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Research Paper

THE RANKS OF THE CLASSES OF THE CHEVALLEY GROUP $G_2(4)$

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Abstract. Let G be a finite simple group and X be a non-trivial conjugacy class of G. The *rank* of X in G, denoted by $rank(G{:}X)$, is defined to be the minimal number of elements of X generating G. In this paper we establish the ranks of all the conjugacy classes of elements for Chevalley group $G_2(4)$ using the structure constants method. The Groups, Algorithms and Programming, GAP [14] is used frequently in our computations.

1. Introduction

Generation of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see Woldar [26] for details). Also Di Martino et al. [17] established a useful connection between generation of

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groups by conjugate elements and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups. Recently more attention was given to the generation of finite groups by conjugate elements. In his PhD Thesis [24], Ward considered generation of a simple group by conjugate involutions satisfying certain conditions.

We are interested in generation of finite simple groups by the minimal number of elements from a given conjugacy class of the group. This motivates the following definition.

Definition 1.1. Let G be a finite simple group and X be a non-trivial conjugacy class of G. The rank of X in G, denoted by $rank(G{:}X)$, is defined to be the minimal number of elements of X generating G.

One of the applications of ranks of conjugacy classes of a finite group is that they are involved in the computations of the covering number of the finite simple group (see Zisser [27]).

In [18, 19, 20], J. Moori computed the ranks of involutry classes of the Fischer sporadic simple group Fi_{22}. He found that $rank(Fi_{22}{:}2B) = rank(Fi_{22}{:}2C) = 3$, while $rank(Fi_{22}{:}2A) \in \{5, 6\}$. The work of Hall and Soicher [15] implies that $rank(Fi_{22}{:}2A) = 6$. Also, in a considerable number of publications (for example but not limited to, see [1, 2, 3, 4, 5, 6, 18] or [21]) Moori, Ali, Motalane and Basheer explored the ranks for various sporadic, classical and alternating simple groups. The authors in [8] established the ranks of all non-trivial conjugacy classes of the group $G_2(3)$. In this article we apply the structure constants method together with some results on generation of simple groups to determine all the ranks of non-trivial classes of elements for the simple Chevalley group $G_2(4)$. The result on the ranks of non-trivial conjugacy classes of the group $G_2(4)$ can be summarized in Theorem 1.2.

Theorem 1.2. *Let G be the Chevalley group $G_2(4)$ and nX be a non-trivial conjugacy class of G. Then*

1. $rank(G{:}2A) = 4$,
2. $rank(G{:}nX) = 3$ *for* $nX \in \{2B, 3A\}$,
3. $rank(G{:}nX) = 2$ *for all* $nX \notin \{1A, 2A, 2B, 3A\}$.

Proof. The proof follows from a sequence of propositions in Section 3. □

2. Preliminaries

Let G be a finite group and $C_1, C_2, \cdots, C_k$ be $k \geq 3$ (not necessarily distinct) conjugacy classes of G with $g_1, g_2, \cdots, g_k$ being representatives for these classes respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \le i \le k-1$, denote by $\Delta_G = \Delta_G(C_1, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, \cdots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$ being the set of complex irreducible characters of G, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{1}$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta_G^*(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad \langle g_1, g_2, \cdots, g_{k-1} \rangle = G. \tag{2}$$

Definition 2.1. If $\Delta_G^*(C_1, C_2, \cdots, C_k) > 0$, the group G is said to be $(C_1, C_2, \cdots, C_k)$-**generated**.

Also if $H \le G$ is any subgroup containing the fixed element $g_k \in C_k$, we let $\Sigma_H(C_1, \cdots, C_k)$ be the total number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ such that $g_1 g_2 \cdots g_{k-1} = g_k$ and $\langle g_1, g_2, \cdots, g_{k-1} \rangle \le H$. The value of $\Sigma_H(C_1, C_2, \cdots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H-conjugacy classes $c_1, c_2, \cdots, c_k$ such that $c_i \subseteq H \bigcap C_i$.

Theorem 2.2. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that* $\gcd(o(g), [N_G(H){:}H]) = 1$. *Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|}, \tag{3}$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See for example Ganief and Moori [11, 12, 13]. □

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta_G^*(C_1, C_2, \cdots, C_k)$, namely $\Delta_G^*(C_1, C_2, \cdots, C_k) \ge \Theta_G(C_1, C_2, \cdots, C_k)$, where

$$\Theta_G(C_1, \cdots, C_k) = \Delta_G(C_1, \cdots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, \cdots, C_k), \tag{4}$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups containing elements of all the classes $C_1, C_2, \cdots, C_k$.

If $\Theta_G > 0$ then certainly G is $(C_1, C_2, \cdots, C_k)$-generated. In the case $C_1 = C_2 = \cdots = C_{k-1} = C$ then G can be generated by $k-1$ elements suitably chosen from C and hence $rank(G{:}C) \leq k-1$.

We now quote some results for establishing generation and non-generation of finite simple groups. These results are also important in determining the ranks of the finite simple groups.

