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Research Paper

PRIME AND PRIMARY IDEALS ON L-ALGEBRAS

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Abstract. In this paper, we review L-algebras and CL-algebras, focusing in particular on the notion of a prime ideal. We establish criteria for a prime ideal and analyze the relationship between maximal and prime ideals. In addition, we define prime ideals in CL-bounded algebras, and state conditions for an ideal to be prime in a self-distributing bounded L-algebra. We introduce the radical of an ideal and show that if its radical is maximal, then it is a prime ideal. Finally, we establish conditions under which every prime ideal in an L-algebra is primary.

1. Introduction

The quantum Yang-Baxter equation (QYBE), formulated by Zhenning Yang and R.J. Baxter in 1967 and 1972, respectively, is a fundamental equation in mathematical physics [8]. The QYBE is closely related to a series of mathematical structures, such as quantum binomial algebras [9, 10], I-type semigroups and Bieberbach groups [11, 26], colorings of plane curves,

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and dyeing of bijective cocycles [8], Semimunipolar small triangular Hopf algebra [11], dynamic system, geometric crystal, etc. Many initial solutions of the QYBE have been discovered and the algebraic structures associated with them have been widely studied. In 2005, W. Rump showed that any set X with a binary operation $\to$ satisfying

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z), \qquad (L)$$

corresponds to the solution of the quantum Yang–Baxter equation if the left multiplication is bijective [16]. He defined and studied L-algebras using the mentioned equation and demonstrated that Hilbert algebras, locales, (left) hoops, (pseudo) MV-algebras, and l-group cones, are L-algebras [17]. Yang and Rump characterized pseudo-MV-algebras and Bosbach's non-commutative bricks as L-algebras. They showed that "an L-algebra is representable as an interval in a lattice-ordered group if and only if it is semi-regular with the smallest element and a bijective negation" [24]. In recent years, many mathematicians have studied the concept of L-algebra and good results have been obtained in this field, which can be found in [1, 2, 4, 12, 15, 23, 20, 21, 22, 25, 27, 28].

Rump introduced the concept of an ideal in an L-algebra, defining a congruence relation through an ideal and constructing the L-algebra $\frac{L}{I}$ [17]. The notion of ideal has been explored across various algebraic structures, including lattices, BL-alebras and MV-algebras. In MV -algebras, ideals are central, while in BL-algebras, the focus has been on filters. In MV-algebras, a prime ideal is defined, while in BL-algebras, a prime filter is defined [6], [3]. Additionally, the concept of a primary ideal in MV-algebras has been defined, and the relationship between prime ideals and primary ideals has been examined [6]. Also, the fundamental concepts of prime ideals, primary ideals, and radical of an ideal in rings in abstract algebra have been carefully defined and studied. Considering that MV-algebras and BL-algebras can be viewed as particular instances of L-algebras, this paper is dedicated to establishing these definitions in L-algebras. Furthermore, it aims to explore the relationships among these notions specifically within the context of CL-algebras.

This paper is organized as follows: In Sec. 2, we recall some definitions and results about L-algebras. In Sec. 3, we show that $\mathcal{I}d(L)$ is an L-algebra and introduce the notion of a prime ideal of L. We also give a necessary and sufficient condition for a prime ideal and study the relation between maximal ideals and prime ideals. In Sec. 4, we discuss primary ideals in bounded CL-algebras and the criteria for an ideal in a bounded self-distributive L-algebra to be primary. We also define the radical of an ideal and show that an ideal is primary if and only if its radical is maximal. Furthermore, we define the primary decomposition and outline the conditions for a prime ideal to be primary.

2. Preliminaries

In this section, we provide a comprehensive summary of important definitions and notable results related to L-algebras. These key concepts and findings will be important for understanding the discussions and studies that will be presented in the following sections of this paper.

An algebraic structure $(L; \to, 1)$ of type $(2, 0)$ is called an *L-algebra* if it satisfies the following conditions

(L_1) $x \to x = x \to 1 = 1$ and $1 \to x = x$,

(L_2) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,

(L_3) if $x \to y = y \to x = 1$, then $x = y$,

for all $x, y, z \in L$ (See [18]).

If the binary operation $\to$ is taken as logically implicative, then there is a partial order on L defined by $x \leqslant y$ if and only if $x \to y = 1$.

Proposition 2.1. [23] *Let L be an L-algebra. Then $x \leq y$ implies $z \to x \leq z \to y$, for any $x, y, z \in L$.*

Proposition 2.2. [23] *For an L-algebra L and any $x, y, z \in L$, the following are equivalent*

(i) $x \leq y \to x$,

(ii) if $x \leq y$, *then* $y \to z \leq x \to z$,

(iii) $((x \to y) \to z) \to z \leq ((x \to y) \to z) \to ((y \to x) \to z)$.

A *KL-algebra* L, is an L-algebra that satisfies the condition (K) for any $x, y \in L$, namely $x \to (y \to x) = 1$. If L is a KL-algebra, the equivalent statements of Proposition 2.2 generally hold. A *CL-algebra* is an L-algebra which for any $x, y, z \in L$ satisfies the condition (C), that is $x \to (y \to z) = y \to (xvz)$. Every CL-algebra is a KL-algebra, since for any $x, y \in L$, we have $x \to (y \to x) = y \to (x \to x) = y \to 1 = 1$.

A subset I of an L-algebra L is called an *ideal of* L if it satisfies the following conditions

(I_1) $1 \in I$,

(I_2) if $x \in I$ and $x \to y \in I$, then $y \in I$,

(I_3) if $x \in I$, then $(x \to y) \to y \in I$,

(I_4) if $x \in I$, then $y \to x \in I$ and $y \to (x \to y) \in I$,

for any $x, y \in L$ (See [18]).

Clearly, $\{1\}$ and L are two trivial ideals of an L-algebra L. An ideal I of L is called a *proper ideal*, if $I \neq L$. The set of all ideals of L is denoted by $\mathcal{I}d(L)$, and the set of all proper ideals of L is denoted by $p\mathcal{I}d(L)$.

Suppose L is a CL-algebra and $\emptyset \neq I \subseteq L$. Then $I \in \mathcal{I}d(L)$ if and only if (I_1) and (I_2) hold (See [7]).

Assume L is an L-algebra and $\emptyset \neq X \subseteq L$. The smallest ideal of L that contains X is called *the generated ideal by* X, denoted $[X)$. Clearly,

$$[X) = \bigcap_{\substack{I \in \mathcal{I}d(L) \\ X \subseteq I}} I.$$

Suppose $(X; \leq)$ is a poset. Then, for any $Y \subseteq X$, we define

$$\uparrow Y = \{x \in X \mid \exists\, y \in Y \text{ s.t } \; y \leq x\} \text{ and } \downarrow Y = \{x \in X \mid \exists\, y \in Y \text{ s.t } \; x \leq y\}.$$

The set Y is called *upset* if $\uparrow Y = Y$ and is called *downset* if $\downarrow Y = Y$. If $y \in X$, then $\downarrow \{y\}$ is denoted by $\downarrow y$. If $(L; \to, 1)$ is an L-algebra and $I \in \mathcal{I}d(L)$, then I is upset.

