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Research Paper

ON PRIME ELEMENT PRINCIPLE IN LE-MODULES

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Abstract. In this article, we have extended the notion of Oka families for commutative rings and modules to le-modules. We have introduced prime element principle, stated as, for a particular family F of submodule elements of an le-module M, submodule element maximal in terms of not belonging to F is prime. As a consequence, many results about the existence of prime elements of le-modules have been established. Furthermore, we have provided a method for constructing prime elements in le-modules and proved the existence of maximal m-systems, which is employed to characterize minimal prime elements.

1. Introduction

In commutative rings and modules, respectively, the existence of prime ideals and prime submodules is important. Lam and Reyes [6] introduced the prime ideal principle (PIP) in rings to club together various results of the maximum implies prime variety. Using the concepts

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of Oka and Ako families, Lam and Reyes [6] examined the PIP in great depth. More results about Oka and Ako families are found in [7], [8] etc.

In [9] R. Nekooei and E. Rostami generalised the notions of Ako and Oka families to modules and by applying these concepts new and elegant proofs for some known results are obtained.

M. Kumbhakar and A. K. Bhuniya introduced and studied the concept of le-modules in series of papers [1], [2], [3], [4], [5].

Here, we delve into an exciting extension of the concept of Oka families, taking beyond their original domain of commutative rings and expand the scope of Oka families to include submodule elements within the context of le-modules.

The cornerstone of this paper is the introduction of the "Prime Element Principle", states that for given family, denoted as F, consisting of submodule elements, "any submodule element that is maximal in terms of not belonging to F is prime". This foundational principle provides us a powerful tool to investigate the existence and properties of prime submodule elements within le-modules.

In Section 2, we briefly introduced the concepts of Oka and strongly Oka families of submodules in le-modules. We demonstrated that every strongly Oka family of submodules in the module M is also Oka, and conversely, this equivalence holds when the underlying ring is a principal ideal ring. Additionally, we introduced more stringent criteria for families of submodule elements, denoted as (P1) and (Q3), and proved that families meeting these criteria are indeed strongly Oka.

In Section 3, we explored some applications of the prime submodule element principle and derived several results concerning the existence of prime elements in le-modules.

In Section 4, we characterized prime elements in terms of m-systems and maximal elements in terms of strong m-systems. We also established the relationship between m-systems and strong m-systems. Furthermore, we provided a method for constructing prime elements in le-modules and proved the existence of maximal m-systems, which we subsequently used to characterize minimal prime elements.

2. Preliminaries

Throughout the paper R denotes a commutative ring with unity 1.

An *le-semigroup* $(M, +, \leq, e)$ which is a complete lattice and also an abelian monoid with 0_M and e are zero and largest elements respectively, that satisfies:

(B) $n + (\vee_{i\in S} n_i) = \vee_{i\in S}(n + n_i)$, for all $n, n_i \in M, i \in S$, an indexed set.

Definition 2.1. Let $(M, +, \leq, e)$ be an le-semigroup and R be a commutative ring with 1. If a mapping $. : R \times M \to M$ satisfies the following, then M is called an le-module and also denoted by ${}_RM$.

$(M1)$ $r(n_1 + n_2) = rn_1 + rn_2$,

$(M2)$ $(r_1 + r_2)n \leq r_1 n + r_2 n$,

$(M3)$ $(r_1 r_2)n = r_1(r_2 n)$,

$(M4)$ $1_R n = n; 0_R n = r0_M = 0_M$,

$(M5)$ $r(\vee_{i\in S} n_i) = \vee_{i\in S}(rn_i)$, where $r, r_1, r_2 \in R$ and $n, n_1, n_2, n_i \in M$ and $i \in S$.

An element $n \in {}_R M$ is called *submodule element* if, for $r \in R$, $n + n \leq n, rn \leq n$. We denote the set of all submodule elements of ${}_R M$ by $Sub(M)$. Note that for each $n \in Sub(M)$, $0_M = 0_R n \leq n$. Proper element $m \in Sub(M)$ is called *prime* if for $r \in R$ and $x \in M$, $rx \leq m$ implies $x \leq m$ or $re \leq m$, i.e., $r \in (m : e)$. We denote the set of all prime submodule elements of M by $Spec(M)$.

For an ideal I of R, we define, $Ie = \vee\{\sum_{i=1}^{r} a_i e : a_i \in I\}$. For $n \in Sub(M)$, we define $(n : e) = \{r \in R | re \leq n\}$ and is an ideal of R.

3. Prime submodule element principle (PSEP)

Definition 3.1. Let M be an le-module. Then $F \subseteq Sub(M)$ with $e \in F$ is called an *Oka family* (resp., *strongly Oka family*) if for $r \in R$, I an ideal of R and $n \in Sub(M)$, $n + re, (n : Rr) \in F$, then $n \in F$ (resp., $n + Ie, (n : I) \in F$, then $n \in F$).