Lemma 2.3 (e.g. see Ali and Moori [3] or Conder et al. [9]). *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \cdots, lX}_{m-times}, (nZ)^m)$-generated.*

Proof. Since G is (lX, mY, nZ)-generated group, it follows that there exists $x \in lX$ and $y \in mY$ such that $xy \in nZ$ and $\langle x, y \rangle = G$. Let $N := \left\langle x, x^y, x^{y^2}, \cdots, x^{y^{m-1}} \right\rangle$. Then $N \trianglelefteq G$. Since G is simple group and N is non-trivial subgroup we obtain that $N = G$. Furthermore we have

$$\begin{aligned} xx^y x^{y^2} x^{y^{m-1}} &= x(yxy^{-1})(y^2xy^{-2}) \cdots (y^{m-1}xy^{1-m}) \\ &= (xy)^m \in (nZ)^m. \end{aligned}$$

Since $x^{y^i} \in lX$ for all i, the result follows. □

Corollary 2.4 (e.g. see Ali and Moori [3]). *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then $rank(G{:}lX) \leq m$.*

Proof. Follows immediately by Lemma 2.3. □

Lemma 2.5 (e.g. see Ali and Moori [3]). *Let G be a finite simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

Proof. Since G is $(2X, mY, nZ)$-generated group, it is also $(mY, 2X, tK)$-generated group. The result follows immediately by Lemma 2.3. □

Corollary 2.6. *If G is a finite simple $(2X, mY, nZ)$-generated group. Then $rank(G{:}mY) = 2$.*

Proof. By Lemma 2.5 and Corollary 2.4 we have $rank(G{:}mY) \leq 2$. But a non-abelian simple group cannot be generated by one element. Thus $rank(G{:}mY) = 2$. □

The following two results are in some cases useful in establishing non-generation for finite groups.

Lemma 2.7 (e.g. see Ali and Moori [3] or Conder et al. [9])**.** *Let G be a finite centerless group. If $\Delta^*_G(C_1, C_2, \cdots, C_k) < |C_G(g_k)|,\ g_k \in C_k$, then $\Delta^*_G(C_1, C_2, \cdots, C_k) = 0$ and therefore G is not $(C_1, C_2, \cdots, C_k)$-generated.*

Proof. We prove the contrapositive of the statement, that is if $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$ then $\Delta^*_G(C_1, C_2, \cdots, C_k) \geq |C_G(g_k)|$, for a fixed $g_k \in C_k$. So let us assume that $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$. Thus there exists at least one $(k-1)$-tuple $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying Equation (2). Let $x \in C_G(g_k)$. Then we obtain

$$x(g_1 g_2 \cdots g_{k-1})x^{-1} = (xg_1x^{-1})(xg_2x^{-1}) \cdots (xg_{k-1}x^{-1}) = (xg_kx^{-1}) = g_k.$$

Thus the $(k-1)$-tuple $(xg_1x^{-1}, xg_2x^{-1}, \cdots, xg_{k-1}x^{-1})$ will generate G. Moreover if x_1 and x_2 are distinct elements of $C_G(g_k)$, then the $(k-1)$-tuples $(x_1g_1x_1^{-1}, x_1g_2x_1^{-1}, \cdots, x_1g_{k-1}x_1^{-1})$ and $(x_2g_1x_2^{-1}, x_2g_2x_2^{-1}, \cdots, x_2g_{k-1}x_2^{-1})$ are also distinct since G is centerless. Thus we have at least $|C_G(g_k)|$ $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ generating G. Hence $\Delta^*_G(C_1, C_2, \cdots, C_k) \geq |C_G(g_k)|$. □

The following result is due to Ree [22].

Theorem 2.8. *Let G be a transitive permutation group generated by permutations $g_1, g_2, \cdots, g_s$ acting on a set of n elements such that $g_1g_2 \cdots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

Proof. See for example Ali and Moori [3]. □

The following result is due to Scott ([9] and [23]).

Theorem 2.9 (Scott's Theorem)**.** *Let $g_1, g_2, \cdots, g_s$ be elements generating a group G such that $g_1g_2 \cdots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([11]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi{\downarrow}^G_{\langle g_i \rangle}, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned} \tag{5}$$

3. The ranks of the classes of $G_2(4)$

In this section we apply the results, discussed in Section 2, to the group $G_2(4)$. We determine the ranks for all its non-trivial conjugacy classes of elements.

The group $G_2(4)$ is a simple group of order $251596800 = 2^{12} \times 3^3 \times 5^2 \times 7 \times 13$. By the Atlas [10], the group $G_2(4)$ has exactly 32 conjugacy classes of its elements and 8 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as in Table 1.

Table 1. Maximal subgroups of $G_2(4)$.

Maximal Subgroup	Order	Index
$J_2 = M_1$	$604800 = 2^7 \times 3^3 \times 5^2 \times 7$	416
$2^{2+8}{:}(A_5 \times 3) = M_2$	$184320 = 2^{12} \times 3^2 \times 5$	1365
$2^{4+6}{:}(A_5 \times 3) = M_3$	$184320 = 2^{12} \times 3^2 \times 5$	1365
$U_3(4){:}2 = M_4$	$124800 = 2^7 \times 3 \times 5^2 \times 13$	2016
$3^{\cdot}L_3(4){:}2 = M_5$	$120960 = 2^7 \times 3^3 \times 5 \times 7$	2080
$U_3(3){:}2 = M_6$	$12096 = 2^6 \times 3^3 \times 7$	20800
$A_5 \times A_5 = M_7$	$3600 = 2^4 \times 3^2 \times 5^2$	69888
$L_2(13) = M_8$	$1092 = 2^2 \times 3 \times 7 \times 13$	230400