Let $(L; \to, 1)$ be an L-algebra. Then, every ideal I of L defines a congruence relation $\sim$ on L as follows

$$x \sim y \;\Leftrightarrow\; \; x \to y,\; y \to x \in I,$$

for any $x, y \in L$ (See [5]). Let $I \in \mathcal{I}d(L)$ and $\dfrac{L}{I} = \{[x] \mid x \in L\}$, where $[x] = \{y \in L \mid x \sim y\}$. Then we define a binary relation $\leq_I$ on $\dfrac{L}{I}$ by $[x] \leq_I [y]$ if and only if $x[x] \to [y] := [x \to y].y \in I$. Thus, $(\dfrac{L}{I}; \leq_I)$ is a poset and $(\dfrac{L}{I}; \to, I)$ is an L-algebra, where for any $[x]$, $[y] \in \dfrac{L}{I}$,

$$[x] \to [y] := [x \to y].$$

Note 2.3. *From now on, we assume that L is an L-algebra unless otherwise stated.*

3. Prime ideals on L-algebras

This section reviews the concept of a Brouwerian semilattice and demonstrates that $\mathcal{I}d(L)$ is an L-algebra. In this section, we recall the notion of a prime element in an L-algebra. Then, since $\mathcal{I}d(L)$ is an L-algebra, using this concept, we introduce the prime element of $\mathcal{I}d(L)$ as the prime ideal of L. In the following, we present and verify a necessary and sufficient condition for a prime ideal. We prove that any maximal ideal is a prime ideal, and we conclude that any proper ideal is contained in a prime ideal. Then, we state another necessary condition for prime ideals.

Definition 3.1. [13] A $\wedge-$semilattice M with the greatest element 1 and a binary operation $\to$ is said to be a *Brouwerian semilattice* if it satisfies

$$x \wedge y \leqslant z \Longleftrightarrow x \leqslant y \to z,$$

for any $x, y, z \in M$.

Proposition 3.2. *$\mathcal{I}d(L)$ is a Brouwerian semilattice where for any $I, J \in \mathcal{I}d(L)$, the meet and join operations are defined as $I \wedge J = I \cap J$ and $I \vee J = [I \cup J)$. The binary operation $\rightarrow$ is defined by*

$$I \rightarrow J = \bigvee \{K \in \mathcal{I}d(L) \mid K \cap I \subseteq J\}.$$

Proof. $\mathcal{I}d(L)$ is a distributive lattice ([22], Theorem 4) where for any $I, J \in \mathcal{I}d(L)$, $I \wedge J = I \cap J$ and $I \vee J = [I \cup J)$. Furthermore, L is the greatest element of $\mathcal{I}d(L)$. Assume $S \in \mathcal{I}d(L)$ and $I \subseteq J \rightarrow S$. Then $I \cap J \subseteq (J \rightarrow S) \cap J$. So

$$I \cap J \subseteq \bigvee \{K \in \mathcal{I}d(L) \mid K \cap J \subseteq S\} \cap J.$$

Therefore,

$$I \cap J \subseteq \bigvee \{K \cap J \mid K \in \mathcal{I}d(L) \text{ and } K \cap J \subseteq S\}.$$

Hence, $I \cap J \subseteq S$. Conversely, if $I \cap J \subseteq S$, then

$$I \subseteq \bigvee \{K \in \mathcal{I}d(L) \mid K \cap J \subseteq S\}.$$

So $I \subseteq J \rightarrow S$. Hence, $\mathcal{I}d(L)$ is a Brouwerian semilattice. □

Corollary 3.3. *$(\mathcal{I}d(L); \rightarrow, L)$ is an L-algebra.*

Proof. Any Brouwerian semilattice is an L-algebra ([22], Example 3), simplify the proof with Proposition 3.2. □

Proposition 3.4. *Let L be a CL-algebra, $x \in L$ and $X \subseteq L$. Then*
(i): $[X) = \{a \in L \mid$ there are $x_1, x_2, ..., x_n \in X$ s.t $x_n \rightarrow (...(x_2 \rightarrow (x_1 \rightarrow a))...) = 1\}$.
(ii): $[x) = [\{x\}) = \{a \in L \mid$ there is $n \in \mathbb{N}$ s.t $x \xrightarrow{n} a := \underbrace{x \rightarrow (...(x \rightarrow (x}_{n-times} \rightarrow a))...) = 1\}$.

Proof. Any CL-algebra is a BCK-algebra (see [7]), making the proof evident (see [14]). □

Lemma 3.5. *Assume $(L; \rightarrow, 1)$ is a KL-algebra. Then the following properties hold for all $x, y, z \in L$*
(K_1) $x \rightarrow y \leqslant (z \rightarrow x) \rightarrow (z \rightarrow y)$.
If L is a CL-algebra, then:
(C_1) $x \leqslant y \rightarrow z$ if and only if $y \leqslant x \rightarrow z$.
(C_2) $x \rightarrow y \leqslant (y \rightarrow z) \rightarrow (x \rightarrow z)$.

Proof. (K_1) : See [12].
(C_1): Let $x, y, z \in L$. Then

$$x \leqslant y \to z \iff x \to (y \to z) = 1 \iff y \to (x \to z) = 1 \iff y \leqslant x \to z.$$

(C_2): By (K), we have $y \to z \leqslant (y \to x) \to (y \to z)$. By (L_2) and (C_1), we get

$$x \to y \leqslant (y \to z) \to (x \to z).$$

□

Proposition 3.6. *Suppose L is a CL-algebra, $I \in \mathcal{I}d(L)$ and $x \in L \setminus I$. Then*

$$[I \cup \{x\}) = \{a \in L \mid \text{ there is } n \in \mathbb{N} \text{ s.t } x \xrightarrow{n} a \in I\}.$$

Proof. Let $A = \{a \in L \mid$ there is $n \in \mathbb{N}$ s.t $x \xrightarrow{n} a \in I\}$. Clearly, $1 \in A$. Let $a, a \to b \in A$. Then there are $n, m \in \mathbb{N}$ such that $x \xrightarrow{n} a \in I$ and $x \xrightarrow{m} (a \to b) \in I$. By (C_2), we have

$$x \to a \leqslant (a \to b) \to (x \to b).$$

By Proposition 2.1 and (C), we get

$$x \to (x \to a) \leqslant x \to ((a \to b) \to (x \to b)) = (a \to b) \to (x \to (x \to b)).$$

So $x \xrightarrow{n} a \leqslant (a \to b) \to (x \xrightarrow{n} b)$. Since I is upset, we conclude $(a \to b) \to (x \xrightarrow{n} b) \in I$. From (K_1),

$$(a \to b) \to (x \xrightarrow{n} b) \leqslant (x \to (a \to b)) \to (x \to (x \xrightarrow{n} b)).$$

Repeatedly,

$$(a \to b) \to (x \xrightarrow{n} b) \leqslant (x \xrightarrow{m} (a \to b)) \to (x \xrightarrow{m+n} b).$$