Proposition 3.2. *In le-module ${}_R M$, every strongly Oka family of $Sub(M)$ is Oka and the converse holds if R is principal ideal ring.*

Proof. Suppose F is *strongly Oka family*. Let $n \in Sub(M), r \in R$ be such that $n + re, (n : Rr) \in F$. For an ideal $I = Rr$ of R, we have $n + Ie = n + (Rr)e = n + R(re) = n + re$ and which implies $n + Ie \in F$. Also, $(n : I) = (n : Rr) \in F$. Thus $n \in F$ and F is Oka. □

Conversely, let R be PIR and F be a Oka family. For an ideal I in R, let $n + Ie, (n : I) \in F$. Now, since $I = Rr$ for some $r \in R$, we have $n + Ie = n + (Rr)e = n + R(re) = n + re$ and $(n : I) = (n : Rr)$. Therefore $n + re, (n : Rr) \in F$ and F being Oka implies $n \in F$. Consequently, F is strongly Oka.

Following is an example of non-Oka family.

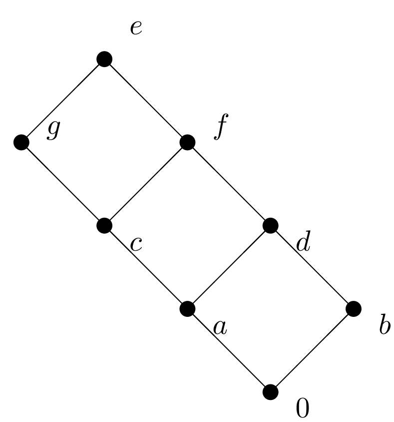
Example 3.3. Let $M = \{S \in \mathcal{P}(\mathbb{Z}) | 0 \in S\}$. Then $(M, +, \leq, \mathbb{Z})$ is an le-module over $\mathbb{Z}$, where $A + B = \{x + y | x \in A \ and \ y \in B\}, nA = \{nx | x \in A\} \ for \ n \in \mathbb{Z}$ and $A \leq B \ if \ A \subseteq B$, for $A, B \in M$. Consider $F = \{\mathbb{Z}, 2\mathbb{Z}\} \subseteq Sub(M)$. For $r = 2 \ and \ n = 4\mathbb{Z}$, we have $n + re = 4\mathbb{Z} + 2\mathbb{Z} = 2\mathbb{Z} \in F$ and $(n : Rr) = (4\mathbb{Z} : 2\mathbb{Z}) = 2\mathbb{Z} \in F$ but $4\mathbb{Z} \notin F$. Thus F is not Oka.

Following is an example of Oka family that is not strongly Oka.

Example 3.4. Consider $R = \{\prod_1^\infty R_i | \ R_i = \mathbb{Z}_2$ for finitely many $i\}$. Then $M = \{S \subseteq P(R) | (0,0,\ldots) \in S\}$ is an le-module over R with respect to operations $A + B = \{x + y | x \in A \ and \ y \in B\}, rA = \{rx | x \in A\}$ for $r \in R$ and $A \leq B$ if $A \subseteq B$. Let $J = \{\prod_{i=1}^\infty S_i \in R | S_i = \{0\}$ if i is odd $\}$ and $K = \{\prod_{i=1}^\infty S_i \in R | S_i = \{0\}$ if i is even$\}$ are ideals in R and hence they are submodule elements in M. Consider $F = \{R, J, K\} \subseteq Sub(M)$. Then F is an Oka family but not strongly Oka.

Remark 3.5. Note that, the condition on R to be PIR is not necessary for the converse of Proposition 3.2, e.g., let $(M, +, \leq, e)$ is an le-module over $\mathbb{Z}[\sqrt{-5}]$, where $M = \{0, a, b, c, d, g, f, e\}$ with $+$ is given in the table, $\leq$ is given in the figure and the scalar multiplication is $rx = x + x + \ldots + x$ ($||r||$ times) for all $x \in M$ and $r \in \mathbb{Z}[\sqrt{-5}]$. Note that $Sub(M) = \{0, a, c, g, e\}$.

+	0	a	b	c	d	g	f	e
0	0	a	b	c	d	g	f	e
a	a	a	e	c	e	g	e	e
b	b	e	e	e	e	e	e	e
c	c	c	e	c	e	g	e	e
d	d	e	e	e	e	e	e	e
g	g	g	e	g	e	g	e	e
f	f	e	e	e	e	e	e	e
e	e	e	e	e	e	e	e	e



M

Let $F \subseteq Sub(M)$ be any family containing e. Now

$$n + re = \begin{cases} n, & \text{if } r = 0, \\ e, & \text{if } r \neq 0, \end{cases} \tag{1}$$

$$(n : Rr) = \begin{cases} e, & \text{if } r = 0, \\ n, & \text{if } r \neq 0. \end{cases} \tag{2}$$

This implies the family F is Oka. Also the family F is strongly Oka, because

$$n + Ie = \begin{cases} n, & \text{if } I = \{0\}, \\ e, & \text{if } I \neq \{0\}, \end{cases} \tag{3}$$

$$(n : I) = \begin{cases} e, & \text{if } I = \{0\}, \\ n, & \text{if } I \neq \{0\}. \end{cases} \tag{4}$$

Thus Oka and strongly Oka are equivalent even though R is not PIR.

