Algebraic Structures and Their Applications Vol. X No. X (20XX) pp XX-XX.

Research Paper

SOME CLASSIFICATIONS OF MONOIDS BY STRONGLY IDEMPOTENT CANCELLATIVE (PWP)

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ABSTRACT. In this paper, we introduce Condition (PWP_{sic}) of acts over monoids and compare this condition with the properties left PP and left PSF in monoid S. At first we give a classification of monoids by this condition of right acts. Also, we give a classification of monoids for which some other properties of their right acts imply Condition (PWP_{sic}) and vice versa. Then a classification of monoids will be given for which all right Rees factor acts of S satisfying some other flatness properties have Condition (PWP_{sic}).

The specific question of when every right S-act satisfying Condition (PWP_{sic}) has certain flatness properties or every (GPW-flat) GP-flat right S-act satisfies Condition (PWP_{sic}), have so far not been considered. In this paper, we will address these problems.

DOI: 10.22034/as.2025.21408.1707

MSC(2010): Primary: 20M30, Secondary: 20M50.

Keywords: Condition (PWP_{sic}), Rees factor act, Regular.

Received: 27 March 2024, Accepted: 01 February 2025.

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1. Introduction and preliminaries

Throughout this paper S will denote a monoid. We refer the reader to [8, 11] for basic results, definitions and terminology relating to semigroups and acts over monoids and to [12, 13], for definitions and results on flatness which are used here.

A monoid S is called *right (left) reversible* if for every $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'(su = s'v)$. A right ideal K of a monoid S is called *left stabilizing* if for every $k \in K$, there exists $l \in K$ such that $lk = k$.

An element s of a monoid S is called *right e-cancellable*, for an idempotent $e \in S$, if $s = es$ and $ker\rho_s \leq ker\rho_e$, i.e. $ts = t's$, $t, t' \in S$, implies $te = t'e$. A monoid S is called *left PP* if every element $s \in S$ is right e-cancellable, for some idempotent $e \in S$. It is easy to see that S is left PP if and only if for every $s \in S$ there exists $e \in E(S)$, such that $ker\rho_s = ker\rho_e$. This is equivalent to the projectivity of every principal left ideal of S. Similarly a right PP monoid is defined. An element $s \in S$ is called *right semi-cancellative* if $ts = t's$, $t, t' \in S$, implies there exists $r \in S$ such that $s = rs$ and $tr = t'r$. A monoid S is called *left PSF* if all principal left ideals of S is strongly flat. It is easy to see that S is left PSF if and only if every element $s \in S$ is right semi-cancellable.

An element $s \in S$ is called *regular*, if $sxs = s$, for some $x \in S$. S is called a *regular monoid* if all its elements are regular. An element s of a monoid S is called *left almost regular* if there exist elements $r, r_1, ..., r_m, s_1, ..., s_m \in S$ and right cancellable elements $c_1, c_2, ..., c_m \in S$ such that

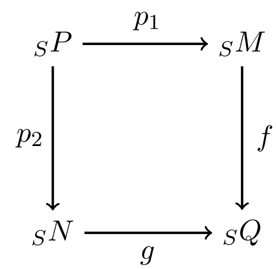
$$\begin{aligned} s_1c_1 &= sr_1, \\ s_2c_2 &= s_1r_2, \\ &\vdots \\ s_mc_m &= s_{m-1}r_m, \\ s &= s_mrs. \end{aligned}$$

If all elements of S are left almost regular, then S is called *left almost regular*. We can see that every left almost regular monoid is left PP ([11, Proposition 4.1.3]).

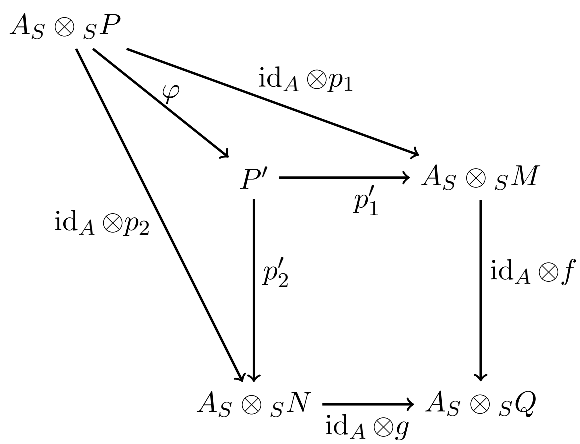
A right S-act is a non-empty set A, usually denoted A_S, on which S acts unitarian from the right, that is, $(as)t = a(st)$ and $a1 = a$, for every $a \in A, s, t \in S$, where 1 is the identity of S. Left S-acts are defined similarly. We say A_S satisfies Condition (P) if $as = a's'$, for $a, a' \in A_S$, $s, s' \in S$, implies the existence of $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs'$.

A right S-act A_S satisfies Condition (E) if $as = as'$, for $a \in A_S$, $s, s' \in S$, implies the existence of $a' \in A_S$ and $u \in S$ such that $a = a'u$ and $us = us'$.