Hence,

$$(x \xrightarrow{m} (a \to b)) \to (x \xrightarrow{m+n} b) \in I.$$

By assumption, we conclude that $x \xrightarrow{m+n} b \in I$. So $b \in A$ and (I_2) holds. Hence, $A \in \mathcal{I}d(L)$. Since $x \to x = 1 \in I$, we get $x \in A$. Let $t \in I$. Then by (K), $t \leq x \to t$. Since I is upset, we get $x \to t \in I$. So $t \in A$. Therefore, $I \cup \{x\} \subseteq A$. Assume $J \in \mathcal{I}d(L)$ and $I \cup \{x\} \subseteq J$. If $a \in A$, then there is $n \in \mathbb{N}$ such that

$$\underbrace{x \to (...(x \to (x}_{n-times} \to a))...) \in I \subseteq J.$$

Since $x \in J$ and by (I_2) we have $a \in J$, it follows that $A \subseteq J$. Therefore, A is the smallest ideal of L containing $I \cup \{x\}$. Thus, $A = [I \cup \{x\})$. □

Definition 3.7. [19] An element $p \in L$ is said to be *prime* if $p < 1$ and for any $x \in L$, either $x \leqslant p$ or $x \to p \leqslant p$.

Definition 3.8. Let $P \in p\mathcal{I}d(L)$. Then P is called a *prime ideal* of L if it is a prime element of L-algebra $(\mathcal{I}d(L); \to, L)$. Equivalently, P is prime if for any $I \in \mathcal{I}d(L)$

$$I \subseteq P \text{ or } I \to P \subseteq P.$$

The set of all prime ideals of L is denoted by $\mathcal{S}pec(L)$.

In a Brouwerian semilattice X, an element $p < 1$ is prime if and only if

$$x \wedge y \leqslant p \implies (x \leqslant p \text{ or } y \leqslant p),$$

holds in X (See [22]).
Note. Assume $P \in \mathcal{I}d(L)$. Then $P \in \mathcal{S}pec(L)$ if and only if $I \cap J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$, for any $I, J \in \mathcal{I}d(L)$.

Example 3.9. Let $(L = \{0, m, n, p, q, r, s, t, u, v, w, 1\}; \leq)$ be a poset with 12 chains as follows.

$$0 \leq m \leq q \leq u \leq 1, 0 \leq m \leq q \leq v \leq 1, 0 \leq m \leq r \leq v \leq 1, 0 \leq n \leq q \leq u \leq 1,$$

$$0 \leq n \leq q \leq v \leq 1, 0 \leq n \leq s \leq u \leq 1, 0 \leq n \leq s \leq w \leq 1, 0 \leq n \leq t \leq v \leq 1,$$

$$0 \leq n \leq t \leq w \leq 1, 0 \leq p \leq r \leq v \leq 1, 0 \leq p \leq t \leq v \leq 1 \text{ and } 0 \leq p \leq t \leq w \leq 1.$$

Define the operation $\to$ on L as follows.

$\to$	0	m	n	p	q	s	r	t	u	v	w	1
0	1	1	1	1	1	1	1	1	1	1	1	1
m	s	1	s	s	1	s	1	s	1	1	s	1
n	r	r	1	r	1	1	r	1	1	1	1	1
p	s	s	s	1	s	s	1	1	s	1	1	1
q	0	r	s	0	1	s	r	s	1	1	s	1
s	r	r	v	r	v	1	r	v	1	v	1	1
r	s	u	s	w	u	s	1	w	u	1	w	1
t	0	0	s	r	s	s	r	1	s	1	1	1
u	0	r	n	0	v	s	r	n	1	v	s	1
v	0	m	s	p	u	s	r	w	u	1	w	1
w	0	0	n	r	n	s	r	v	s	v	1	1
1	0	m	n	p	q	s	r	t	u	v	w	1

Then $(L;\to,1)$ is a L-algebra. Moreover,

$\mathcal{I}d(L)=\{I_1=\{1\},I_2=\{u,s,w,1\},I_3=\{v,1\},I_4=\{q,n,t,v,u,s,w,1\},I_5=\{r,v,1\},I_6=L\}$

and $\mathcal{S}pec(L)=\{I_2,I_4,I_5\}$.

Example 3.10. Suppose $L=\{1,-4,-9,-14,\dots\}$ and for any $x,y\in L$

$$x\to y=min\{y-x+1,1\}.$$

Then L is an L-algebra. If $I\in\mathcal{I}d(L)$ and $I\neq\{1\}$, then there is $x\in I$ such that $x\neq 1$. Let $y\in L$. If $x\leqslant y$, then $x\to y=1\in I$. By (I_2), we have $y\in I$. If $2x-1\leqslant y<x$, then $y-x+1\lneqq 1$. So $x\to y=y-x+1$. On the other hand, $x\leqslant y-x+1$. Since $x\in I$ and I is upset, we conclude $x\to y\in I$. By (I_2), we have $y\in I$. If $3x-2\leqslant y<2x-1$, then $y-x+1<x<1$. So $x\to y=y-x+1$. Since $3x-2\leqslant y$, we have $2x-1\leqslant y-x+1$. As we proved in the previous step, $2x-1\in I$. Hence, $x\to y\in I$. By (I_2), we conclude that $y\in I$. Continuing this process shows that $y\in I$ for any $y\in L$. Thus, $I=L$. Therefore, $\mathcal{S}pec(L)=\{\{1\}\}$.

Assume $I\in p\mathcal{I}d(L)$. I is a *maximal ideal* of L if it is not contained in any other proper ideal of L. The set of all maximal ideals of L is denoted $\mathcal{M}ax(L)$.

Proposition 3.11. *Let $I\in p\mathcal{I}d(L)$. Then there is $M\in\mathcal{M}ax(L)$ such that $I\subseteq M$.*

Proof. Consider

$$\Omega=\{P\in p\mathcal{I}d(L)\mid I\subseteq P\}.$$

Since $I\in\Omega$, we get $\Omega\neq\emptyset$. Let $\{I_i\}_{i\in\Delta}$ be a chain in a partially ordered set $(\Omega,\subseteq)$. Put $J=\bigcup_{i\in\Delta}I_i$. Clearly, $J\in p\mathcal{I}d(L)$ and $I\subseteq J$. Obviously, $J\in\Omega$. Hence, J is an upper bound for $\{I_i\}_{i\in\Delta}$. By Zorn's Lemma, Ω has a maximal element, denoted M. Clearly, $M\in\mathcal{M}ax(L)$. Moreover, $I\subseteq M$. □

Theorem 3.12. $\mathcal{M}ax(L)\subseteq\mathcal{S}pec(L)$.