Definition 3.6. Let M be an le-module, $F \subseteq Sub(M)$ and $e \in F$. Then

(1) F is a semi-filter, if for $n_1, n_2 \in Sub(M)$, $n_1 \in F$ and $n_1 \leq n_2$ implies $n_2 \in F$.
(2) F is a filter, if it is a semi-filter and $n_1, n_2 \in F$ implies $n_1 \wedge n_2 \in F$.
(3) F is monoidal, if $n_1, n_2 \in F$ implies $(n_1 : e)n_2, (n_2 : e)n_1 \in F$.

We denote the complement of F by F' and is given by $F' = \{n \in Sub(M) : n \notin F\}$, and $Max(F')$ denotes the set of all maximal submodule elements of F'.

Theorem 3.7. *(Prime submodule element principle) For a given Oka family F of $Sub(M)$ in an le-module M, every element of $Max(F')$ is prime.*

Proof: Suppose that $n \in Max(F')$. As $e \in F$, we have $n \neq e$. Let $rn_1 \leq n$, for $n_1 \in M$ with $n_1 \nleq n$ and $r \in R$ with $r \notin (n : e)$. As $n \in Max(F')$, $n + re$, $(n : Rr) \in F$. As $n + re$, $(n : Rr) \in F$, and F being Oka implies $n \in F$, which is not true. Therefore, n is prime.

We say that le-module M satisfy the property $(*)$, if $(x + x_1 : e) = (x : e) + (x_1 : e)$, where $x, x_1 \in Sub(M)$.

Theorem 3.8. *Let M be an le-module and $F \subseteq Sub(M)$. For $x, y, n, n_1 \in Sub(M)$, consider the following:*

(1) *(P1) F is a monoidal filter.*
(2) *(Q1) F is a monoidal semi-filter.*
(3) *(P2) F is monoidal and for $n_1 \in F$ and $(n : e)n \leq n_1 \leq n$, implies $n \in F$.*
(4) *(Q2) F is monoidal and for $n_1 \in F$, $(n : e)^{t-1}n \leq n_1 \leq n$ for some $t \geq 1$, implies $n \in F$.*

(5) *(P3) If $n+x,\ n+y \in F$, then $n+(x:e)y \in F$ amd $n+(y:e)x \in F$.*

(6) *(Q3) If $x, y \in F$ and $(x:e)y \leq n \leq x \wedge y$ or $(y:e)x \leq n \leq x \wedge y$, then $n \in F$.*

Then $(P_1) \Leftrightarrow (Q_1)$, $(P_2) \Leftrightarrow (Q_2)$, $(P_3) \Rightarrow (Q_3)$, $(Q_1) \Rightarrow (Q_2)$, $(P_3) \Rightarrow$ (Strongly Oka) and if M satisfies property $(*)$, then $(P_2) \Rightarrow (P_3)$ and $(Q_3) \Rightarrow (P_3)$.

Proof. $(P_1) \Rightarrow (Q_1)$ Follows from definition.

$(Q_1) \Rightarrow (P_1)$ Suppose that (Q_1) is true. If $n, n_1 \in F$ and as F is monoidal, $(n:e)n_1 \in F$. Now, as $(n:e)n_1 \leq n \wedge n_1$ and F being a semifilter, $n \wedge n_1 \in F$. Therefore F satisfies (P_1).

$(P_2) \Rightarrow (Q_2)$ Suppose that (P_2) holds and $(n:e)^{t-1}n \leq n_1 \leq n$ for some $n_1 \in F$ and $t \geq 1$. For $n \notin F$, we have $(n:e)^{1-1}n + n_1 = n + n_1 = n \notin F$. Also, for every $l \geq t$, $(n:e)^{l-1}n \leq (n:e)^{t-1}n \leq n_1$ and so $(n:e)^{l-1}n + n_1 = n_1 \in F$. Now, choose the largest integer s with $1 \leq s \leq t$ such that $n_1 + (n:e)^{s-1}n \notin F$. Note that $((n_1 + (n:e)^{s-1}n) : e)(n_1 + (n:e)^{s-1}n) \leq n_1 + (n:e)^s n \leq n_1 + (n:e)^{s-1}n$.
By assumption on s, $n_1 + (n:e)^s n \in F$ and by (P_2), $n_1 + (n:e)^{s-1}n \in F$, a contradiction. Hence, $n \in F$ and (Q_2) holds.

$(Q_2) \Rightarrow (P_2)$ Follows immediately if we put $t = 2$ in (Q_2).

$(P_3) \Rightarrow (Q_3)$ Suppose that (P_3) holds. Let $x, y \in F$ with $(y:e)x \leq n \leq x \wedge y$. Now, as $n \leq x, y$, $x = n + x$, $y = n + y \in F$. Again, as $(y:e)x \leq n$, $n = n + (y:e)x \in F$ by (P_3). On the similar lines, $(x:e)y \leq n \leq x \wedge y$ implies $n \in F$.