In [12], the notation $P(M, N, f, g, Q)$ was introduced to denote the pullback diagram



of homomorphisms $f : {}_SM \to {}_SQ$ and $g : {}_SN \to {}_SQ$ in the category of left S-acts. Tensoring such a diagram by A_S produces the outer square in the diagram



(in the category of sets) that may or may not be a pullback diagram, depending on whether or not the mapping φ is bijective. Here,

$$P' = \{(a \otimes m, a' \otimes n) \in (A_S \otimes {}_SM) \times (A_S \otimes {}_SN) | a \otimes f(m) = a' \otimes g(n)\}$$

with p_1' and p_2' the restrictions of the projections is the pullback of mappings $id_A \otimes f$ and $id_A \otimes g$ in the category of sets, and the mapping φ, obtained via the universal property of pullbacks, is given by

$$\varphi(a \otimes (m, n)) = (a \otimes m, a \otimes n),$$

for all $a \in A_S$ and $(m, n) \in {}_SP$.

We recall from [1, 12, 13] that:

The S-act A_S is *weakly pullback flat*, if the corresponding φ is bijective for every pullback diagram $P(S, S, f, g, S)$.

The S-act A_S is *weakly kernel flat* if the corresponding φ is bijective for every pullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S.

The S-act A_S is *principally weakly kernel flat* if the corresponding φ is bijective for every pullback diagram $P(Ss, Ss, f, f, S)$, where $s \in S$.

The S-act A_S is *translation kernel flat* if the corresponding φ is bijective for every pullback diagram $P(S,S,f,f,S)$.

The S-act A_S is *weakly homoflat*, if for all elements $s,t \in S$, all homomorphisms $f : {}_S(Ss \cup St) \to {}_SS$, all $a,a' \in A_S$, if $af(s) = a'f(t)$, then there exist $a'' \in A_S$, $u,v \in S$, $s',t' \in \{s,t\}$ such that $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes {}_S(Ss \cup St)$ and $f(us') = f(vt')$.

The S-act A_S is *principally weakly homoflat*, if for all $a,a' \in A_S$, $s \in S$,

$$as = a's \Rightarrow (\exists a'' \in A_S)(\exists u,v \in S)(a = a''u, a' = a''v, us = vs).$$

Recall from [7] that A_S is called ETF if every $e \in E(S)$ acts injectivity on A_S, that is, if $ae = a'e$, $a,a' \in A_S$, then $a = a'$. It is easy to see that A_S is ETF if and only if for every $a \in A_S, e \in E(S)$, $ae = a$. Also an act A_S is called *strongly torsion free* or STF if every $s \in S$ acts injectively on A_S, that is, if $as = a's$, $a,a' \in A_S$, then $a = a'$.

We use the following abbreviations,

- weak pullback flatness := WPF,
- weak kernel flatness := WKF,
- principal weak kernel flatness := $PWKF$,
- translation kernel flatness := TKF,
- weak homoflatness := (WP),
- principal weak homoflatness := (PWP).

2. General properties

In this section we introduce Condition (PWP_{sic}) and give some properties of it.

Definition 2.1. An element $s \in S$ is called *right idempotent-cancellative* if $ts = t's$, $t,t' \in S$, implies there exists $e \in E(S)$, such that $s = es$ and $te = t'e$.

Definition 2.2. A right S-act A_S is called *strongly idempotent-cancellative-(PWP)* or satisfies Condition (PWP_{sic}) if $as = a's$, for all $a,a' \in A_S$ and $s \in S$, implies $ae = a'e$ and $es = s$, for some $e \in E(S)$.

It is easy to see that if S is left PP, then S_S satisfies Condition (PWP_{sic}). Also if S_S satisfies Condition (PWP_{sic}), then S is left PSF. The following examples show that the converse of these relations is not true, but the converse holds for any regular monoid S.

Example 2.3. Suppose $S_1 = (\mathbb{N} \setminus \{1\}, max)$ and $S_2 =< a >$ be an infinite monogenic semigroup. Let $T = S_1 \cup S_2$ with the multiplication

$$x * y = \begin{cases} max\{x, y\}, & \text{if } x, y \in S_1, \\ xy, & \text{if } x, y \in S_2, \\ y = y * x, & \text{if } x \in S_1, y \in S_2, \end{cases}$$

and let $S = T^1$. It is easy to see that S is a monoid which satisfies Condition (PWP_{sic}), but for $y \in S_2$ there exist no idempotent $e \in E(S)$ such that $ker\rho_y = ker\rho_e$, and so S is not left PP.

Example 2.4. Let $(I, \leq)$ be a totally ordered set which has no maximum element. Consider the commutative monoid

$$S = \{x_i^m | i \in I, m \in \mathbb{N}\} \cup \{1\},$$

such that

$$x_i^m x_j^n = \begin{cases} x_j^n, & \text{if } i < j, \\ x_i^{m+n}, & \text{if } i = j. \end{cases}$$

Clearly S is left PSF but it doesn't satisfy Condition (PWP_{sic}). Indeed, for $i < j < k, x_i^m x_k^t = x_j^n x_k^t$, but there exist no idempotent $e \in E(S)$ such that $x_i^m e = x_j^n e$.

Now we establish some general properties.

Proposition 2.5. *The following statements hold:*

(1) Θ_S *satisfies Condition* (PWP_{sic}).