In this section we let $G = G_2(4)$. By the electronic Atlas of Wilson [25], G has two generators a and b in terms of 6×6 matrices over $\mathbb{F}_4$. For the sake of computations with GAP, we also considered a permutation representation for G in terms of 416 points, that is the action on the conjugates of J_2. Generators g_1 and g_2 for this permutation representation can be taken as follows:

$$\begin{aligned} g_1 = {} & (1,2)(3,5)(4,7)(6,10)(8,13)(9,12)(14,20)(15,22)(16,24)(17,26)(18,27)(19,29) \\ & (21,32)(25,36)(28,40)(30,43)(33,47)(34,49)(35,50)(37,53)(38,55)(39,57) \\ & (41,60)(44,62)(45,64)(46,66)(48,69)(51,72)(52,74)(54,77)(56,79)(58,82) \\ & (59,84)(61,87)(63,90)(65,92)(67,95)(68,97)(70,100)(71,101)(73,104)(75,107) \\ & (78,111)(80,109)(81,114)(83,117)(85,120)(86,122)(88,125)(91,128)(93,131) \\ & (94,132)(96,135)(98,138)(102,140)(103,142)(105,143)(106,144)(108,147) \\ & (110,150)(112,153)(113,155)(115,156)(116,158)(118,133)(119,161)(121,164) \\ & (123,166)(124,167)(126,170)(127,171)(129,174)(130,176)(134,181)(136,184) \\ & (139,186)(141,188)(146,192)(148,195)(149,196)(151,179)(152,200)(154,203) \end{aligned}$$

$$(159,208)(160,209)(162,212)(163,211)(168,217)(169,219)(172,223)(173,225)$$
$$(175,228)(177,230)(178,232)(180,235)(182,238)(185,241)(187,244)(189,193)$$
$$(190,248)(191,250)(194,252)(197,254)(198,255)(199,257)(201,220)(202,259)$$
$$(204,262)(205,214)(206,240)(207,264)(210,267)(213,221)(215,263)(216,274)$$
$$(218,276)(222,224)(226,280)(227,282)(229,247)(231,266)(233,287)(234,289)$$
$$(236,277)(237,291)(239,294)(242,284)(243,278)(245,301)(246,302)(249,304)$$
$$(251,307)(253,308)(256,311)(258,313)(260,275)(265,319)(268,322)(269,324)$$
$$(270,326)(271,327)(272,328)(273,330)(279,333)(283,336)(285,339)(286,340)$$
$$(288,299)(290,343)(292,345)(293,347)(295,349)(297,351)(298,352)(300,354)$$
$$(303,356)(305,359)(309,362)(312,365)(317,368)(318,370)(320,373)(321,350)$$
$$(323,375)(329,377)(331,338)(332,376)(334,379)(335,363)(337,348)(341,384)$$
$$(342,383)(344,353)(346,387)(355,392)(358,395)(360,396)(361,389)(364,398)$$
$$(366,381)(367,397)(371,399)(372,380)(374,401)(378,403)(385,405)(386,393)$$
$$(391,394)(402,407)(404,411)(406,408)(413,414),$$

$$g_2 = (1,3,6,11,17)(2,4,8,14,21)(5,9,15,23,34)(7,12,18,28,41)(10,16,25,37,54)$$
$$(13,19,30,44,63)(20,31,45,65,93)(22,33,48,70,43)(24,35,51,73,105)$$
$$(26,38,56,80,113)(27,39,58,83,118)(29,42,61,88,126)(32,46,67,96,136)$$
$$(36,52,75,108,148)(40,59,85,121,165)(47,68,98,139,101)(49,71,102,141,143)$$
$$(53,76,109,149,197)(55,78,60,86,123)(57,81,115,157,206)(62,89,82,116,159)$$
$$(64,91,129,175,100)(66,94,133,180,236)(69,99,140,187,245)(72,103,107,146,193)$$
$$(74,106,145,191,114)(77,110,151,199,155)(79,112,154,204,164)$$
$$(84,119,162,213,271)(87,124,168,218,267)(90,127,172,224,122)$$
$$(92,130,177,231,285)(95,134,182,239,295)(97,137,185,242,298)$$
$$(111,152,201,258,314)(117,160,210,268,323)(120,163,214,272,329)$$
$$(125,169,220,241,297)(128,173,226,281,334)(131,178,233,288,342)$$
$$(132,179,234,290,138)(135,183,166,215,273)(142,189,247,223,279)$$
$$(144,190,249,305,360)(147,194,238,293,322)(150,198,256,212,270)$$
$$(153,202,260,316,367)(156,205,263,244,300)(158,207,265,320,365)$$
$$(161,211,269,325,308)(167,216,228,250,306)(170,221,277,328,302)$$

$$(171,222,278,332,326)(174,227,283,337,382)(176,229,284,338,362)$$
$$(181,237,292,346,209)(184,240,296,350,389)(186,243,299,353,391)$$
$$(188,246,252,192,251)(195,230,235,287,200)(196,253,309,363,392)$$
$$(203,261,276,331,378)(208,266,321,374,254)(217,275,313,294,348)$$
$$(225,262,317,369,356)(232,286,341,327,376)(248,303,357,394,370)$$
$$(255,310,280,304,358)(257,312,366,351,390)(259,315,347,388,274)$$
$$(264,318,371,400,354)(282,335,380,368,289)(291,344,385,311,364)$$
$$(301,355,393,407,411)(307,361,397,409,359)(319,372,398,375,349)$$
$$(324,339,383,404,412)(330,352,373,387,406)(333,377,402,336,381)$$
$$(343,379,345,386,403)(395,408,401,410,414)(399,405,413,415,416)$$

with $o(g_1) = 2,\ o(g_2) = 5$ and $o(g_1g_2) = 13$.