$(Q_3) \Rightarrow (P_3)$ Suppose that (Q3) holds and M satisfies the property $(*)$. Now, let $n+x, n+y \in F$. By the property $(*)$, $(n+x:e)(n+y) = ((n:e)+(x:e))(n+y) = (n:e)(n+y) + (x:e)(n+y) \leq n + (x:e)y \leq (n+x) \wedge (n+y)$. Then $(n+x:e)(n+y) \leq n+(x:e)y \leq (n+x) \wedge (n+y)$. Therefore, by (Q_3), $n + (x:e)y \in F$. Similarly, $n + (y:e)x \in F$. Consequently, (P_3) holds.

$(Q_1) \Rightarrow (Q_2)$ Suppose that Q_1 holds. Let $n_1 \in F$ and $(n:e)^{t-1}n \leq n_1 \leq n$ for some $t \geq 1$. As $n_1 \leq n$ and F being a monoidal semi-filter, $n \in F$. Therefore, Q_2 holds.

$(P_2) \Rightarrow (P_3)$ Suppose that P_2 holds and $n+x, n+y \in F$ for $n, x, y \in Sub(M)$.
Then $n + (x:e)y \leq (n+x) \wedge (n+y) \leq n+x, n+y$. —(1)
Also, $((n + (x:e)y) : e) \leq ((n+x):e)$.—(2)
From (1) and (2), we have $((n+(x:e)y):e)(n+(x:e)y) \leq ((n+x):e)(n+y)$.—(3)
Now, by the property $(*)$,
$((n+x):e)(n+y) = ((n:e)+(x:e))(n+y) \leq n + (x:e)y$. —(4)
From (3) and (4), we get $((n+(x:e)y):e)(n+(x:e)y) \leq n + (x:e)y$.

Now, as F is monoidal and $n+x, n+y \in F$ implies $((n+x):e)(n+y) \in F$ and again by (P_2), $n + (x:e)y \in F$. Similarly, $n + (y:e)x \in F$. Therefore, (P_3) holds.

$(P_3) \Rightarrow$ (Strongly Oka) Suppose that (P_3) holds and $n + Ie, (n:I) \in F$ for $n \in Sub(M)$ and ideal I of R. Now, as $x = (n:I) \in Sub(M)$ and $In \leq n$, implies $n \leq x$. Therefore, $n + x = x = (n:I) \in F$. As, $n+x, n+Ie \in F$, by (P_3), $n + (x:e)Ie \in F$ and $n + (Ie:e)x \in F$.

Now, $(x:e)Ie = I(x:e)e \leq Ix \leq n$. Therefore, $n+(x:e)Ie = n \in F$. Consequently, F is strogly Oka. □

Next, we will apply the above result to form families of $Sub(M)$, namely strongly Oka.

Proposition 3.9. *Let M be a multiplication le-module and $\{l_i | i \in S$, an indexed set$\} \subseteq Spec(M)$. Then $F = \{n \in Sub(M) | n \nleq l_i,\ i \in S\}$ possesses the property (P_1). Particularly, when M possesses property $(*)$, F is strongly Oka.*

Proof. Note that F is a semifilter with $e \in F$. As $(Q_1) \iff (P_1)$, it is enough to show that F is monoidal. For $a, b \in F$, $(a:e)b \notin F$, implies $(a:e)b \leq l_i$ for some i. As l_i is prime and $b \nleq l_i$, we have $(a:e) \subseteq (l_i:e)$. Now $a = (a:e)e \leq (l_i:e)e = l_i$, so $a \leq l_i$, a contradiction. Therefore, $(a:e)b \in F$. On the similar lines, $(b:e)a \in F$. Hence, F is monoidal and (Q_1) is true.

$(P_1) \Rightarrow (P_3)$ holds by Theorem 3.8 as M possesses the property $(*)$. To prove F is strongly Oka, suppose that, $n+Ie, (n:I) \in F$, for $n \in Sub(M)$ and ideal I of R. Now, as F is semifilter and $(n:I) \leq n+(n:I)$ implies that $n+(n:I) \in F$. Therefore, $n+Ie \in F$, $n+(n:I) \in F$ and hence, by (P_3), $n+(Ie:e)(n:I) \in F$. But $(Ie:e)(n:I) \leq n$ implies that $n+(Ie:e)(n:I) = n \in F$. Hence F is strongly Oka. □

Proposition 3.10. *Let M be an le-module and $\{n_i | i \in S\} \subseteq Max(M)$. Then $F = \{n \in Sub(M) | n \notin \{n_i | i \in S\}\}$ possesses the property (Q_3). Particularly, F is strongly Oka if M possesses the property $(*)$.*

Proof. Clearly, $e \in F$. Let $n \in Sub(M)$ be such that $(a:e)b \leq n \leq a \wedge b$, where $a, b \in F$. If $n \notin F$, then $n = n_i$ for some i, where n_i is maximal. Now, as $n \leq a$, $n \leq b$ and $n = n_i$ is maximal, implies $a = b = e$. Therefore, $(a:e)b = (e:e)e \leq n$, and hence $e \leq n$, a contradiction. Therefore $n \in F$. On the similar lines, $(b:e)a \leq n \leq a \wedge b$ implies $n \in F$. Consequently, F satisfies the property (Q_3).