(2) *Let* A_S *be an act satisfying Condition* (PWP_{sic}). *Then every subact of* A_S *satisfies Condition* (PWP_{sic}).

(3) S_S *satisfies Condition* (PWP_{sic}) *if and only if every element of* S *is right idempotent-cancellative, equivalently*

$$(\forall x, y, s \in S)(xs = ys \Rightarrow (\exists e \in E(S))(xe = ye \wedge es = s)).$$

(4) *Let* $A = \dot{\bigcup}_{i\in I} A_i$, *where every* A_i, $i \in I$, *is a subact of* A. *Then* A *satisfies Condition* (PWP_{sic}) *if and only if* A_i *satisfies Condition* (PWP_{sic}), *for every* $i \in I$.

(5) *Let* $\{A_i \mid i \in I\}$, *is a chain of subacts of* A. *Then* $A = \bigcup_{i\in I} A_i$, *satisfies Condition* (PWP_{sic}) *if and only if* A_i *satisfies Condition* (PWP_{sic}), *for every* $i \in I$.

(6) *A right* S*-act* A_S *is* STF *if and only if* A_S *is* ETF *and satisfies Condition* (PWP_{sic}).

Proof. The proof of parts (1), (2), (3), (4) and (5) are straightforward.

(6). Suppose that A_S be STF. Then A_S is ETF, by definition. Now let $as = a's$, for $a, a' \in A_S$, $s \in S$. By assumption, $a = a'$. Thus $1s = s, a1 = a'1$, and so A_S satisfies

Condition (PWP_{sic}). Conversely let $as = a's$, for $a, a' \in A_S$, $s \in S$. By assumption, there exists $e \in E(S)$ such that $es = s$ and $ae = a'e$. Since A_S is ETF, we have $a = a'$, and so A_S is STF. □

As defined in [11], a subset $U \neq \emptyset$ of a right S-act A_S is said to be a generating set of A_S if every element $a \in A_S$ can be presented as $a = us$, for some $u \in U$ and $s \in S$. In other words, U is a set of generating elements for A_S if $< U >:= \bigcup_{u \in U} uS = A_S$ where $uS = \{us\ s \in S\}$. We say that a right S-act A_S is finitely generated if $A_S =< U >$, for some U, $|U| < \infty$. We call A_S a cyclic S-act if $A_S =< u >$, where $u \in A_S$.

Here we give a criterion for a cyclic right S-act to satisfy Condition (PWP_{sic}).

Proposition 2.6. *Let ρ be a right congruence on monoid S. Then the right act S/ρ satisfies Condition (PWP_{sic}) if and only if, for all $x, y, s \in S$, $(xs)\rho(ys)$ implies $(xe)\rho(ye)$ and $es = s$, for some $e \in E(S)$.*

Proof. **Necessity**. Let $(xs)\rho(ys)$, for $x, y, s \in S$. Then $[x]_\rho s = [y]_\rho s$. By assumption, there exists $e \in E(S)$ such that $[x]_\rho e = [y]_\rho e$, $es = s$ and so $(xe)\rho(ye)$.

Sufficiency. Let $[x]_\rho s = [y]_\rho s$, for $x, y, s \in S$. Then $(xs)\rho(ys)$. By assumption, there exists $e \in E(S)$ such that $es = s$, $(xe)\rho(ye)$, and so $[x]_\rho e = [y]_\rho e$. Thus S/ρ satisfies Condition (PWP_{sic}).□

Proposition 2.7. *Let $w \in S$ and $\rho = \rho(w, 1)$. Then the right act S/ρ satisfies Condition (PWP_{sic}) if and only if for all $x, y, s \in S$ and non-negative integers m, n,*

$$(w^m xs = w^n ys \Rightarrow (\exists p, q \in \mathbb{N}_0)(\exists e \in E(S))(w^p xe = w^q ye \wedge es = s)).$$

Proof. **Necessity**. Let $w^m xs = w^n ys$, for $x, y, s \in S$, $m, n \in \mathbb{N}_0$. It follows by [11, Corollary 3.8.7], $(xs)\rho(ys)$, and so by Proposition 2.6, there exists an idempotent $e \in E(S)$ such that $es = s$ and $(xe)\rho(ye)$. Then, by [11, Corollary 3.8.7], there exist $p, q \in \mathbb{N}_0$, with $w^p xe = w^q ye$, as required.

Sufficiency. Let $(xs)\rho(ys)$, $x, y, s \in S$. By [11, Corollary 3.8.7], there exist $m, n \in \mathbb{N}_0$, such that $w^m xs = w^n ys$. Hence, by assumption, there exist $e \in E(S)$, $p, q \in \mathbb{N}_0$, with $w^p xe = w^q ye$ and $es = s$. Then $(xe)\rho(ye)$, by [11, Corollary 3.8.7], and so S/ρ satisfies Condition (PWP_{sic}), by Proposition 2.6. □

In the previous proposition if $w = 1$, then $S/\rho = S/\rho(w,1) = S/\rho(1,1) = S/\Delta_S \cong S_S$. Thus S_S satisfies Condition (PWP_{sic}) if and only if for $x, y, s \in S$, $xs = ys$ implies the existence $e \in E(S)$ such that $xe = ye$ and $es = s$. Hence the assertion (3) of Proposition 2.5 is a corollary of the previous proposition.