In Table 2, we list the values of h for all the non-identity conjugacy classes and all the maximal subgroups of $G_2(4)$. To calculate these values, we have used Equation (3).

We start our investigation on the ranks of the non-trivial classes of $G_2(4)$ by looking at the classes of involutions $2A$ and $2B$. It is well-known that two involutions generate a dihedral group. Thus the lower bound for the rank of a class of involutions in a finite simple group G is 3.

Proposition 3.1. *$rank(G : 2A) \neq 3$.*

Proof. We show that G is not $(2A, 2A, 2A, nX)$-generated for all the conjugacy classes nX of G.

We start by showing that the group $G = G_2(4)$ is not $(2A, 2A, 2A, nX)$-generated for $nX \in \{2A, 2B, 3A, 4C\}$. The direct computation with GAP gives $\Delta_{G_2(4)}(2A, 2A, 2A, 3A) = 0$. Also for $nX \in \{2A, 2B, 4C\}$ we have

$$\begin{aligned}
\Delta_G(2A, 2A, 2A, 2A) &= 28931 < 61440 = |C_G(g)|,\ g \in 2A,\\
\Delta_G(2A, 2A, 2A, 2B) &= 480 < 3840 = |C_G(g)|,\ g \in 2B,\\
\Delta_G(2A, 2A, 2A, 4C) &= 192 < 512 = |C_G(g)|,\ g \in 4C.
\end{aligned}$$

Using Lemma 2.7 we conclude that G is not $(2A, 2A, 2A, nX)$-generated for $nX \in \{2A, 2B, 4C\}$.

Secondly, we show that the group G is not $(2A, 2A, 2A, nX)$-generated for any $nX \in \{3B, 4A, 4B, 6A\}$. For this purpose we use Scott's Theorem. Let d_{nX} be the codimension of the fixed point space of $\langle g \rangle$ on $\mathbb{V}$, where $g \in nX$ and $\mathbb{V}$ is an irreducible module for

TABLE 2. The values $h(g, M_i),\ 1 \leq i \leq 8$ for non-identity classes and maximal subgroups of $G_2(4)$

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
$2A$	32	21	85	96	160	320	256	0
$2B$	16	25	21	32	32	80	256	320
$3A$	56	105	42	0	1	280	336	0
$3B$	5	12	15	6	10	10	21	30
$4A$	16	13	5	0	16	64	0	0
$4B$	0	1	5	0	16	16	0	0
$4C$	0	5	5	16	0	0	0	0
$5A$	1	0	5	6	10	0	13	0
$5B$	1	0	5	6	10	0	13	0
$5C$	6	5	0	1	0	0	13	0
$5D$	6	5	0	1	0	0	13	0
$6A$	8	9	10	0	1	8	16	0
$6B$	1	4	3	2	2	2	1	2
$7A$	3	0	0	0	1	3	0	9
$8A$	4	1	1	0	4	8	0	0
$8B$	0	1	1	4	0	0	0	0
$10A$	2	1	0	1	0	0	1	0
$10B$	2	1	0	1	0	0	1	0
$10C$	1	0	1	2	2	0	1	0
$10D$	1	0	1	2	2	0	1	0
$12A$	4	1	2	0	1	4	0	0
$12B$	0	1	2	0	1	4	0	0
$12C$	0	1	2	0	1	4	0	0
$13A$	0	0	0	1	0	0	0	1
$13B$	0	0	0	1	0	0	0	1
$15A$	1	0	2	0	1	0	1	0
$15B$	1	0	2	0	1	0	1	0
$15C$	0	2	0	1	0	0	1	0
$15D$	0	2	0	1	0	0	1	0
$21A$	0	0	0	0	1	0	0	0
$21B$	0	0	0	0	1	0	0	0

$G_2(4)$. Let χ be the third irreducible character of G as in the Atlas, where we can see that $\deg(\chi) = 78$. Then using Equation (5) with respect to the character χ, we obtain $d_{2A} = 32$, $d_{3B} = 44$, $d_{4A} = 52$, $d_{4B} = 54$ and $d_{6A} = 58$. If G is $(2A, 2A, 2A, nX)$-generated group then we must have $3 \times d_{2A} + d_{nX} \geq 2 \times 78$. However, it is clear that

$$3 \times d_{2A} + d_{nX} \quad = \quad \begin{cases} 96 + 44 = 140 < 156 & \text{if } nX = 3B, \\ 96 + 52 = 148 < 156 & \text{if } nX = 4A, \\ 96 + 54 = 150 < 156 & \text{if } nX = 4B, \\ 96 + 58 = 154 < 156 & \text{if } nX = 6A. \end{cases}$$

Thus G is not a $(2A, 2A, 2A, nX)$-generated group for any $nX \in \{3B, 4A, 4B, 6A\}$.

Now we handle the cases $(2A, 2A, 2A, 13X)$ and $(2A, 2A, 2A, 21X)$ for $X \in \{A, B\}$.