Proof. Let $M\in\mathcal{M}ax(L)$ and $M\notin\mathcal{S}pec(L)$. Then there are $I,J\in\mathcal{I}d(L)$ such that $I\nsubseteq M$ and $J\nsubseteq M$ and $I\cap J\subseteq M$. Clearly, $I,J\in p\mathcal{I}d(L)$. Since $M\in\mathcal{M}ax(L)$, we get $M\nsubseteq I$ and $M\nsubseteq J$. Since $M\subseteq J\vee M$ and $M\in\mathcal{M}ax(L)$, we have $J\vee M=M$ or $J\vee M=L$. If $J\vee M=M$, then $J\subseteq M$, which leads to a contradiction. Thus, $J\vee M=L$, and similarly, $I\vee M=L$. $\mathcal{I}d(L)$ is distributive ([22], Theorem 4), hence

$$M\vee(I\cap J)=(M\vee I)\cap(M\vee J)=L\cap L=L.$$

Additionally, since $I\cap J\subseteq M$, it follows that $M\vee(I\cap J)=M$. It is a contradiction. So $M\in\mathcal{S}pec(L)$. Therefore, $\mathcal{M}ax(L)\subseteq\mathcal{S}pec(L)$. □

Corollary 3.13. *If $I \in p\mathcal{I}d(L)$, then there is $P \in \mathcal{S}pec(L)$ such that $I \subseteq P$.*

The following example demonstrates that $\mathcal{M}ax(L)$ is not always equal to $\mathcal{S}pec(L)$.

Example 3.14. Let $(L = \{0, a, b, c, 1\}; \leq)$ be a poset, where $0 \leq a \leq b, c \leq 1$. Define the operation $\to$ on L as follows.

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then L is an L-algebra and

$$\mathcal{I}d(L) = \{I_1 = \{1\}, I_2 = \{1, b\}, I_3 = \{1, c\}, I_4 = \{a, b, c, 1\}, I_5 = L\}.$$

Moreover, $\mathcal{M}ax(L) = \{I_4\}$ and $\mathcal{S}pec(L) = \{I_2, I_3, I_4\}$. Clearly, $\mathcal{M}ax(L) \neq \mathcal{S}pec(L)$.

Proposition 3.15. *Let $P \in p\mathcal{I}d(L)$ such that for any $x, y \in L$ either $x \to y \in P$ or $y \to x \in P$. Then $P \in \mathcal{S}pec(L)$.*

Proof. Suppose $I, Q \in \mathcal{I}d(L)$ such that $I \cap Q \subseteq P$ and $Q \nsubseteq P$. We demonstrate that $I \subseteq P$. Let $x \in I$. We know that there is $y \in Q$ such that $y \notin P$. Since $x \in I$, by (I_4), we have $(y \to x) \to x \in I$. Since $y \in Q$, by (I_3), it follows that $(y \to x) \to x \in Q$. Therefore, $(y \to x) \to x \in I \cap Q$, which leads to

$$(y \to x) \to x \in P. \tag{1}$$

Similarly,

$$(x \to y) \to y \in P. \tag{2}$$

By assumption, we have $x \to y \in P$ or $y \to x \in P$. If $x \to y \in P$, then by (I_2) and (2), we conclude $y \in P$ and this is a contradiction. So $y \to x \in P$. Thus, by (I_2) and (1), we conclude that $x \in P$, resulting in $I \subseteq P$. □

The following example demonstrates that the converse of the Proposition 3.15 is not always true.

Example 3.16. Consider Example 3.9. Then $I_5 \in \mathcal{S}pec(L)$ but $m \to t = s \notin I_5$ and $t \to m = 0 \notin I_5$.

Proposition 3.17. *If L is a chain, then $p\mathcal{I}d(L) = \mathcal{S}pec(L)$.*

Proof. Let $I \in p\mathcal{I}d(L)$. Since L is a chain, we have $x \leqslant y$ or $y \leqslant x$ for any $x, y \in L$. So $x \rightarrow y = 1 \in I$ or $y \rightarrow x = 1 \in I$. By Proposition 3.15, we conclude that $I \in \mathcal{S}pec(L)$. □

The following example indicates that the converse of the Proposition 3.17 is not always valid.

Example 3.18. Let $(L = \{0, a, b, 1\}; \leqslant)$ be a poset where $0 \leqslant a, b \leqslant 1$. Define the operation $\rightarrow$ on L as follows

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	0	1
b	0	0	1	1
1	0	a	b	1

Then $(L; \rightarrow, 1)$ is an L-algebra and $p\mathcal{I}d(L) = \mathcal{S}pec(L)$ but L is not a chain.

Proposition 3.19. *Let L be a CL-algebra and $M \in p\mathcal{I}d(L)$. Then the following statements are equivalent:*
(i) $M \in \mathcal{M}ax(L)$.
(ii) $[M \cup \{x\}) = L$, for any $x \in L \setminus M$.

Proof. $(i \Rightarrow ii)$ Suppose $M \in \mathcal{M}ax(L)$ and $x \in L \setminus M$. Then $M \subsetneq [M \cup \{x\}) \subseteq L$. Since $M \in \mathcal{M}ax(L)$, we have $[M \cup \{x\}) = L$.
$(ii \Rightarrow i)$ Consider $M \subseteq Q \subseteq L$. If $M \neq Q$, then there exists $x \in Q$ such that $x \notin M$. By assumption, we get $[M \cup \{x\}) = L$. Moreover, since $M \cup \{x\} \subseteq Q$, we have $L = [M \cup \{x\}) \subseteq Q$. So $Q = L$. Hence, $M \in \mathcal{M}ax(L)$.

□

Proposition 3.20. *Let L be a bounded CL-algebra and $x \in L$. Then the following conditions are equivalent.*
(i) $M \in \mathcal{M}ax(L)$.
(ii) $x \notin M$ if and only if there is $n \in \mathbb{N}$ such that $x \overset{n}{\rightarrow} 0 \in M$.

Proof. $(i \Rightarrow ii)$ If $M \in \mathcal{M}ax(L)$ and $x \notin M$, then by Proposition 3.19, we have $[M \cup \{x\}) = L$. So $0 \in [M \cup \{x\})$. By Proposition 3.6, there is $n \in \mathbb{N}$ such that $x \overset{n}{\rightarrow} 0 \in M$. Conversely, assume there is $n \in \mathbb{N}$ such that $x \overset{n}{\rightarrow} 0 \in M$. If $x \in M$, then by (I_2), we get $0 \in M$. So $M = L$ and it is a contradiction. Hence, $x \notin M$.
$(ii \Rightarrow i)$ Consider $J \in \mathcal{I}d(L)$ and $M \subsetneq J$. Then there is $x \in J$ such that $x \notin M$. By

assumption, there is $n \in \mathbb{N}$ such that $x \xrightarrow{n} 0 \in M$. Since $M \subset J$, we have $x \xrightarrow{n} 0 \in J$. Since $x \in J$, by (I_2), we get $0 \in J$. So $J = L$. Therefore, $M \in \mathcal{M}ax(L)$. □

In the following example, we show that condition (C) in Proposition 3.20 is necessary.

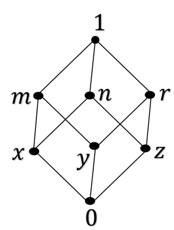
Example 3.21. Consider Example 3.9. Then $I_5 \in \mathcal{M}ax(L)$ and $w \in L \setminus I_5$ but for any $n \in \mathbb{N}$ we have $w \xrightarrow{n} 0 = 0$ and $0 \notin I_5$. So condition (C) in Proposition 3.20 is necessary.

