $\mathcal{A}$ be an implicative Hilbert algebra. It is easy to see that $\mathcal{A}$ is an implicative aGE algebra. By Theorem 4.17, $\mathcal{A}$ is commutative. The converse follows from Lemma 4.1 (vi). □

Remark 4.19. For pre-Hilbert algebras, the implicative property is not equivalent to the commutative property.

We now present several old or new characterizations of Tarski algebras.

Theorem 4.20. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$. *The following are equivalent:*

(i) $\mathcal{A}$ *is a Tarski algebra;*

(ii) $\mathcal{A}$ *satisfies (Re), (M), (pimpl), (Com);*

(iii) $\mathcal{A}$ *is a commutative pre-Hilbert algebra;*

(iv) $\mathcal{A}$ *is a commutative Hilbert algebra;*

(v) $\mathcal{A}$ *is an implicative Hilbert algebra;*

(vi) $\mathcal{A}$ *satisfies (Re), (An), (B), (im);*

(vii) $\mathcal{A}$ *is an implicative BCC algebra;*

(viii) $\mathcal{A}$ *is an implicative BCK algebra;*

(ix) $\mathcal{A}$ *is an implicative aBE algebra;*

(x) $\mathcal{A}$ *is an implicative aGE algebra;*

(xi) $\mathcal{A}$ *is a commutative GE algebra.*

Proof. (i) $\Longrightarrow$ (ii). By definition.

(ii) $\Longrightarrow$ (iii). From Lemma 3.1 (i) we see that $\mathcal{A}$ also satisfies (K) and (p-1). Hence $\mathcal{A}$ is a commutative pre-Hilbert algebra.

(iii) $\Longrightarrow$ (iv). Since (M) and (Com) imply (An), see Lemma 4.1 (v), we conclude that $\mathcal{A}$ is a Hilbert algebra. Obviously, $\mathcal{A}$ is commutative.

(iv) $\Longrightarrow$ (v). By Proposition 4.18.

(v) $\Longrightarrow$ (vi) and (vi) $\Longrightarrow$ (vii). Obvious.

(vii) $\Longrightarrow$ (viii). By definition, (Re), (M), (L), (An), (B), (im) hold in $\mathcal{A}$. By Lemma 2.2 (i), (M) + (B) imply (*) and (**). By Lemma 2.2 (v), (M) + (L) + (**) imply (K). Applying Lemma 4.1 (iv), we conclude that $\mathcal{A}$ satisfies (C). Then (Ex) holds in $\mathcal{A}$, by Lemma 2.2 (iv). Thus $\mathcal{A}$ is a BCK algebra.

(viii) $\Longrightarrow$ (ix). Obvious.

(ix) $\Longrightarrow$ (x). By Lemma 4.1 (iii), (im) implies (pi). From Corollary 3.8 it follows that $\mathcal{A}$ is a GE algebra. Then (x) holds.

(x) $\Longrightarrow$ (xi). From Theorem 4.17 we see that $\mathcal{A}$ is a commutative GE algebra.

(xi) $\Longrightarrow$ (i). By Proposition 4.5, $\mathcal{A}$ is a commutative Hilbert algebra. Then $\mathcal{A}$ satisfies (Re), (M), (pimpl), (Com). Thus, $\mathcal{A}$ is a Tarski algebra. $\square$

Remark 4.21. *By Theorem 4.20,* **T** = **im-H** = **im-BCK** = **im-BCC** = **im-aGE** = **im-aBE** = **com-GE** = **com-H** = **com-pre-H** = **im-aRML** *+ (B) =* **im-pre-BCK** *+ (An).*

From Remarks 4.11, 4.12 and 4.21 we obtain the hierarchy between **im-RML** and **T**, in the next Figure 3.