The case $(2A, 2A, 2A, 13X)$, $X \in \{A, B\}$:

We can see from Table 2 that M_4 is the only maximal subgroup of G with a nonempty intersection with the classes $2A$ and $13X$, $X \in \{A, B\}$, of $G_2(4)$. The direct computations with GAP yield $\Delta_G(2A, 2A, 2A, 13X) = 169$, $X \in \{A, B\}$ and $\Sigma(M_4) = 169$. We further find that $\Sigma(M_{41}) = 169$ where M_{41} is the largest maximal subgroup of M_4 isomorphic to $PSU(3, 4)$. Other maximal subgroups of M_4 isomorphic to groups $(2^{2 \cdot} 2^4){:}(3 \times D_{10})$, $A_5 \times D_{10}$ and $5^2{:}D_{12}$ have orders not divisible by 13 while the remaining subgroup isomorphic to 13:6 does not have involutions fusing into the class $2A$ of G. Since none of the maximal subgroups of M_{41} has the required fusion it follows that $\Sigma^*(M_{41}) = \Sigma(M_{41}) = 196$ and consequently $\Sigma^*(M_4) = \Sigma(M_4) - 1 \cdot \Sigma^*(M_{41}) = 196 - 196 = 0$. Therefore for $X \in \{A, B\}$ we have

$$\begin{aligned} \Delta_G^*(2A, 2A, 2A, 13X) \quad &= \quad \Delta_G(2A, 2A, 2A, 13X) - 1 \cdot \Sigma^*(M_4) - 1 \cdot \Sigma^*(M_{41}) \\ &= \quad 196 - 0 - 196 = 0, \end{aligned}$$

showing that G is not $(2A, 2A, 2A, 13X)$-generated.

The case $(2A, 2A, 2A, 21X)$, $X \in \{A, B\}$:

We can see from Table 2 that M_5 is the only maximal subgroup of G with a nonempty intersection with the classes $2A$ and $21X$, $X \in \{A, B\}$, of $G_2(4)$. The direct computations with GAP yield $\Delta_G(2A, 2A, 2A, 21X) = 441$, $X \in \{A, B\}$ and $\Sigma(M_5) = 441$. We further find that $\Sigma(M_{51}) = 441$ and $\Sigma(M_{54}) = 0$ where M_{51} and M_{54} are the maximal subgroups of M_5 isomorphic to $SL(3, 4)$ and $PSL(3, 2){:}S_3$ respectively. Other maximal subgroups of M_5 isomorphic to groups $(3^{\cdot} A_6){:}2$, $(4^{2 \cdot}(A_4 \times S_3)$, $(3^2{:}3){:}QD_{16}$ and $A_5 \times S_3$ respectively have orders not divisible by 21 and hence no class of elements of order 21. Furthermore we find $\Sigma(M_{516}) = \Sigma(M_{517}) = \Sigma(M_{518}) = 0$ for maximal subgroups of M_{51} isomorphic to the group $3 \times PSL(3, 2)$. The remaining maximal subgroups of M_{51} isomorphic to the groups $3 \times (2^4{:}A_5)$, $3^{\cdot} A_6$ and $(3^2{:}3){:}Q_8$ do not have the required fusions. Thus $\Sigma^*(M_{51}) = \Sigma(M_{51}) = 441$ and

$\Sigma^*(M_5) = \Sigma(M_5) - 1 \cdot \Sigma^*(M_{51}) = 441 - 441 = 0$. Therefore

$$\begin{aligned}\Delta^*_G(2A, 2A, 2A, 21X) &= \Delta_G(2A, 2A, 2A, 21X) - 1 \cdot \Sigma^*(M_5) - 1 \cdot \Sigma^*(M_{51})\\ &= 441 - 0 - 441 = 0,\end{aligned}$$

showing that G is not $(2A, 2A, 2A, 21X)$-generated for $X \in \{A, B\}$.

Finally, in a similar way to the two cases above we show that the group G is not generated for the remaining classes of G. Let

$$T := \{5A, 5B, 5C, 5D, 6B, 7A, 8A, 8B, 10A, 10B, 10C, 10D, 12A, 12B, 12C, 15A, 15B, 15C, 15D\}.$$

For all the classes $nX \in T$ we give in Table 3 the information about the structure constant $\Delta_G(2A, 2A, 2A, nX) := \Delta(G)$, $\Sigma^*_{H_i}(2a, 2a, 2a, nx) := \Sigma^*(H_i)$ for H_i a subgroup of G, $|H_i|$ and the structures of the groups that these triples generate.

We observe from Table 3 that for each nX in the first column the total number of triples generating a proper subgroup(s) i.e., $\Sigma^*(H_i)$ is equal to $\Delta(G)$. Thus $\Delta^*_G(2A, 2A, 2A, nX) = 0$, hence G is not $(2A, 2A, 2A, nX)$-generated for all $nX \in T$. This completes the proof that G is not $(2A, 2A, 2A, nX)$-generated for all the conjugacy classes nX of G. Hence we have $rank(G{:}2A) \neq 3$. □

Proposition 3.2. $rank(G{:}2A) = 4$.

Proof. We see from Table 2 that only the maximal subgroup M_5 has a nonempty intersection with the 5-tuple $(2A, 2A, 2A, 2A, 21A)$. The computations with GAP yield $\Delta_G(2A, 2A, 2A, 2A, 21A) = 1194669$ and $\Sigma(M_5) = 157437$. Thus,

$$\Delta^*_G(2A, 2A, 2A, 2A, 21A) \geq \Delta_G(2A, 2A, 2A, 2A, 21A) - 1 \cdot \Sigma(M_5) = 1194669 - 157437 = 1037232$$

showing that G is $(2A, 2A, 2A, 2A, 21A)$-generated. Since G cannot be generated by two involutions and by Proposition 3.1 $rank(G{:}2A) \neq 3$, it follows that $rank(G{:}2A) = 4$. □

Proposition 3.3. $rank(G{:}2B) = 3$.