Particular case follows on similar lines as that of Proposition 3.9. □

Theorem 3.11. *Let M be an le-module and S be a non-empty multiplicatively closed subset of R. If le-module M possesses the property $(*)$, then*
$F = \{n \in Sub(M) | (n:e) \cap S \neq \varphi\}$ satisfies $(P1)$. Consequently, F is strongly Oka.

Proof. Note that, $e \in F$. As $P_1 \iff Q_1$, we only need to show that F satisfies the property (Q_1). For $n, n_1 \in F$, $(n:e) \cap S \neq \varphi \neq (n_1:e) \cap S$. Let $r \in (n:e) \cap S$ and $s \in (n_1:e) \cap S$. Then, $rse \leq rn_1 \leq (n:e)n_1$. Then $rs \in ((n:e)n_1:e) \cap S$, and hence $((n:e)n_1:e) \cap S \neq \varphi$.

Similarly, $((n_1 : e)n : e) \cap S \neq \varphi$. Therefore, $(n : e)n_1, (n_1 : e)n \in F$ and F is monoidal. Also, F is a semifilter, and consequently, F satisfies (Q_1).

Particular case follows on similar lines as that of Proposition 3.9. □

Corollary 3.12. *Let M be an le- module with the property $(*)$ and S be a multiplicative closed subset of R. Then $n \in Sub(M)$ maximal satisfying the property $(n : e) \cap S = \varphi$, is prime.*

Proof. Follows from Theorem 3.11 and Theorem 3.7. □

4. Applications

Here, we use Theorem 3.7 and Theorem 3.8 to examine families of submodule elements that exhibit the properties (P_1) and (Q_1). Some of the conclusions we make are well-established in le-module theory. For $F = \{e\}$, F becomes an Oka family and which implies $Max(M) \subseteq Spec(M)$.

Proposition 4.1. *Let M be an le-module and $\{a_i | i \in S\} \subseteq R$. Then for some finite subset J of S, $F = \{n \in Sub(M) | \prod_{j \in J} a_j \subseteq (n : e)\ \}$ satisfies (P_1).*

Proof. Note that $e \in F$. By Theorem 3.8, we need to show that F satisfies (Q_1) only. For $n, n_1 \in F$, $(n : e)(n_1 : e) \leq ((n : e)n_1 : e)$ and $(n : e)(n_1 : e) \subseteq ((n_1 : e)n : e)$. So $(n : e)n_1$, $(n_1 : e)n \in F$ and F is monoidal. Now, it is easy to observe that, F is a semifilter and satisfies (Q_1). □

A nonzero ideal in a commutative ring R is called *essential* if it intersects every nonzero ideal nontrivially.

Definition 4.2. An element $n \in Sub(M)$ is called *essential* if for $x \in Sub(M)$, $n \wedge x = 0_M$ implies $x = 0_M$.

Theorem 4.3. *Let M be a faithful multiplication le-module. Then $n \in Sub(M)$ is essential if and only if $(n : e)$ is an essential ideal of R.*

Proof. Suppose first that $n \in Sub(M)$ is essential. Then there exists an ideal I of R such that $n = Ie$. Suppose $I \cap J = (0)$ for some ideal J of R. Then we have $n \wedge Je = Ie \wedge Je = (I \cap J)e = (0)e = 0_M$, and hence $Je = 0_M$. But M is faithful implies that $J = (0)$. Hence I is essential.

Conversely, let K be an essential ideal. Let $k \in Sub(M)$ with $(Ke) \wedge k = 0_M$. Then $k = Ie$ for some ideal I of R and hence $(K \cap I)k \leq (Ke) \wedge k = 0_M$. As M is faithful, we have

$K \cap I = (0)$. Since K is essential, we have $I = (0)$ and which implies $k = Ie = (0)e = 0_M$. It follows that $Ke \in Sub(M)$ is essential. □

Proposition 4.4 ([6]). *Let R be a reduced ring. Then the family F of essential ideals in R has the (strongest) property (P_1). In particular, an ideal in R maximal with respect to being non-essential in R is prime.*

Corollary 4.5. *Let M be a faithful multiplication le-module over a reduced ring R. Then a family F of essential elements of $Sub(M)$ is a (P_1)-family. In particular, an element of $Sub(M)$ maximal with respect to being non-essential, is prime.*

Proof. Follows from Proposition 4.3 and Proposition 4.4. □

Theorem 4.6. *Let M be a faithful multiplication le-module. Then a semifilter $F \subseteq Sub(M)$ is strongly Oka if and only if (P_3) holds.*

Proof. If F satisfies (P_3), then by the Theorem 3.8, F is strongly Oka.

Conversely, let F is strongly Oka and for $n, a, b \in Sub(M)$, $n + a, n + b \in F$. Put $x = n + (a : e)b$. Observe that, $n + b \leq (x : (a : e))$, and as F is a semifilter, $(x : (a : e)) \in F$. Since $x + (a : e)e = x + a = n + (a : e)b + a = n + a \in F$, $x + (a : e)e \in F$. Now, since F is strongly Oka, $x \in F$. Consequently, F is a (P_3)-family. □

Definition 4.7. An element $n \in Sub(M)$ of an le-module M is said to be *small* if for $l \in Sub(M)$ with $n + l = e$, then $l = e$.