Proposition 2.8. *The following statements hold:*

(1) *Every act satisfying Condition (PWP_{sic}) is principally weakly flat.*
(2) *If S is a left PP monoid then every principally weakly flat act satisfies Condition (PWP_{sic}).*

Proof. (1). Suppose that A_S satisfies Condition (PWP_{sic}). Let $a \otimes s = a' \otimes s$ in $A \otimes_S S$, for $a, a' \in A_S$, $s \in S$. Then, by [11, Proposition 2.5.13], $as = a's$, and so by assumption, there exists $e \in E(S)$ such that $es = s$ and $ae = a'e$. Thus

$$a \otimes s = a \otimes es = ae \otimes s = a'e \otimes s = a' \otimes es = a' \otimes s,$$

in $A \otimes_S Ss$. Hence A_S is principally weakly flat.

(2). It follows from [11, Theorem 3.10.16].□

3. Classification by condition (PWP_{sic}) of right acts

In this section we give a classification of monoids when acts with other properties satisfy Condition (PWP_{sic}) and vice versa. We also give some classifications of monoids when all their acts satisfy Condition (PWP_{sic}).

Note that, in act categories we call a monomorphism $f : A_S \to B_S$ an embedding of A_S into B_S. If there exists such a monomorphism we say that A_S can be embedded into B_S or B_S contains (an isomorphic copy of) A_S or B_S is an extension of A_S. If $A_S \subseteq B_S$ is a subact, then the restriction $(id_{B_S})|_{A_S}$ is called the inclusion or natural embedding of A_S into B_S.

Proposition 3.1. *The following statements are equivalent:*

(1) *all right S-acts satisfy Condition (PWP_{sic});*
(2) *all finitely generated right S-acts satisfy Condition (PWP_{sic});*
(3) *all right S-acts generated by at most two elements satisfy Condition (PWP_{sic});*
(4) *all right S-acts generated by exactly two elements satisfy Condition (PWP_{sic});*
(5) *all cyclic right S-acts satisfy Condition (PWP_{sic});*
(6) *all monocyclic right S-acts satisfy Condition (PWP_{sic});*
(7) *all monocyclic right S-acts of the form $S/\rho(s, s^2)$, $s \in S$ satisfy Condition (PWP_{sic});*
(8) *for all $x, y, s \in S$ there exists $e \in E(S)$ such that $es = s$, $(xe)\rho(xs, ys)(ye)$;*
(9) *for all $x, y, s \in S$, there exists $r \in S$ such that $rs = s, (xr)\rho(xs, ys)(yr)$;*

(10) *all right Rees factor acts of S satisfy Condition (PWP_{sic});*

(11) *all right Rees factor acts of S of the form S/sS, $s \in S$ satisfy Condition (PWP_{sic});*

(12) *all divisible right S-acts satisfy Condition (PWP_{sic});*

(13) *all principally weakly injective right S-acts satisfy Condition (PWP_{sic});*

(14) *all finitely generated weakly injective right S-acts satisfy Condition (PWP_{sic});*

(15) *all weakly injective right S-acts satisfy Condition (PWP_{sic});*

(16) *all injective right S-acts satisfy Condition (PWP_{sic});*

(17) *all cofree right S-acts satisfy Condition (PWP_{sic});*

(18) *S is regular.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$, $(3) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7)$, $(5) \Rightarrow (10) \Rightarrow (11)$ and $(8) \Rightarrow (9)$ are obvious.

Since Cofree $\Rightarrow$ Injective $\Rightarrow$ weakly injective $\Rightarrow$ finitely generated weakly injective $\Rightarrow$ principally weakly injective $\Rightarrow$ divisible, implications $(1) \Rightarrow (12) \Rightarrow (13) \Rightarrow (14) \Rightarrow (15) \Rightarrow (16) \Rightarrow (17)$ are obtained immediately.

$(4) \Rightarrow (18)$. Let $s \in S$. If $sS = S$ then there exists $x \in S$ such that $sx = 1$, and so $sxs = s$. Thus s is regular. Now suppose that $sS \neq S$. Set

$$A_S = S \overset{sS}{\amalg} S = \{(l,x) |\ l \in S \setminus sS\} \,\dot{\cup}\, sS \,\dot{\cup}\, \{(t,y) |\ t \in S \setminus sS\}.$$

Obviously $A_S = (1,x)S \cup (1,y)S =< (1,x),(1,y) >$. Since A_S is generated by two elements, then, by assumption, A_S satisfies Condition (PWP_{sic}). Thus the equality $s = (1,x)s = (1,y)s$ implies the existence of $e \in E(S)$ such that $es = s$ and $(1,x)e = (1,y)e$. The second equality implies $e \in sS$, and so there exists $x \in S$ such that $e = sx$. Hence $s = es = sxs$, that is s is a regular element. Thus S is regular.