Proof. Since by Proposition 18 of [7] the group G is $(2B, 3B, 13A)$-generated, it follows by applications of Lemma 2.3 that G is $(2B, 2B, 2B, (13A)^3)$-generated; that is $(2B, 2B, 2B, 13A)$-generated group. Thus, $rank(G{:}2B) \leq 3$. We know from the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ that G cannot be $(2B, 2B, nX)$-generated for any conjugacy class nX of G. Therefore we deduce that $rank(G{:}2B) = 3$. □

TABLE 3. The groups generated by $(x, y, z) \in (2A)^3$ where xyz is a fixed element in nX

nX	$\Delta(G)$	$\Sigma^*(M)$	Structure description	Order
5A	1925	800	A_5	60
		1125	$2^4{:}D_{10}$	160
5B	1925	800	A_5	60
		1125	$2^4{:}D_{10}$	160
5C	300	300	$PSU(3,4)$	62400
5D	300	300	$PSU(3,4)$	62400
6B	192	192	$3^2{:}S_3$	54
7A	441	147	$PSL(3,2)$	168
		294	$SL(3,4)$	60480
8A	128	128	$(4^2{:}3){:}2$	96
8B	384	384	$2^{4\cdot}(2^4{:}D_{10})$	2560
10A	80	80	$5^2{:}S_3$	150
10B	80	80	$5^2{:}S_3$	150
10C	200	200	$2^5{:}A_5$	1920
10D	200	200	$2^5{:}A_5$	1920
12A	384	384	$3^{\cdot}A_6$	1080
12B	384	384	$3^{\cdot}A_6$	1080
12C	384	384	$3^{\cdot}A_6$	1080
15A	375	150	$SL(3,4)$	60480
		225	$3^{\cdot}A_6$	1080
15B	375	150	$SL(3,4)$	60480
		225	$3^{\cdot}A_6$	1080
15C	135	135	$PSU(3,4)$	62400
15D	135	135	$PSU(3,4)$	62400

Lemma 3.4. *The group G is not $(3A, 3A, nX)$-generated for all conjugacy classes nX of $G_2(4)$ with $n \in \{2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 15, 21\}$.*

Proof. The group G cannot be generated by the triples $(3A, 3A, 3A)$ and $(3A, 3A, 3B)$ since each of these triples violate the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. We calculate $\Delta^*_G(3A, 3A, nX) = \Delta_G(3A, 3A, nX) = 0$ for any non-trivial class $nX \in \{4B, 4C, 6B, 8A, 8B, 10A, 10B, 10C, 10D, 12A, 12B, 12C, 15A, 15B, 15C, 15D, 21A, 21B\}$. By

Propositions 19, 20 and 21 of [7] the group G is not $(3A,3A,nX)$-generated for $nX \in \{5A,5B,5C,5D,7A,13A,13B\}$. Lastly we look at the remaining cases, namely $\Delta_G(3A,3A,4A)$ and $\Delta_G(3A,3A,6A)$. For these cases we apply Ree's Theorem for the action of G on conjugates of $M_1 = J_2$. Now, if G is $(3A,3A,nX)$-generated for $nX \in \{4A,6A\}$, we should have $c_{3A}+c_{3A}+c_{nX} \leq 418$. From Table 5 we see that $c_{3A}+c_{3A}+c_{4A} = 176+176+110 = 462$ and $c_{3A}+c_{3A}+c_{6A} = 176+176+96 = 448$. In both cases we find that $c_{3A}+c_{3A}+c_{nX} > 418$. It follows by Ree's Theorem 2.8 that G is not $(3A,3A,nX)$-generated for $nX \in \{4A,6A\}$. □

Proposition 3.5. *$rank(G{:}3A) \neq 2$.*

Proof. The proof follows immediately from Lemma 3.4. □

Remark 3.6. An alternative way to show that two elements from the class $3A$ cannot generate G is by using a theorem by Brauer (see for example [16]), which states that if χ is a character of a group G such that $\langle \chi, \mathbf{1}\rangle = 0$ and if $H, K \leq G$ such that $\left\langle \chi\downarrow_H^G, \mathbf{1}_H\right\rangle + \left\langle \chi\downarrow_K^G, \mathbf{1}_K\right\rangle > \left\langle \chi\downarrow_{H\bigcap K}^G, \mathbf{1}_{H\bigcap K}\right\rangle$, then $\langle H, K\rangle < G$. Now let $G = G_2(4)$, $\chi \in \mathrm{Irr}(G)$ such that $\deg(\chi) = 65$, $H = \langle x\rangle$ and $K = \langle y\rangle$, where $x, y \in 3A$ and $x \neq y$. Then $\langle \chi, \mathbf{1}\rangle = 0$ and $H \bigcap K = \{1_G\}$. Moreover, $\left\langle \chi\downarrow_H^G, \mathbf{1}_H\right\rangle = \left\langle \chi\downarrow_K^G, \mathbf{1}_K\right\rangle = 35$ and $\left\langle \chi\downarrow_{\{1_G\}}^G, \mathbf{1}_{\{1_G\}}\right\rangle = 65$. Since $\left\langle \chi\downarrow_H^G, \mathbf{1}_H\right\rangle + \left\langle \chi\downarrow_K^G, \mathbf{1}_K\right\rangle = 35 + 35 = 70 > 65 = \left\langle \chi\downarrow_{\{1_G\}}^G, \mathbf{1}_{\{1_G\}}\right\rangle$, it follows by Brauer's Theorem that $\langle x, y\rangle < G$. Hence two elements from class $3A$ cannot generate the group $G_2(4)$, meaning that $rank(G{:}3A) \neq 2$.