4. Primary ideals on bounded CL-algebras

In this section, we introduce primary ideals in bounded CL-algebras and state the conditions under which an ideal qualifies as primary in a self-distributive finite L-algebra. Furthermore, we introduce the notion of a radical of an ideal and show that an ideal is primary if and only if its radical is maximal. We also define a primary decomposition of an ideal and state the necessary and sufficient condition for a prime ideal to be primary.

Definition 4.1. Assume L is a bounded CL-algebra and $I \in p\mathcal{I}d(L)$. The ideal I is called a *primary* ideal of L if there exists $n \in \mathbb{N}$ such that $a \xrightarrow{n} 0 \in I$ or there exists $m \in \mathbb{N}$ such that $(a \to 0) \xrightarrow{m} 0 \in I$, for any $a \in L$.

Example 4.2. Assume $(L = \{0, x, y, z, m, n, r, 1\}, \leq)$ is a poset with the following Hasse diagram.



We define the operation $\to$ on L as follows.

$\to$	0	x	y	z	m	n	r	1
0	1	1	1	1	1	1	1	1
x	r	1	r	r	1	1	r	1
y	n	n	1	n	1	n	1	1
z	m	m	m	1	m	1	1	1
m	z	n	r	z	1	n	r	1
n	y	m	y	r	m	1	r	1
r	x	x	m	n	m	n	1	1
1	0	x	y	z	m	n	r	1

Then L is a bounded CL-algebra, and $I = \{y, m, r, 1\}$ is a primary ideal. Furthermore, $\{r, 1\} \in p\mathcal{I}d(L)$. For any $s \in \mathbb{N}$:

$$z \xrightarrow{s} 0 = m \notin \{r, 1\} \text{ and } (z \to 0) \xrightarrow{s} 0 = z \notin \{r, 1\}.$$

Thus, $\{r, 1\}$ is not a primary ideal.

An L-algebra L is called *local* if it has a unique maximal ideal.

Example 4.3. Consider Example 3.14. In this case, $\mathcal{M}ax(L) = \{I_4\}$. Therefore, L is local.

Proposition 4.4. *Presume L is a bounded CL-algebra and $I \in p\mathcal{I}d(L)$. If $\frac{L}{I}$ is local, then I is a primary ideal.*

Proof. Assume $\mathcal{M}ax(\frac{L}{I}) = \{J\}$ and $a \in L$. Then there exists $M \in \mathcal{I}d(L)$ such that $I \subseteq M$ and $J = \frac{M}{I}$, where

$$\frac{M}{I} = \{[t] \in \frac{L}{I} \mid t \in M\}.$$

Consider $M^* \in p\mathcal{I}d(L)$ with $M \subseteq M^*$. If $M \neq M^*$, then there exists $t \in M^*$ such that $t \notin M$. Therefore, we have $[t] \in \frac{M^*}{I}$, but $[t] \notin \frac{M}{I}$. This implies that $\frac{M}{I} \subsetneq \frac{M^*}{I}$. Since J is a maximal ideal, we conclude that $\frac{M^*}{I} = \frac{L}{I}$. Consequently, $[0] \in \frac{M^*}{I}$, which means $0 \in M^*$, leading to a contradiction. Thus, we must have $M = M^*$, confirming that $M \in \mathcal{M}ax(L)$. If $a \notin M$ and $[I \cup \{a\}) \neq L$, then by Proposition 3.11, there exists $M^{**} \in \mathcal{M}ax(L)$ such that $[I \cup \{a\}) \subseteq M^{**}$. Therefore, we have $I \subseteq M^{**}$. Clearly, $\frac{M^{**}}{I} \in \mathcal{M}ax(\frac{L}{I})$. Since J is unique, it follows that $\frac{M^{**}}{I} = \frac{M}{I}$. Thus, we conclude that $a \in M$ which leads to another contradiction. Hence, we must have $[I \cup \{a\}) = L$. So $0 \in [I \cup \{a\})$. By Proposition 3.6, there exists $n \in \mathbb{N}$ such that $a \xrightarrow{n} 0 \in I$. Let $a \in M$. Since $0 \notin M$, by (I_2), we conclude that $a \to 0 \notin M$. Therefore, according to the above explanations, there exists $m \in \mathbb{N}$ such that $(a \to 0) \xrightarrow{m} 0 \in I$. □

Corollary 4.5. *Let L be a bounded CL-algebra and $M \in \mathcal{M}ax(L)$. Then M is a primary ideal.*

An L-algebra L is called simple if $\mathcal{I}d(L) = \{\{1\}, L\}$.

In the following, we state the necessary and sufficient conditions for an ideal to be primary.

Theorem 4.6. *Assume L is a bounded CL-algebra. For any $a \in L$ and $M \in \mathcal{M}ax(L)$, we have either $a \in M$ or $a \to 0 \in M$. Then, $I \in p\mathcal{I}d(L)$ is a primary ideal if and only if $\frac{L}{I}$ is local.*

Proof. Suppose I is a primary ideal. Since $I \in p\mathcal{I}d(L)$, by Proposition 3.11, we conclude that there exists $M_1 \in \mathcal{M}ax(L)$ such that $I \subseteq M_1$. Clearly, $\frac{M_1}{I} \in \mathcal{M}ax(\frac{L}{I})$. If $I = M_1$, then $\frac{L}{I}$ is simple, meaning it has a unique maximal ideal; hence, $\frac{L}{I}$ is local.
Now, suppose $I \subsetneq M_1$. Then there exists $c \in M_1$ such that $c \notin I$. Since I is a primary ideal,

$$\text{there is } n \in \mathbb{N} \text{ such that } c \xrightarrow{n} 0 \in I, \qquad (*)$$

or

$$\text{there is } m \in \mathbb{N} \text{ such that } (c \to 0) \xrightarrow{m} 0 \in I. \qquad (**)$$

If condition $(*)$ is satisfied, then by Proposition 3.6, $0 \in [I \cup \{c\})$. So $[I \cup \{c\}) = L$. On the other hand, $I \subset M_1$. So $L = [I \cup \{c\}) \subseteq [M_1 \cup \{c\}) = M_1$. Hence, $L \subseteq M_1$, which leads to a contradiction. Therefore, $I = M_1$. Consequently, $\frac{L}{I}$ is a local CL-algebra.
If condition $(**)$ is satisfied, we will show that $\frac{M_1}{I}$ is the unique maximal ideal of $\frac{L}{I}$. Since $M_1 \in \mathcal{M}ax(L)$, , it follows that $\frac{M_1}{I} \in \mathcal{M}ax(\frac{L}{I})$. Now, suppose $\frac{M_2}{I} \in \mathcal{M}ax(\frac{L}{I})$. Then we have $I \subsetneq M_2 \in \mathcal{M}ax(L)$. If $c \notin M_2$, then by assumption, $c \to 0 \in M_2$. By $(**)$, we have $0 \in [I \cup \{c \to 0\})$. According to the above explanations, we conclude that $L \subseteq M_2$, which leads to a contradiction. Therefore, $c \in M_2$. Thus, for any $c \in M_1 \setminus I$, if condition $(**)$ is satisfied, then $c \in M_2$. This implies that $M_1 \subseteq M_2$. Since $M_1, M_2 \in \mathcal{M}ax(L)$, we conclude that $M_2 = M_1$. So $\frac{L}{I}$ has a unique maximal ideal. Therefore, $\frac{L}{I}$ is local.
Conversely, by Proposition 4.4, the proof is obvious. □

In the following example, we will show that the condition of Theorem 4.6 is necessary.