$(7) \Rightarrow (18)$. By Proposition 2.8, part (1), all monocyclic right S-acts of the form $S/\rho(s,s^2), s \in S$, are principally weakly flat. Therefore by [11, Theorem 4.6.6], S is a regular monoid.

$(11) \Rightarrow (18)$. By Proposition 2.8, part (1), all right Rees factor acts of S of the form S/sS, $s \in S$, are principally weakly flat. It follows by [11, Theorem 4.6.6] that S is a regular monoid.

$(9) \Rightarrow (18)$. We show that all cyclic right S-acts are principally weakly flat. Let ρ be a right congruence on S and $[x]_\rho s = [y]_\rho s$, for $x,y,s \in S$. Then $(xs)\rho(ys)$, and so $\rho(xs,ys) \subseteq \rho$. Now, by assumption, there exists $r \in S$ such that $rs = s$ and $(xr)\rho(xs,ys)(yr)$, and so $(xr)\rho(yr)$. Thus

$$[x]_\rho \otimes s = [x]_\rho \otimes rs = [x]_\rho r \otimes s = [xr]_\rho \otimes s = [yr]_\rho \otimes s$$
$$= [y]_\rho r \otimes s = [y]_\rho \otimes rs = [y]_\rho \otimes s.$$

in $S/\rho \otimes Ss$. Hence all cyclic right S-acts are principally weakly flat and so by [11, Theorem 4.6.6], S is regular.

(17) $\Rightarrow$ (18). Since every act can be embedded into a cofree act, by assumption, every S-act is a subact of act which satisfying Condition (PWP_{sic}). By Proposition 2.5, part (2), all right S-acts satisfy Condition (PWP_{sic}). It follows, by Proposition 2.8, that all right S-acts are principally weakly flat. Thus by [11, Theorem 4.6.6], S is a regular monoid.

(18) $\Rightarrow$ (1). By [11, Theorem 4.6.6], all right S-acts are principally weakly flat. Since every regular monoid is left PP, by Proposition 2.8, part (2), all right S-acts satisfy Condition (PWP_{sic}).

(6) $\Rightarrow$ (8). Let $x, y, s \in S$. By assumption $S/\rho(xs, ys)$ satisfies Condition (PWP_{sic}). Since $(xs)\rho(xs, ys)(ys)$, by Proposition 2.6, (8) is satisfied.$\square$

Note that, by Proposition 3.1, if S is not regular, then there exists at least one cofree right S-act which does not satisfy Condition (PWP_{sic}).

Recall, from [12, 2, 3] that a right S-act A_S satisfies Condition (E') if $as = as'$ and $sz = s'z$, for $a \in A_S$, $s, s', z \in S$, imply that there exist $a' \in A_S$ and $u \in S$ such that $a = a'u$ and $us = us'$. A right S-act A_S satisfies Condition (EP) if $as = at$, for $a \in A_S$, $s, t \in S$, implies that there exist $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. A right S-act A_S satisfies Condition $(E'P)$ if $as = at$ and $sz = tz$, for $a \in A_S$, $s, t, z \in S$, imply that there exist $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. It is obvious that $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$.

Proposition 3.2. *The following statements are equivalent:*

(1) *all right S-acts satisfy Condition (PWP_{sic});*
(2) *all right S-acts satisfying Condition (E), satisfy Condition (PWP_{sic});*
(3) *all right S-acts satisfying Condition (E'), satisfy Condition (PWP_{sic});*
(4) *all right S-acts satisfying Condition $(E'P)$, satisfy Condition (PWP_{sic});*
(5) *all right S-acts satisfying Condition (EP), satisfy Condition (PWP_{sic});*
(6) *S is regular.*

Proof. Since $(E) \Rightarrow (E') \Rightarrow (E'P)$ and $(E) \Rightarrow (EP) \Rightarrow (E'P)$, implications (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (6). Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Hence $sxs = s$, and so s is regular. Now suppose that $sS \neq S$. Set

$$A_S = S \overset{sS}{\amalg} S = \{(l, x) |\ l \in S \setminus sS\} \,\dot{\cup}\, sS \,\dot{\cup}\, \{(t, y) |\ t \in S \setminus sS\}.$$

Obviously

$$B_S = \{(l, x) |\ l \in S \setminus sS\} \,\dot{\cup}\, sS \cong S_S \cong \{(t, y) |\ t \in S \setminus sS\} \,\dot{\cup}\, sS = C_S.$$

Moreover B_S and C_S are subacts of A_S, where generated by $(1, x)$ and $(1, y)$, respectively. A_S is generated by $(1, x)$ and $(1, y)$, because $A_S = B_S \cup C_S$. By the above isomorphisms, B_S and C_S satisfy Condition (E), and so A_S satisfies Condition (E). By assumption, A_S satisfies Condition (PWP_{sic}). Hence the equality $(1, x)s = (1, y)s$ implies there exists $e \in E(S)$ such that $(1, x)e = (1, y)e$ and $es = s$. From the equality $(1, x)e = (1, y)e$ we get $e \in sS$. Thus there exists $x \in S$ such that $e = sx$. Therefore $s = es = sxs$, and so s is regular. Thus S is regular.