Proposition 3.7. *$rank(G{:}3A) = 3$.*

Proof. The direct computations with GAP yield $\Delta_G(3A,3A,3A,13A) = 169$. We can see from Table 2 that no maximal subgroup of G has a nonempty intersection with the quadruple $(3A,3A,3A,13A)$. Thus $\Delta_G^*(3A,3A,3A,13A) = \Delta_G(3A,3A,3A,13A) = 169$ showing that G is $(3A,3A,3A,13A)$-generated. Since by Proposition 3.5 $rank(G{:}3A) \neq 2$ we conclude that $rank(G{:}3A) = 3$. □

Proposition 3.8. *$rank(G{:}3B) = 2$.*

Proof. By Proposition 8 of [7] the group G is $(2B,3B,13B)$-generated. It follows by applications of Lemma 2.5 that G is $(3B,3B,(13B)^2)$-generated; that is $(3B,3B,13A)$-generated group. Thus $rank(G{:}3B) = 2$ and the proof is complete. □

Proposition 3.9. *$rank(G{:}4A) = 2$.*

Proof. We can see from Table 2 that only the maximal subgroups M_2 and M_3 have a nonempty intersection with the triple $(4A, 4A, 8B)$. We calculate $\Delta_G(4A, 4A, 8B) = 272$ and $\Sigma(M_i) = 8 + 8 = 16$ for $i \in \{2, 3\}$. With $h = 1$ we obtain

$$\Delta^*_G(4A, 4A, 8B) \geq \Delta_G(4A, 4A, 8B) - 1 \cdot \Sigma(M_2) - 1 \cdot \Sigma(M_3) = 272 - 16 - 16 = 240.$$

Since $\Delta^*_G(4A, 4A, 8B) > 0$ we conclude that $G_2(4)$ is $(4A, 4A, 8B)$-generated and hence $rank(G : 4A) = 2$. □

Note: We deal with the rank of class $4B$ collectively with other classes in Proposition 3.19.

Proposition 3.10. *$rank(G{:}4C) = 2$.*

Proof. We can see from Table 2 that no maximal subgroup of G has a nonempty intersection with the triple $(2B, 4C, 21B)$. Therefore $\Delta^*_G(2B, 4C, 21B) = \Delta_G(2B, 4C, 21B) = 126$. Since $\Delta^*_G(2B, 4C, 21B) > 0$ we conclude that G is $(2B, 4C, 21B)$-generated. Application of Theorem 2.5 shows that G is $(4C, 4C, 21A)$-generated and hence $rank(G : 4C) = 2$. □

Proposition 3.11. *$rank(G{:}5X) = 2$ for any $X \in \{A, B, C, D\}$.*

Proof. We have from Propositions 33 and 36 of [7] that the group G is $(5A, 5A, 5C)$-, $(5B, 5B, 5C)$-, $(5C, 5C, 5C)$- and $(5D, 5D, 5D)$-generated. It follows immediately that $rank(G{:}5X) = 2$ for $X \in \{A, B, C, D\}$ and the proof is complete. □

Remark 3.12. The above result can also be deduced from Proposition 12 (ii) of [7] as follows: Since G is $(2B, 5X, 7A)$-generated for all $X \in \{A, B, C, D\}$ then by Theorem 2.5 G is $(5X, 5X, 7A)$-generated, showing that $rank(G{:}5X) = 2$.

Proposition 3.13. *$rank(G{:}7A) = 2$.*

Proof. Since by Proposition 43 of [7] the group G is $(7A, 7A, 7A)$-generated, it follows immediately that $rank(G{:}7A) = 2$. □

Proposition 3.14. *$rank(G{:}13X) = 2$ for $X \in \{A, B\}$.*

Proof. By Proposition 45 of [7] the group G is $(13X, 13X, 13B)$-generated for $X \in \{A, B\}$. Therefore $rank(G{:}13X) = 2$ for $X \in \{A, B\}$. □

Lemma 3.15. *The group G is $(2A, nX, 21A)$-generated for $nX \in \{8B, 10A, 10B, 15C, 15D\}$.*

Proof. By Table 2 no maximal subgroup of G will make a contribution in the calculations of $\Delta_G^*(2A, nX, 21A)$ for $nX \in \{8B, 10A, 10B, 15C, 15D\}$. Thus $\Delta_G^*(2A, nX, 21A) = \Delta_G(2A, nX, 21A)$. Computations with GAP yield

$$\begin{aligned}\Delta_G^*(2A, nX, 21A) &= \Delta_G(2A, nX, 21A)\\ &= \begin{cases} 168 & \text{for } nX \in \{8B, 10A, 10B\},\\ 273 & \text{for } nX \in \{15C, 15D\},\end{cases}\end{aligned}$$

and the results follows. □

Proposition 3.16. *$rank(G{:}nX) = 2$ for $nX \in \{8B, 10A, 10B, 15C, 15D\}$.*

Proof. Since by Lemma 3.15, G is $(2A, nX, 21A)$-generated group, it follows by applications of Lemma 2.5 that G is $(nX, nX, (21A)^2)$-generated; that is $(nX, nX, 21B)$-generated group for $nX \in \{8B, 10A, 10B, 15C, 15D\}$. Thus $rank(G{:}nX) = 2$. □