Example 4.7. Suppose $(L = \{0, a, b, c, 1\}, \leq)$ with the relation $\leqslant$ such that $0 \leq a, b \leq c \leq 1$. We define the operation $\to$ on L as follows.

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then L is a bounded CL-algebra, $M = \{b, c, 1\} \in \mathcal{M}ax(L)$, $a \notin M$ and $a \to 0 = 0 \notin M$. Let $I = \{1\}$. If $x = 0$, then $x \to 0 = 1 \in I$ and if $x \neq 0$, then $(x \to 0) \to 0 = 1 \in I$. So I is primary. On the other hand, $\{[1], [a], [c]\}, \{[1], [b], [c]\} \in \mathcal{M}ax(\frac{L}{I})$. So $\frac{L}{I}$ is not local.

Definition 4.8. [19] An L-algebra L is called *self-distributive* if the binary operation $\to$ is left distributive on itself, that is $x \to (y \to z) = (x \to y) \to (x \to z)$, for any $x, y, z \in L$.

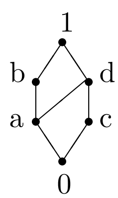
Example 4.9. Consider Example 4.2. Then L is a self-distributive L-algebra.

In any self-distributive L-algebra, by (L_2), we have

$$x \to (y \to z) = (x \to y) \to (x \to z) = (y \to x) \to (y \to z) = y \to (x \to z).$$

Thus, we conclude that: $x \to (y \to z) = y \to (x \to z)$, for any $x, y, z \in L$. Therefore, any self-distributive L-algebra is a CL-algebra. However, the following example demonstrates that a CL-algebra is not necessarily self-distributive.

Example 4.10. Assume $(L = \{0, a, b, c, d, 1\}, \leq)$ is a poset with the following Hasse diagram. Define the operation $\to$ on L as follows.



$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then L is a CL-algebra. Since

$$a \to (a \to c) = a \to d = 1 \neq d = a \to c = 1 \to (a \to c) = (a \to a) \to (a \to c),$$

we conclude that L is not self-distributive.

Corollary 4.11. *Let L be a bounded self-distributive and $I \in p\mathcal{I}d(L)$. Then I is a primary ideal if and only if $\frac{L}{I}$ is local.*

Proof. Suppose $M \in \mathcal{M}ax(L)$ and $a \notin M$. Then, by Proposition 3.19, we have $[M \cup \{a\}) = L$. Since $0 \in [M \cup \{a\})$, there exists $n \in \mathbb{N}$ such that $a \overset{n}{\to} 0 \in M$. Given that L is self-distributive, we have $a \to 0 \in M$. Hence, by Theorem 4.6, we conclude $\frac{L}{I}$ is local. $\square$

Definition 4.12. Let L be an L-algebra and $I \in p\mathcal{I}d(L)$. The intersection of all maximal ideals of L containing I is called *radical* of I, denoted by $Rad(I)$. Then

$$Rad(I) = \bigcap_{\substack{M \in \mathcal{M}ax(L) \\ I \subseteq M}} M.$$

Example 4.13. Suppose $(L = \{0, a, b, c, d, 1\}, \leq)$ is a poset with 3 chains as follows.

$$0 \leqslant a \leqslant b \leqslant 1, 0 \leqslant c \leqslant d \leqslant 1, 0 \leqslant a \leqslant d \leqslant 1.$$

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	c	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	0	b	b	c	1	1
1	0	a	b	c	d	1

Then

$$\mathcal{I}d(L) = \{I_1 = \{1\}, I_2 = \{1, b\}, I_3 = \{1, d\}, I_4 = \{1, c, d\}, I_5 = \{a, b, d, 1\}, I_6 = L\}.$$

and

$$Rad(I_1) = I_3, Rad(I_2) = I_5, Rad(I_3) = I_3, Rad(I_4) = I_4, Rad(I_5) = I_5.$$

Proposition 4.14. *Let L be an L-algebra and $I, J \in p\mathcal{I}d(L)$. Then*
$(i) : I \subseteq Rad(I)$.
(ii) : If $M \in \mathcal{M}ax(L)$, then $Rad(M) = M$.
(iii) : If L is finite, $Rad(I) = I$ and $I \in \mathcal{S}pec(L)$, then $I \in \mathcal{M}ax(L)$.
$(iv) : Rad(Rad(I)) = Rad(I)$.
$(v) : Rad(I \cap J) = Rad(I) \cap Rad(J)$.

Proof. $(i), (ii)$: The proof is clear.
(iii) : Since L is finite, we conclude that $\mathcal{M}ax(L)$ is finite. Thus, $Rad(I) = M_1 \cap \cdots \cap M_n$. By assumption, $Rad(I) = I$. Therefore, $M_1 \cap (\bigcap_{i \neq 1} M_i) \subseteq I$. Since $I \in \mathcal{S}pec(L)$, we have either $M_1 \subseteq I$ or $(\bigcap_{i \neq 1} M_i) \subseteq I$. If $M_1 \subseteq I$, then $I \in \mathcal{M}ax(L)$. Assume instead that $(\bigcap_{i \neq 1} M_i) \subseteq I$. By continuing this process, we conclude that there exists $1 \leqslant t \leqslant n$ such that $M_t \subseteq I$. Thus, $I \in \mathcal{M}ax(L)$.
(iv) : Consider $S \in \mathcal{M}ax(L)$ and $Rad(I) \subseteq S$. From (i), we conclude that $I \subseteq S$. Let $J \in \mathcal{M}ax(L)$ such that $I \subseteq J$. For any $x \in Rad(I)$, we have $x \in \bigcap_{\substack{K \in \mathcal{M}ax(L) \\ I \subseteq K}} K$. Therefore, $x \in J$, which implies that $Rad(I) \subseteq J$. Hence,

$$\bigcap_{\substack{S \in \mathcal{M}ax(L) \\ Rad(I) \subseteq S}} S = \bigcap_{\substack{J \in \mathcal{M}ax(L) \\ I \subseteq J}} J.$$

Thus, $Rad(Rad(I)) = Rad(I)$.
(v) : If $I \subseteq T \in \mathcal{M}ax(L)$, then $I \cap J \subseteq T$. If $J \subseteq S \in \mathcal{M}ax(L)$, then $I \cap J \subseteq S$. Let

$x \in Rad(I \cap J)$ and assume $I \subseteq T \in \mathcal{M}ax(L)$ and $J \subseteq S \in \mathcal{M}ax(L)$. Then $x \in T$ and $x \in S$. Therefore,

$$x \in \Big(\bigcap_{\substack{T \in \mathcal{M}ax(L) \\ I \subseteq T}} T\Big) \cap \Big(\bigcap_{\substack{S \in \mathcal{M}ax(L) \\ J \subseteq S}} S\Big).$$