Proposition 4.8. *Let M be a faithful multiplication le-module. Then $F = \{n \in Sub(M) | n$ is small $\} \cup \{e\}$ satisfies (P_3). In particular, submodule element maximal with respect to not being small, is prime.*

Proof. Let $n + a, n + b \in F$. Since $n + (a : e)b \leq n + a$ and we have $n + (a : e)b \in F$. On the same lines, $n + (b : e)a \in F$. Therefore F satisfies (P_3). The particular case follows from Theorem 3.7. □

5. m-systems in le-modules

In [10], Sudarshana and Sivakumar introduced the notion of a strong m-system in modules and established the existence of unique maximal submodule through strong m-system.

Here, we have extended this concept to le-modules and characterized prime submodule elements by using m-system, and maximal elements using strong m-system. We also established a connection between m-systems and strong m-system. Moreover, we have introduced a method for obtaining prime elements in le-modules and established the existence of maximal m-system. These maximal m-system is effectively used to characterize minimal prime elements.

Definition 5.1. Let M be an le-module. A non-empty subset S of $\subseteq M - \{0_M\}$ is called an *m-system* if, for ideal I of R, and for $n, k \in Sub(M)$, $n + k \in S$ and $n + Ie \in S$, then $n + Ik \in S$.

Theorem 5.2. *Let M be an le-module. Then non-zero $p \in Sub(M)$ is prime if and only if $S = \{x \in M : x \nleq p\}$ is an m-system.*

Proof. Suppose that $p \in Sub(M)$ is prime. Let I be an ideal of R, and $n, k \in Sub(M)$ with $n + k \in S$ and $n + Ie \in S$. If $n + Ik \notin S$, then $n + Ik \leq p$. Hence $n \leq p$, $Ik \leq p$ and p being prime implies $k \leq p$ or $Ie \leq p$. Therefore $n + k \leq p$ or $n + Ie \leq p$, a contradiction. Hence, $n + Ik \in S$ and consequently, S is an m-system in M.

Conversely, suppose that $S = \{x \in M : x \nleq p\}$ is an m-system. Let $Ik \leq p$, where I is an ideal of R and $k \in Sub(M)$. If $k \nleq p$ and $Ie \nleq p$, then $k \in S$ and $Ie \in S$. Thus, $Ik \in S$, a contradiction. Consequently, $p \in Sub(M)$ is prime. □

Proposition 5.3. *Let M be an le-module and S be an m-system. Then $p \in Sub(M)$ maximal with respect to $p \notin S$ is prime.*

Proof. Suppose that $In \leq p$, where I is an ideal of R and $n \in Sub(M)$. If $n \nleq p$ and $Ie \nleq p$, then by the maximality of p, $p + Ie \in S$ and $p + n \in S$. Now, being S an m-sytem, we have $p + In \in S$. Since, $In \leq p$, therefore $p + In \leq p$ and $p \in S$, a contradiction. Consequently, p is prime. □

Lemma 5.4. *Let M be an le-module and I be an ideal of R. Then for $n, m \in Sub(M)$, $I(n + m) = In + Im$.*

Proof. For $n, m \in Sub(M)$, we have $n \leq n+m$ and $m \leq n+m$. Then $In \leq I(n+m)$ and $Im \leq I(n + m)$ and hence $In + Im \leq I(n + m) + I(n + m) = I(n + m)$.

Now, $r(n+m) = rn + rm$ for all $r \in R$. Then $\sum_{i=0}^{k} r_i(n+m) = \sum_{i=0}^{k} r_i n + \sum_{i=0}^{k} r_i m$ and which implies $\vee\{\sum_{i=0}^{k} r_i(n+m)\} \leq \vee\{\sum_{i=0}^{k} r_i n\} + \vee\{\sum_{i=0}^{k} r_i m\}$. Therefore $I(n+m) \leq In + Im$. Hence $I(n+m) = In + Im$. □

Theorem 5.5. *Let M be an le-module over R. A proper element $p \in Sub(M)$ is unique maximal if and only if for an ideal I of R and for $n, m \in Sub(M)$ with $n \wedge m = 0_M$ and $I(n+m) \leq p$, either $n+m \leq p$ or $n + Ie \leq p$.*

Proof. Suppose that for ideal I of R and for $n, m \in Sub(M)$ with $n \wedge m = 0$ and $I(n+m) \leq p$, either $n+m \leq p$ or $n + Ie \leq p$. Also, suppose there exists $n \in Sub(M)$ with $p \leq n < e$.

If we take $I = (0)$ and $m = 0_M$, in given condition, then by assumption $n \leq p$ and therefore, $p = n$. Hence p is a maximal element of M.