Lemma 3.17. *The group G is $(2A, nX, 13A)$-generated for $nX \in \{6A, 8A, 12X\}$, $X \in \{A, B, C\}$.*

Proof. By Table 2 no maximal subgroup of G will make a contribution in the calculations of $\Delta_G^*(2A, nX, 13A)$ for $nX \in \{6A, 8A, 12X\}$, $X \in \{A, B, C\}$. Thus $\Delta_G^*(2A, nX, 13A) = \Delta_G(2A, nX, 13A)$. Computations with GAP yield

$$\begin{aligned}\Delta_G^*(2A, nX, 13A) &= \Delta_G(2A, nX, 13A)\\ &= \begin{cases} 13 & \text{for } nX = 6A,\\ 52 & \text{for } nX = 12A,\\ 104 & \text{for } nX \in \{8A, 12B, 12C\},\end{cases}\end{aligned}$$

and the results follows. □

Proposition 3.18. *$rank(G{:}nX) = 2$ for $nX \in \{6A, 8A, 12X\}$ for $X \in \{A, B, C\}$.*

Proof. Since by Lemma 3.17, G is $(2A, nX, 13A)$-generated group for all $nX \in \{6A, 8A, 12X\}$, $X \in \{A, B, C\}$. It follows by applications of Lemma 2.5 that G is $(nX, nX, (13A)^2)$-generated; that is $(nX, nX, 13B)$-generated group for all $nX \in \{6A, 8A, 12X\}$, $X \in \{A, B, C\}$. Thus $rank(G{:}nX) = 2$ for $nX \in \{6A, 8A, 12A, 12B, 12C\}$. □

Proposition 3.19. *Let $T := \{4B, 6B, 10C, 10D, 15A, 15B, 21A, 21B\}$. Then $rank(G{:}nX) = 2$ for all $nX \in T$.*

TABLE 4. Some information on the classes $nX \in T$

	$\Delta(G)$	$\Sigma(M_5)$	$\Delta^*(G) \geq$
$4B$	105	0	105
$6B$	5712	357	5355
$10C$	3528	189	3339
$10D$	3528	189	3339
$15A$	4368	0	4368
$15B$	4368	0	4368
$21A$	3360	0	3360
$21B$	3024	0	3024

Proof. The aim here is to show that G is a $(2B, nX, 21B)$-generated group for any $nX \in T$. We first note that M_5 is the only maximal subgroup of G that contain elements of order 21. From Table 2 we have $h(g, M_5) = 1$ for a fixed $g \in 21B$. For all the classes $nX \in T$ we give in Table 4 the computations obtained for $\Delta_G(2B, nX, 21B) := \Delta(G)$, $\Sigma_{M_5}(2B, nX, 21B) := \Sigma(M_5)$ and finally $\Delta^*_G(2A, nX, 21B) := \Delta^*(G)$, where from Equation 4 we know that:

$$\Delta^*(G) \geq \Delta_G(2B, nX, 21B) - 1 \cdot \Sigma(M_5).$$

The last column of Table 4 shows that G is $(2B, nX, 21B)$-generated group for all $nX \in T$. It follows by Lemma 2.5 that G is $(nX, nX, (21B)^2)$-generated, i.e., G is $(nX, nX, 21A)$-generated. Hence $rank(G{:}nX) = 2$ for all $nX \in T$. □

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TABLE 5. Cycle structures of all conjugacy classes of $G_2(4)$

nX	Cycle Structure	c_i
$1A$	1^{416}	416
$2A$	$1^{32}2^{192}$	224
$2B$	$1^{16}2^{200}$	216
$3A$	$1^{56}3^{120}$	176
$3B$	$1^{5}3^{137}$	142
$4A$	$1^{6}2^{8}4^{96}$	110
$4B$	$2^{16}4^{96}$	112
$4C$	$2^{16}4^{96}$	112
$5A$	$1^{1}5^{83}$	84
$5B$	$1^{1}5^{83}$	84
$5C$	$1^{6}5^{82}$	88
$5D$	$1^{6}5^{82}$	88
$6A$	$1^{8}2^{24}3^{8}6^{56}$	96
$6B$	$1^{1}2^{2}3^{5}6^{66}$	74
$7A$	$1^{3}7^{59}$	62
$8A$	$1^{4}2^{6}4^{4}8^{48}$	62
$8B$	$4^{8}8^{48}$	56
$10A$	$1^{2}2^{2}5^{6}10^{38}$	48
$10B$	$1^{2}2^{2}5^{6}10^{38}$	48
$10C$	$1^{1}5^{3}10^{40}$	44
$10D$	$1^{1}5^{3}10^{40}$	44
$12A$	$1^{4}2^{2}3^{4}4^{12}6^{2}12^{28}$	52
$12B$	$2^{4}4^{12}6^{4}12^{28}$	48
$12C$	$2^{4}4^{12}6^{4}12^{28}$	48
$13A$	13^{32}	32
$13B$	13^{32}	32
$15A$	$1^{1}5^{11}15^{24}$	36
$15B$	$1^{1}5^{11}15^{24}$	36
$15C$	$3^{(}2)5^{1}15^{27}$	30
$15D$	$3^{(}2)5^{1}15^{27}$	30
$21A$	$3^{1}7^{8}21^{17}$	26
$21B$	$3^{1}7^{8}21^{17}$	26

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