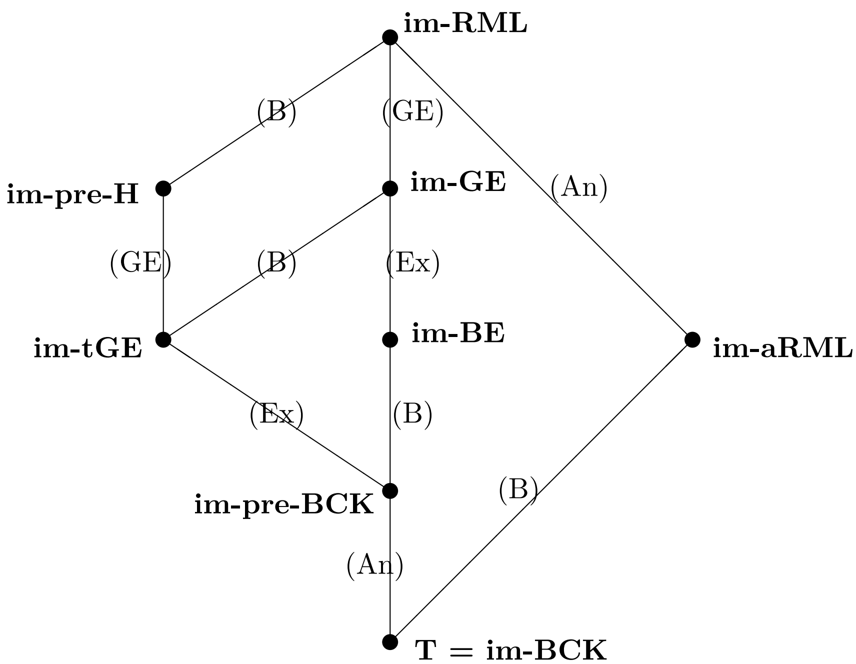

FIGURE 3.

Proposition 4.22. *We have*

(i) ***im-tGE*** = ***im-GE*** ∩ ***im-pre-H***,

(ii) ***im-pre-BCK*** = ***im-pre-H*** ∩ ***im-BE*** = ***im-tGE*** ∩ ***im-BE***,

(iii) ***T*** = ***im-pre-H*** ∩ ***im-aRML*** = ***im-GE*** ∩ ***im-aRML***.

Proof. (i) Since **tGE** = **GE** ∩ **pi-pre-H** (see Proposition 3.22) and (im) implies (pi), we deduce that (i) holds.

(ii) We obtain **im-pre-H** ∩ **im-BE** = [**im-RML** + (B)] ∩ [**im-RML** + (Ex)] = **im-RML** + (B) + (Ex) = **im-pre-BCK**. Hence we get **im-tGE** ∩ **im-BE** = **im-pre-BCK**, since **im-tGE** ⊂ **im-pre-H**.

(iii) We have **im-pre-H** ∩ **im-aRML** = [**im-RML** + (B)] ∩ [**im-RML** + (An)] = **im-RML** + (B) + (An) = **T** by Theorem 4.20. Moreover, **T** = **im-aGE** = **im-RML** + (GE) + (An) = [**im-RML** + (GE)] ∩ [**im-RML** + (An)] = **im-GE** ∩ **im-aRML**. □

5. Summary and future work

In this article, we introduced and studied implicative pre-Hilbert algebras. We obtained their properties and characterizations. By Theorem 4.6, an implicative pre-Hilbert algebra is equivalent to an implicative pre-BCC algebras. For any Hilbert algebra, the implicative property is equivalent to the commutative property (see Proposition 4.18). We gave several examples of the algebras considered here. Moreover, we established some characterizations of Tarski algebras. In particular, we showed that Tarski algebras coincide with commutative pre-Hilbert algebras and with implicative antisymmetric GE algebras. Finally, we presented (see Figure 3) the interrelationships between different classes of implicative algebras.

The results obtained in the paper can be a starting point for future research. We suggest the following topics:

(1) Studying the exchange pre-Hilbert algebras, that is, pre-Hilbert algebras satisfying (Ex).

(2) Describing the deductive systems, the congruences, the quotient algebras, etc. of pre-Hilbert algebras.