For uniqueness, suppose $t \in Sub(M)$ is maximal element. If $I = (0)$ and $m = 0_M$, then $(0)(t + 0_M) \leq p$. Again by assumption, $t \leq p$. Hence $p \in Sub(M)$ is unique maximal.

The converse is obvious. □

Definition 5.6. A non-empty subset S of $M - \{0_M\}$ is called a *strong m-system if it satisfies semifilter condition for set of submodule elements in S, for an ideal I of R and for $n, m \in Sub(M)$ with $n \wedge m = 0_M$, if $n + m \in S$ and $n + Ie \in S$, then $I(n+m) \in S$.*

Proposition 5.7. *In an le-module M, every strong m-system is m-system.*

Proof. Suppose that S is a strong m-system. Let I be an ideal of R, and $n, m \in Sub(M)$ with $n + m \in S$ and $n + Ie \in S$.

Case 1) $n \wedge m = 0_M$. Since S is a strong m-system, by definition, $I(n+m) \in S$. By Lemma 5.4, we have $I(n+m) = In + Im \leq n + Im$ and therefore, $n + Im \in S$.

Case 2) $n \wedge m \neq 0_M$. Let $n_1 = n + m$ and $m_1 = 0_M$. Then, we have $n_1 \wedge m_1 = 0_M$, $n_1 + m_1 \in S$ and $n_1 + Ie \in S$. Since S is a strong m-system, $I(n_1 + m_1) \in S$. Now, since $I(n+m) \leq n + Im$, $n + Im \in S$. Therefore S is an m-system. □

Following example illsutrate that the converse of the above theorem need not be true in general.

Example 5.8. Let M be the collection of all subsets of $\mathbb{Z}_6$ containing 0. Then $(M, +, \leq, \mathbb{Z}_6)$ is an le-module over $\mathbb{Z}_6$ with respect to the operations $A + B = \{a + b | a \in A \ and \ b \in B\}, rA = \{ra | a \in A\} \ for \ r \in R$ and $A \leq B \ if \ A \subseteq B$. Let $S = \{\{0,1\}, \{0,5\}, \langle 3 \rangle, \mathbb{Z}_6\}$. By taking $n, m \ and \ I$ from the set $\{\{0\}, \langle 2 \rangle, \langle 3 \rangle, \mathbb{Z}_6\}$ such that $n + m, n + Ie \in S$ implies $n + Im \in S$ and therefore S is a m-system. But S is not a strong system. For $n = \langle 3 \rangle$ and $m = \langle 2 \rangle$, $n \wedge m = \{0\}$ and $n + m = \langle 3 \rangle + \langle 2 \rangle = \mathbb{Z}_6 \ and \ n + Ie = \langle 3 \rangle + \langle 2 \rangle \mathbb{Z}_6 = \mathbb{Z}_6 \in S$ but $I(n+m) = \langle 2 \rangle \mathbb{Z}_6 = \langle 2 \rangle \notin S$.

Theorem 5.9. *Let M be an le-module and $p \in Sub(M)$. Then p is unique maximal if and only if $S = \{x \in M : x \nleq p\}$ is a strong m-system.*

Proof. Let $S = \{x \in M : x \nleq p\}$ be a strong m-system. Suppose $I(n+m) \leq p$, where I is an ideal of R and $n, m \in Sub(M)$ with $n \wedge m = 0_M$.

If $n + m \nleq p$ and $n + Ie \nleq p$, then $n + m \in S$ and $n + Ie \in S$. As S a strong m-system, we have $I(n+m) \in S$ and which implies $I(n+m) \nleq p$, a contradiction. Therefore either $n + m \leq p$ or $n + Ie \leq p$ and by Theorem 5.5, p is unique maximal.

The converse follows immediately. □

In Proposition 5.3, we have proved that, in an m-system S, if $p \in Sub(M)$ maximal with respect to the property that $p \notin S$, then p is prime. Following theorem shows that a similar result holds for strong m-system.

Theorem 5.10. *Let M be an le-module, S be a strong m-system in M and $n \in Sub(M)$ with $n \notin S$. Then $n \leq p$, where $p \in Sub(M)$ is unique maximal with $p \notin S$.*

Proof. Let $S_n = \{m \in Sub(M) : n \leq m \text{ and } m \notin S\}$. Note that $S_n \neq \varphi$, as $n \in S_n$. Then

by the Zorn's lemma, S_n possesses a maximal element p with $n \leq p$ and $p \notin S$. Now, we will show that p is unique.

Suppose that $I(n_1 + n_2) \leq p$, where I is an ideal of R and $n_1, n_2 \in Sub(M)$ with $n_1 \wedge n_2 = 0_M$. If $n_1 + n_2 \nleq p$ and $n_1 + Ie \nleq p$, then as p is maximal, $p + n_1 + n_2, p + n_1 + Ie \in Sub(M)$ with $p + n_1 + n_2 \in S$ and $p + n_1 + Ie \in S$. Also, $p + n_1 + n_2 + 0_M \in S$ and $p + n_1 + Ie + 0_M \in S$. Since $p + n_1 + Ie \in S$ and $n_2 \in Sub(M)$, this implies $p + n_1 + n_2 + Ie \in S$. Note that $(p + n_1 + n_2) \wedge 0_M = 0_M$. Since S is a strong m-system, $I(p + n_1 + n_2 + 0_M) \in S$, i.e., $I(p + n_1 + n_2) \in S$. But $I(p + n_1 + n_2) \leq Ip + I(n_1 + n_2) \leq p$, a contradiction. Therefore either $n_1 + n_2 \leq p$ or $n_1 + Ie \leq p$ and by Theorem 5.5, $p \in Sub(M)$ is unique maximal with $n \leq p$ and $p \notin S$. □

Following result establishes the existence of maximal m-system in le-module.