(6) $\Rightarrow$ (1). It is true, by Proposition 3.1. □

Note that the above proof implies that the previous theorem is true for finitely generated right S-acts and right S-acts generated by at most (exactly) two elements. Thus if S is not regular, then there exists at least one right S-act generated by at most (exactly) two elements satisfying Condition (E), but does not satisfy Condition (PWP_{sic}).

Recall, from [14, 6, 5, 4] that a right S-act A satisfies Condition (PF''), if for every $a, a' \in A$ and $s, s', t, t', z, w \in S$, $as = a's'$, $at = a't'$, and $sz = tw = t'w = s'z$ imply $a = a''u$, $a' = a''v$, for some $a'' \in A$, $u, v \in S$ with $us = vs'$ and $ut = vt'$. A right S-act A satisfies Condition (P'), if for every $a, a' \in A$ and $t, t', z \in S$, $at = a't'$ and $tz = t'z$ imply $a = a''u$, $a' = a''v$, for some $a'' \in A$, $u, v \in S$ with $ut = vt'$. A right S-act A satisfies Condition (P_E), if for every $a, a' \in A$ and $s, s' \in S$, $as = a's'$ implies $ae = a''ue$, $a'f = a''vf$, $es = s$, $fs' = s'$, for some $a'' \in A$, $u, v \in S$, $e, f \in E(S)$ with $us = vs'$. A right S-act A satisfies Condition (PWP_E), if for every $a, a' \in A$ and $s \in S$, $as = a's$ implies $ae = a''ue$, $a'f = a''vf$, $es = s = fs$, for some $a'' \in A$, $u, v \in S$, $e, f \in E(S)$ with $us = vs$.

Proposition 3.3. *Suppose that (U) be a property of S-acts which implies principally weak flatness and S_S satisfies the property (U). Then the following statements are equivalent:*

(1) *all right S-acts satisfying property (U), satisfy Condition (PWP_{sic});*
(2) *all finitely generated right S-acts satisfying property (U), satisfy Condition (PWP_{sic});*
(3) *all cyclic right S-acts satisfying property (U), satisfy Condition (PWP_{sic});*
(4) *S_S satisfies Condition (PWP_{sic}).*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Since S_S is a cyclic act satisfying property (U), by assumption S_S satisfies Condition (PWP_{sic}).

(4) $\Rightarrow$ (1). Suppose that A_S be a right S-act satisfying property (U). Let $as = a's$, for $a, a' \in A_S$ and $s \in S$. Since S_S satisfies Condition (PWP_{sic}), S is left PSF. Also, by assumption, A_S is principally weakly flat, and so, by [17, Lemma 1.3], there exists $u \in S$ such

that $au = a'u$ and $us = s$. Since S_S satisfies Condition (PWP_{sic}), we get $e \in E(S)$ such that $ue = e$ and $es = s$. Thus $ae = aue = a'ue = a'e$, and so A_S satisfies Condition (PWP_{sic}). □

Note that property (U) in the above proposition can be any property as free, projective generator, projective, strongly flat, equalizer flat, WPF, (PF''), WKF, $PWKF$, TKF, (WP), (P), (P'), (PWP), flat, weakly flat, (P_E), (PWP_E), principally weakly flat.

Proposition 3.4. *The following statements are equivalent:*

(1) *all right S-acts satisfy Condition (PWP_{sic});*
(2) *all generator right S-acts satisfy Condition (PWP_{sic});*
(3) *all finitely generated generator right S-acts satisfy Condition (PWP_{sic});*
(4) *all generator right S-acts generated by at most three elements satisfy Condition (PWP_{sic});*
(5) *$S \times A_S$ satisfies Condition (PWP_{sic}) for every right S-act A_S;*
(6) *$S \times A_S$ satisfies Condition (PWP_{sic}) for every finitely generated right S-act A_S;*
(7) *$S \times A_S$ satisfies Condition (PWP_{sic}) for every right S-act A_S generated by at most two elements;*
(8) *$S \times A_S$ satisfies Condition (PWP_{sic}) for every generator right S-act A_S;*
(9) *$S \times A_S$ satisfies Condition (PWP_{sic}) for every finitely generated generator right S-act A_S;*
(10) *$S \times A_S$ satisfies Condition (PWP_{sic}) for every generator right S-act A_S generated by at most three elements;*
(11) *a right S-act A_S satisfies Condition (PWP_{sic}) if $Hom(A_S, S_S) \neq \emptyset$;*
(12) *a finitely generated right S-act A_S satisfies Condition (PWP_{sic}) if $Hom(A_S, S_S) \neq \emptyset$;*
(13) *a right S-act A_S generated by at most three elements satisfies Condition (PWP_{sic}) if $Hom(A_S, S_S) \neq \emptyset$;*
(14) *S is regular.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4), (5) ⇒ (6) ⇒ (7), (1) ⇒ (8) ⇒ (9) ⇒ (10) and (1) ⇒ (11) ⇒ (12) ⇒ (13) are obvious.

(1) ⇔ (14). It follows from proposition 3.1.