Hence, $Rad(I \cap J) \subseteq Rad(I) \cap Rad(J)$.
Conversely, assume $y \in Rad(I) \cap Rad(J)$ and $I \cap J \subseteq M \in \mathcal{M}ax(L)$. Let $I \nsubseteq M$ and $J \nsubseteq M$. Thus, $M \vee I \neq M$ and $M \vee J \neq M$. Since $M \in \mathcal{M}ax(L)$, we have $M \vee I = M \vee J = L$. Therefore,

$$L = L \cap L = (M \vee I) \cap (M \vee J) = M \vee (I \cap J) = M.$$

This leads to a contradiction. Thus, either $I \subseteq M$ or $J \subseteq M$, which implies $y \in M$. Hence, $y \in Rad(I \cap J)$. Therefore, $Rad(I) \cap Rad(J) = Rad(I \cap J)$. □

In the following example, we demonstrate that the converse of Proposition 4.14(ii) does not hold in general.

Example 4.15. Consider Example 4.10. We have $Rad(\{1\}) = \{1\}$, but $\{1\} \notin \mathcal{M}ax(L)$.

Theorem 4.16. *Let L be a bounded CL-algebra, $I \in p\mathcal{I}d(L)$, and $a \in L$. For any $M \in \mathcal{M}ax(L)$, we have either $a \in M$ or $a \to 0 \in M$. The following conditions are equivalent.*
(i) : *I is primary.*
(ii) : $Rad(I) \in \mathcal{M}ax(L)$.
(iii) : *$Rad(I)$ is primary.*

Proof. $(i \Rightarrow ii)$: Assume I is a primary ideal. By Theorem 4.6, $\frac{L}{I}$ is local. Let $\mathcal{M}ax(\frac{L}{I}) = \{\frac{M^*}{I}\}$. Therefore, $I \subseteq M^* \in \mathcal{M}ax(L)$. Since

$$Rad(I) = \bigcap_{\substack{M \in \mathcal{M}ax(L) \\ I \subseteq M}} M,$$

we have $Rad(I) \subseteq M^*$. If $Rad(I) \neq M^*$, then there exists an element $t \in M^*$ such that $t \notin Rad(I)$. Consequently, we have $[t] \in \frac{M^*}{I}$ and $[t] \notin \frac{Rad(I)}{I}$.
On the other hand, we have:

$$Rad\Big(\frac{Rad(I)}{I}\Big) = \bigcap_{\frac{M}{I} \in \mathcal{M}ax(\frac{L}{I}),\, \frac{Rad(I)}{I} \subseteq \frac{M}{I}} \frac{M}{I} = \frac{M^*}{I}.$$

Moreover, by Proposition 4.14(iv), it follows that $Rad(\frac{Rad(I)}{I}) = \frac{Rad(I)}{I}$. Therefore, we conclude that $\frac{Rad(I)}{I} = \frac{M^*}{I}$. This leads to a contradiction. Thus, we establish that $Rad(I) = M^*$, which implies that $Rad(I) \in \mathcal{M}ax(L)$.

$(ii \Rightarrow iii)$: Let $Rad(I) \in \mathcal{M}ax(L)$. If $a \notin Rad(I)$, then we have $[Rad(I) \cup \{a\}) = L$. Thus, $0 \in [Rad(I) \cup \{a\})$. By Proposition 3.6, there exists $n \in \mathbb{N}$ such that $a \overset{n}{\to} 0 \in Rad(I)$. If $a \in Rad(I)$, then by (I_2), $a \to 0 \notin Rad(I)$. Therefore, $0 \in [Rad(I) \cup \{a \to 0\})$, and there exists $m \in \mathbb{N}$ such that $(a \to 0) \overset{m}{\longrightarrow} 0 \in Rad(I)$. Hence, we conclude that $Rad(I)$ is a primary ideal.
$(iii \Rightarrow i)$: Let $Rad(I)$ be primary. Furthermore, let $\frac{M_1}{I}, \frac{M_2}{I} \in \mathcal{M}ax(\frac{L}{I})$. Since $I \subseteq M_1 \in \mathcal{M}ax(L)$ and $I \subseteq M_2 \in \mathcal{M}ax(L)$, we have $Rad(I) \subseteq M_1$ and $Rad(I) \subseteq M_2$. Hence, $\frac{M_1}{Rad(I)}$ and $\frac{M_2}{Rad(I)}$ are two maximal ideals of $\frac{L}{Rad(I)}$. Therefore, $\frac{M_1}{Rad(I)} = \frac{M_2}{Rad(I)}$. Hence, $M_1 = M_2$. This implies that $\mathcal{M}ax(\frac{L}{I}) = \frac{M_1}{I}$. Consequently, we conclude that I is primary. □

Proposition 4.17. *Let L be a bounded CL-algebra and $I \in p\mathcal{I}d(L)$. If any ideal containing I is a prime ideal, then I is primary.*

Proof. If $I \in \mathcal{M}ax(L)$, then by Corollary 4.5, I is primary. Assume $I \notin \mathcal{M}ax(L)$. Then by Proposition 3.11, there exists $M_1 \in \mathcal{M}ax(L)$ such that $I \subseteq M_1$. Thus, $\frac{M_1}{I} \in \mathcal{M}ax(L)$. Let $\frac{M_2}{I}$ be another maximal ideal of $\frac{L}{I}$. Then $I \subseteq M_1 \cap M_2 \in \mathcal{I}d(L)$. By assumption, $S = M_1 \cap M_2 \in \mathcal{S}pec(L)$. Since $M_1 \cap M_2 \subseteq S$, we conclude that $M_1 \subseteq S$ or $M_2 \subseteq S$. Therefore, either $M_1 = M_1 \cap M_2$ or $M_2 = M_1 \cap M_2$. Hence, $\frac{M_1}{I} \subseteq \frac{M_2}{I}$ or $\frac{M_2}{I} \subseteq \frac{M_1}{I}$. Thus, $\frac{L}{I}$ has a unique maximal ideal, and by Proposition 4.4, I is primary. □

In the following example, we demonstrate that if I is a primary ideal of L, it does not necessarily follow that I is a prime ideal.

Example 4.18. Consider Example 3.14. Then L is a CL-algebra. Moreover, $\frac{L}{I_1}$ is local. By Proposition 4.4, I_1 is a primary ideal, but $I_1 \notin \mathcal{S}pec(L)$.

In the following example, we demonstrate that the condition stated in Proposition 4.17 is necessary, and if I is a prime ideal of L, it does not necessarily imply that I is a primary ideal.

Example 4.19. Suppose $(L = \{0, a, b, c, 1\}, \leq)$ is a poset, where $0 \leq a, b \leq c \leq 1$. Define the operation $\to$ on L as follows.