(3) Studying more deeply the proper generalizations of Hilbert algebras, namely the pi-BE, pi-pre-BCC, pi-pre-Hilbert, pi-pre-BCK algebras.

6. Acknowledgements

The author would like to thank the referees for their valuable suggestions and comments.

References

[1] R. K. Bandaru, A. Borumand Saeid and Y. B. Jun, *On GE-algebras*, Bull. Sect. Logic, **50** (2021) 81-96.

[2] A. Borumand Saeid, H. S. Kim and A. Rezaei, *On BI-algebras*, Analele Stiint. ale Univ. Ovidius Constanta Ser. Mat., **25** (2017) 177-194.

[3] A. Borumand Saeid, A. Rezaei, R. K. Bandaru and Y. B. Jun, *Voluntary GE-filters and further results of GE-filters in GE-algebras*, J. Algebraic Syst., **10** (2022) 31-47.

[4] D. Busneag and S. Rudeanu, *A glimpse of deductive systems in algebra*, Cent. Eur. J. Math., **8** (2010) 688-705.

[5] A. Diego, *Sur les algébras de Hilbert*, Collection de Logigue Mathématique, Ser. A, No 21, Gauthier-Villars, Paris, (1966).

[6] L. Henkin, *An algebraic characterization of quantifilers*, Fundam. Math., **37** (1950) 63-74.

[7] A. Iorgulescu, *New generalizations of BCI, BCK and Hilbert algebras-Part I, II*, J. Mult.-Valued Log. Soft Comput., **27** (2016) 353-456.

[8] K. Iséki, *An algebra related with a propositional calculus*, Proc. Jpn. Acad., **42** (1966) 26-29.

[9] K. Iséki, S. Tanaka, *An introduction to the theory of BCK-algebras,* Math. Japon., **23** (1978) 1-26.

[10] Y. B. Jun, M. S. Kang, *Fuzzifications of generalized Tarski filters in Tarski algebras*, Comput. Math. with Appl., **61** (2011) 1-7.

[11] H. S. Kim, Y. H. Kim, *On BE-algebras*, Scient. Math. Japon., **66** (2007) 113-116.

[12] Y. Komori, *The class of BCC-algebras is not a variety*, Math. Japon., **29** (1984) 391-394.

[13] C. A. Meredith, *Formal Logics*, 2nd ed, Oxford, 1962.

[14] A. Monteiro, *Lectures on Hilbert and Tarski algebras,* Insitituto de Mathemática, Universuidad Nacional del Sur, Bahía Blanca, Argentina, 1960.

[15] S. Tanaka, *A new class of algebras*, Math. Seminar Notes, **3** (1975) 37-43.

[16] A. Walendziak, *The implicative property for some generalizations of BCK algebras*, J. Mult.-Valued Log. Soft Comput., **31** (2018) 591-611.

[17] A. Walendziak, *The property of commutativity for some generalizations of BCK algebras*, Soft Comput., **23** (2019) 7505-7511.

[18] A. Walendziak, *On implicative BE algebras,* Ann. Univ. Mariae Curie-Skłodowska Lub., Sec. A., **76** (2022) 45-54.

[19] A. Walendziak, *On implicative and positive implicative GE algebras*, Bull. Sect. Logic, **52** (2023) 497-515.

[20] A. Walendziak, *On pre-Hilbert and positive implicative pre-Hilbert algebras*, Bull. Sect. Logic, **53** (2024) 345-364.

[21] A. Walendziak, *Connections between GE algebras and pre-Hilbert algebras,* Alg. Struc. Appl., **11** (2024) 229-241.

[22] Z. Zelent, *Transitivity of implicative aBE algebras,* Ann. Univ. Mariae Curie-Skłodowska Lub., Sec. A., **76** (2022) 55-58.

Andrzej Walendziak

Institute of Mathematics,

Faculty of Exact and Natural Sciences,

University of Siedlce, PL-08110 Siedlce, Poland.

`walent@interia.pl`