(7) ⇒ (1). Suppose that A_S be a right S-act and $as = a's$ for $a, a' \in A_S$ and $s \in S$. Let $B_S = aS \cup a'S$. It is obvious that B_S is generated by at most two elements. By assumption, $S \times B_S$ satisfies Condition (PWP_{sic}). Now $(1, a)s = (1, a')s$ implies the existence of $e \in E(S)$ such that $(1, a)e = (1, a')e$ and $es = s$. Hence $ae = a'e$ and $es = s$, and so A_S satisfies Condition (PWP_{sic}).

(2) $\Rightarrow$ (5). Let A_S be a right S-act. Consider the map $\pi : S \times A_S \to S_S$ where $\pi(s, a) = s$, for $s \in S, a \in A_S$. Obviously π is an epimorphism. It follows from [11, Theorem 2.3.16] that $S \times A_S$ is a generator in **Act-S**. Thus, by assumption, $S \times A_S$ satisfies Condition (PWP_{sic}).

(13) $\Rightarrow$ (4). Suppose that A_S be a generator right S-act generated by at most three elements. Since A_S is a generator, there exists an epimorphism $\pi : A_S \to S_S$, by [11, Theorem 2.3.16]. Thus $Hom(A_S, S_S) \neq \emptyset$, and so by assumption, A_S satisfies Condition (PWP_{sic}).

(4) $\Rightarrow$ (2). Suppose that A_S be a generator right S-act and $as = a's$, for $a, a' \in A_S$ and $s \in S$. Since A_S is a generator, there exists an epimorphism $\pi : A_S \to S_S$, by [11, Theorem 2.3.16]. Let $\pi(a'') = 1$. Set $B_S = aS \cup a'S \cup a''S$. It is obvious that $\pi|_{B_S}$ is an epimorphism from B_S onto S_S, and so B_S is a generator. Since B_S is generated by at most three elements, by assumption, B_S satisfies Condition (PWP_{sic}), and so $as = a's$, for $a, a' \in B_S$, implies the existence of $e \in E(S)$ such that $ae = a'e$ and $es = s$. Thus A_S satisfies Condition (PWP_{sic}).

(10) $\Rightarrow$ (2). Applying the proof of (4) $\Rightarrow$ (2), $S \times B_S$ satisfies Condition (PWP_{sic}). Since $as = a's$ in A_S, $(1, a)s = (1, a')s$, in $S \times B_S$. Thus there exists $e \in E(S)$, such that $es = s$ and $(1, a)e = (1, a')e$, and so $ae = a'e$. Hence A_S satisfies Condition (PWP_{sic}). □

Proposition 3.5. *The following statements are equivalent:*

(1) *all torsion free right S-acts satisfy Condition (PWP_{sic});*
(2) *all torsion free finitely generated right S-acts satisfy Condition (PWP_{sic});*
(3) *all torsion free cyclic right S-acts satisfy Condition (PWP_{sic});*
(4) *all torsion free right Rees factor acts of S satisfy Condition (PWP_{sic});*
(5) *S is left almost regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). By Proposition 2.8, part (1), all torsion free right Rees factor acts of S are principally weakly flat. It follows from [11, Theorem 4.6.5] that S is left almost regular.

(5) $\Rightarrow$ (1). By [11, Theorem 4.6.5], all torsion free right S-acts are principally weakly flat. On the other hand by dual of [11, Proposition 4.1.3] every left almost regular monoid is a left PP monoid. Now by Proposition 2.8, part (2), all torsion free right S-acts satisfy Condition (PWP_{sic}). □

Recall from [18] that the right S-act A_S is called $\mathfrak{R}$-*torsion free* if for any $a, b \in A_S$ and for any right cancellable element $c \in S$, $ac = bc$ and $a\mathfrak{R}b$ imply that $a = b$.

Proposition 3.6. *The following statements are equivalent:*

(1) *all $\mathfrak{R}$-torsion free right S-acts satisfy Condition (PWP_{sic});*
(2) *all $\mathfrak{R}$-torsion free finitely generated right S-acts satisfy Condition (PWP_{sic});*

(3) *all $\mathfrak{R}$-torsion free right S-acts generated by at most two elements satisfy Condition* (PWP_{sic})*;*

(4) *all $\mathfrak{R}$-torsion free right S-acts generated by exactly two elements satisfy Condition* (PWP_{sic})*;*

(5) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$, and so $sxs = s$. Thus s is a regular element. Now assume that $sS \neq S$. Set

$$A_S = S \overset{sS}{\amalg} S = \{(l, x) |\ l \in S \setminus sS\} \mathbin{\dot\cup} sS \mathbin{\dot\cup} \{(t, y) |\ t \in S \setminus sS\}.$$

Using the same proof to that of Proposition 3.2 (2 $\Rightarrow$ 6), A_S satisfies Condition (E) and is generated with two elements $(1, x)$ and $(1, y)$. By [18, Proposition 1.2], A_S is $\mathfrak{R}$-torsion free, and so by assumption, A_S satisfies Condition (PWP_{sic}). Again the same proof to that of Proposition 3.2 (2 $\Rightarrow$ 6), implies that s is a regular element. Thus S is regular.