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then L is a bounded CL-algebra and

$$\mathcal{I}d(L) = \{I_1 = \{1\}, I_2 = \{1, c\}, I_3 = \{1, b, c\}, I_4 = \{1, a, c\}, I_5 = L\},$$

and

$$\mathcal{M}ax(L) = \{I_3, I_4\},\ \mathcal{S}pec(L) = \{I_1, I_3, I_4\}.$$

For any $M \in \mathcal{M}ax(L)$ and for any $x \in L$, we have $x \in M$ or $x \to 0 \in M$.
Since $Rad(I_1) = I_2 \notin \mathcal{M}ax(L)$, by Theorem 4.16, I_1 is not a primary ideal. Therefore, I_1 is a prime ideal but not a primary ideal.

Proposition 4.20. *Let L be a chain bounded CL-algebra and $I \in p\mathcal{I}d(L)$. Then I is primary.*

Proof. Suppose $a \in L$. Then either $a \leqslant a \to 0$ or $a \to 0 \leqslant a$.
(i): If $a \leqslant a \to 0$, then by Proposition 2.1, we have $a \to a \leqslant a \to (a \to 0)$. This implies $1 \leqslant a \overset{2}{\to} 0$. Therefore, $a \overset{2}{\to} 0 \in I$.
(ii): If $a \to 0 \leqslant a$, then by Proposition 2.2, we have $a \to 0 \leqslant (a \to 0) \to 0$. Thus, we can conclude that

$$(a \to 0) \to (a \to 0) \leqslant (a \to 0) \to ((a \to 0) \to 0).$$

This implies $1 \leqslant (a \to 0) \overset{2}{\to} 0$. Hence, $(a \to 0) \overset{2}{\to} 0 \in I$. Therefore, we conclude that I is a primary ideal. □

Definition 4.21. Let L be a bounded CL-algebra, $I \in p\mathcal{I}d(L)$, and let $\{J_i\}_{i=1}^n$ be a finite set of primary ideals. The collection $\{J_i\}_{i=1}^n$ is called a primary decomposition of I if

$$I = J_1 \cap J_2 \cap \cdots \cap J_n.$$

Example 4.22. Consider Example 4.19. Then $I_3 \cap I_4$ is a primary decomposition of I_2.

Theorem 4.23. *Let L be a finite bounded CL-algebra. Then any ideal of L has a primary decomposition if and only if any prime ideal of L is primary.*

Proof. Suppose any ideal of L has a primary decomposition, and let $I \in \mathcal{S}pec(L)$. Furthermore, let $I = J_1 \cap J_2 \cap \cdots \cap J_n$ be a primary decomposition of I, where $a \in L$. Then we have $J_1 \cap (\bigcap_{i\neq 1} J_i) \subseteq I$. Since $I \in \mathcal{S}pec(L)$, we conclude that either $J_1 \subseteq I$ or $\bigcap_{i\neq 1} J_i \subseteq I$. Assume $J_1 \subseteq I$. Since J_1 is a primary ideal, it follows that there exists $n \in \mathbb{N}$ such that either $a \overset{n}{\to} 0 \in J_1$ or there exists $m \in \mathbb{N}$ such that $(a \to 0) \overset{m}{\to} 0 \in J_1$. Given that $J_1 \subseteq I$, we conclude that I is primary. If instead $\bigcap_{i\neq 1} J_i \subseteq I$, then by continuing this process, we find that L is primary.
Conversely, assume that any prime ideal of L is primary and $I \in p\mathcal{I}d(L)$. Then, by Proposition 3.11, there exists $M \in \mathcal{M}ax(L)$ such that $I \subseteq M$. By Proposition 3.12, we have $M \in \mathcal{S}pec(L)$.

Suppose $A = \{J_k\}_{k\in\Lambda}$ is the set of all prime ideals including I. Then $A \neq \emptyset$ is a finite set. If $I \in \mathcal{S}pec(L)$, then $I = \bigcap_{k\in\Lambda} J_k$. Let $I \notin \mathcal{S}pec(L)$. Clearly, $I \subseteq \bigcap_{k\in\Lambda} J_k$. We demonstrate that $\bigcap_{k\in\Lambda} J_k \subseteq I$. We show that if $S \in \mathcal{I}d(L)$ and $S \nsubseteq I$, then $S \neq \bigcap_{k\in\Lambda} J_k$. Since $I \notin \mathcal{S}pec(L)$, there exist $T, V \in \mathcal{I}d(L)$ such that $T \nsubseteq I$ and $V \nsubseteq I$, but $T \cap V \subseteq I$. Let $T \cap V = I$. Since $S \nsubseteq I$, we can conclude that $S \nsubseteq T$ or $S \nsubseteq V$. Suppose $S \nsubseteq V$. If $V \in \mathcal{S}pec(L)$, then it follows that $S \nsubseteq \bigcap_{k\in\Lambda} J_k$. Otherwise, we continue this process sequentially. Given that L is a finite L-algebra, we conclude that $\mathcal{I}d(L)$ is also finite. Therefore, there exists a $V^* \in \mathcal{S}pec(L)$ such that $I \subseteq V^*$ and $S \nsubseteq V^*$. Consequently, it follows that $S \nsubseteq \bigcap_{k\in\Lambda} J_k$. Hence, we have $\bigcap_{k\in\Lambda} J_k \subseteq I$.

Let $T \cap V \subsetneq I$. If $S \subseteq V \vee I$ and $S \subseteq T \vee I$, since $\mathcal{I}d(L)$ is distributive, we conclude that

$$S \subseteq (V \vee I) \cap (T \vee I) = (V \cap T) \vee I = I,$$

which leads to a contradiction. Therefore, either $S \nsubseteq V \vee I$ or $S \nsubseteq T \vee I$. Consider the case where $S \nsubseteq V \vee I$. If $V \vee I \in \mathcal{S}pec(L)$, then it follows that $S \nsubseteq \bigcap_{k\in\Lambda} J_k$. Otherwise, the process continues. Since L is finite, we conclude that this process stops at an ideal. Hence, $\bigcap_{k\in\Lambda} J_k \subseteq I$. Therefore, $I = \bigcap_{k\in\Lambda} J_k$. By assumption, any prime ideal of L is primary, so J_k is a primary ideal for any $k \in \Lambda$. Since L is finite, we conclude that Λ is finite. Hence, $\{J_k\}_{k\in\Lambda}$ is a primary decomposition of I. □

5. Conclusions and future works

In this paper, we studied L-algebras and CL-algebras and introduced a fundamental concept in L-algebras, that is, the prime ideal. The study of prime and maximal ideals of L-algebras was a topic of interest. Furthermore, we showed that $\mathcal{I}d(L)$ is an L-algebra. We gave a necessary and sufficient condition for a prime ideal and investigated the relationship between maximal ideals and prime ideals. Additionally, we defined primary ideals in bounded CL-algebras and stated the criteria for an ideal in a bounded self-distributive L-algebra to be primary. We also introduced the radical of an ideal, demonstrating that an ideal is primary if and only if its radical is maximal. Finally, we present the primary decomposition and establish conditions for an L-algebra, under which any prime ideal becomes a primary ideal. The main theorem at the end of this article is related to finite CL-algebras. We are very interested in investigating these results in infinite CL-algebras, which could be a topic for future work.

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