Lemma 5.11. *Let M be an le-module, $n \in M$ and S be an m-system in M satisfying the property $m \nleq n$ for all $m \in S$. There is an m-system T of M with $S \subseteq T$ and which is maximal with respect to the property $t \nleq n$ for all $t \in T$.*

Proof. Let $U = \{V \subseteq M : V \text{ is an } m\text{-system of } M, S \subseteq V \text{ and } v \nleq n \text{ for all } v \in V\}$. Note that U is non-empty, as $S \in U$ and U is a partial ordered set under set inclusion. For any chain C in U, the union A of all elements of C is an upper bound of C. Also A is m-system of M with $x \nleq n$ for all $x \in A$. Then, by the Zorn's lemma, U posseses a maximal element T, i.e., T is an m-system of M, $S \subseteq T$ and $t \nleq n$ for all $t \in T$. □

The following result provide a way to construct prime elements in le-modules.

Lemma 5.12. *Let M be an le-module over R, $n \in M$ and S be an m-system of M. If $x \nleq n$ for all $x \in S$, then there exists an element $p \in M$ maximal with respect to $x \nleq p$ for all $x \in S$ and such p is prime.*

Proof. By the Zorn's lemma, such an element $p \in M$ exists. The proof of remaining part follows from Proposition 5.3. □

Definition 5.13. Let M be an le-module over R and $x \in M$. A prime element p of M is said to be a minimal prime over x if there is no prime $q \in M$ such that $a \leq q < p$.

Note that a minimal prime over 0_M is nothing but a minimal prime element.

An element $n \in M$ is said to be completely join prime, if $x \leq \vee_{i \in I} x_i$ implies $x \leq x_j$ for some $j \in I$, where I is an indexed set.

Theorem 5.14. *Let M be an le-module over R with each element of M is completely join prime and $x \in M$. A prime element p of M is minimal prime over x if and only if $S = \{z \in M : z \nleq p\}$ is a maximal m-system of M with $z \nleq x$ for all $z \in S$.*

Proof. Suppose that $S = \{z \in M : z \nleq p\}$ is a maximal m-system of M with $z \nleq x$ for all $z \in S$. Then by Lemma 5.12, there is a prime element $q \geq x$ maximal with respect to the property $z \nleq q$ for all $z \in S$. Therefore, by Theorem 5.2, the set $S_1 = \{z \in M : z \nleq q\}$ is an m-system.

As $q \geq x$, we have $z \nleq x$ for any $z \in S_1$. But S is a maximal m-system with the property that $z \nleq x$ for all $z \in S$, hence we must have $S_1 \subseteq S$. Now, if $y \in S$ then $y \nleq q$ and hence $y \in S_1$. Consequently, we have $S = S_1$. Now, let $t \leq p$, i.e., $t \notin S$. Then $t \notin S_1$ and it implies that $t \leq q$ and further implies $p \leq q$. Similarly, we have $q \leq p$ and hence $p = q$.

Now, we show that p is minimal prime. Suppose that $p' \in M$ is prime with $x \leq p' < p$. Then by Theorem 5.2, $S_2 = \{z \in M : z \nleq p'\}$ is m-system with $z \nleq x$ for all $z \in S_2$ and $S \subseteq S_2$, this contradicts to the maximality of S. Hence p is a minimal prime element of M with $x \leq p$.

Conversely, suppose that $p \in M$ is minimal prime with $x \leq p$. Then by Theorem 5.2, S is a m-system with $z \nleq x$ for all $z \in S$. By Lemma 5.11, there is a maximal m-system T with $S \subseteq T$ and $z \nleq x$ for all $z \in T$. We show that $T = U = \{z \in M : z \nleq p'\}$, where $p' = \vee(M - S)$. Let $y \in U = \{z \in M : z \nleq \vee(M - S)\}$. This gives $y \nleq \vee(M - S)$, i.e., $y \in S$

and $U \subseteq T$. Now, if $t \in T$, then $t \notin M - S$ and $t \nleq \vee(M - S)$. As each element of M is completely join prime, we have $t \in U$ and therefore $T = U$.

By the first part, as T is a maximal m-system of M with respect to $z \nleq x$ for all $z \in S$, we conclude that p' is minimal prime with $x \leq p'$. Clearly, $S \subseteq T = U$ gives that $p' \leq p$ and since p is minimal, we must have $p = p'$. Hence $S = T = U$ is the required maximal m-system of M with $z \nleq x$ for all $z \in S$. □

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