(5) $\Rightarrow$ (1). Note that every $\mathfrak{R}$-torsion free right S-act is principally weakly flat, by [18, Proposition 4.5]. Moreover every regular monoid is left PP. Hence by Proposition 2.8, part (2), principal weak flatness equivalent to Condition (PWP_{sic}). Thus all $\mathfrak{R}$-torsion free right S-acts satisfy Condition (PWP_{sic}). □

Definition 3.7. The principal left ideal Ss, $s \in S$, is called finitely definable if there exist $(u_1, v_1), ..., (u_n, v_n) \in S \times S$ such that:

(i) $u_i s = v_i s$ $(1 \leq i \leq n)$,

(ii) if $xs = ys$, $x, y \in S$, then there exist $s_1, s_2, ..., s_n \in S$ such that

$$\begin{aligned} x &= s_1 u_1, \\ s_1 v_1 &= s_2 u_2, \\ &\vdots \\ s_n v_n &= y. \end{aligned}$$

If S is a monoid, the cartesian product S^I is a right and left S-act with the action componentwise, where I is a nonempty set. For more information the reader is referred to [17].

Proposition 3.8. [17, Proposition 2.2] *The following statements are equivalent:*

(1) *S^I_S is principally weakly flat, for each nonempty set I;*

(2) *for any $a \in S$, the mapping $f_a : S^I_S \otimes Sa \to (Sa)^I$ given by $f_a((s_i)_I \otimes sa) = (s_i sa)_I$ is injective, for each nonempty set I;*

(3) *all principal left ideals of S are finitely definable;*

(4) $S_S^{S\times S}$ *is principally weakly flat.*

We recall from [10], if A_S is an S-act and its second dual exists, then the mapping $\varphi_A : A_S \to Hom(Hom(A_S, S_S), {}_SS) = (A_S)^{**}$ defined by $\varphi_A(a)(f) = f(a)$, for every $a \in A$ and $f \in Hom(A_S, S_S)$, is Homomorphism. An act A_S is called *torsionless* if φ_A is injective. Also an act A_S, $|A_S| > 1$, is torsionless if and only if for every $x, y \in A_S$, $x \neq y$, there exists $f \in hom(A_S, S_S)$ such that $f(x) \neq f(y)$.

Proposition 3.9. *The following statements are equivalent:*

(1) *all torsionless right S-acts satisfy Condition (PWP_{sic});*
(2) *all torsionless right S-acts are principally weakly flat and S_S satisfies Condition (PWP_{sic});*
(3) *all torsionless right S-acts are principally weakly flat and S is left PSF;*
(4) *S_S^I satisfies Condition (PWP_{sic}), for any nonempty set I;*
(5) *$S_S^{S\times S}$ satisfies Condition (PWP_{sic});*
(6) *S_S satisfies Condition (PWP_{sic}) and S_S^I is principally weakly flat, for any nonempty set I;*
(7) *S is left PP.*

Proof. (1) ⇒ (2). By assumption and Proposition 2.8, all torsionless right S-acts are principally weakly flat. Moreover, by [10, Lemma 2.3], S_S is a torsionless act and so, by assumption, S_S satisfies Condition (PWP_{sic}).

(2) ⇒ (3). It is obvious by definition.

(3) ⇒ (7). It follows, from [10, Lemma 2.5], that for each nonempty set I, S^I is torsionless. Then it is principally weakly flat, by assumption. Now, by [17, Proposition 1.5], S is left PP.

(1) ⇒ (4). It follows, from [10, Lemma 2.5], that for each nonempty set I, S^I is torsionless, and so by assumption, S_S^I satisfies Condition (PWP_{sic}).

(4) ⇒ (5). It is obvious.

(5) ⇒ (6). By Proposition 2.8, $S_S^{S\times S}$ is principally weakly flat. It follows, from Proposition 3.8, that S_S^I is principally weakly flat, for every nonempty set I. Now we prove S_S satisfies Condition (PWP_{sic}). Let $xs = ys$, for $x, y, s \in S$. Set $|S \times S| = I$. Suppose that for $i_0 \in I$, $x_{i_0} = x, y_{i_0} = y$ and for $i \in I \setminus \{i_0\}$, $x_i = y_i = 1$. Then we have $(x_i)_I s = (y_i)_I s$. Since $S_S^{S\times S}$ satisfies Condition (PWP_{sic}), there exists $e \in E(S)$ such that $(x_i)_I e = (y_i)_I e$ and $es = s$. Now $x_i e = y_i e$, for every $i \in I$, and so $xe = x_{i_0} e = y_{i_0} e = ye$. Therefore S_S satisfies Condition (PWP_{sic}).

(6) ⇒ (7). Since S_S satisfies Condition (PWP_{sic}), S is left PSF, and so, by [17, Proposition 1.5], S is left PP.

(7) $\Rightarrow$ (1). Assume that A_S be a torsionless right S-act. By [10, Proposition 2.6], there exists the nonempty set I such that A_S can be embedded into S_S^I. Thus there exists subact B_S of S_S^I such that $A_S \cong B_S$. Since S is a left PP monoid, by [17, Proposition 1.5], S_S^I is principally weakly flat. It follows, from Proposition 2.8, that S_S^I satisfies Condition (PWP_{sic}). On the other hand every subact of an act satisfying Condition (PWP_{sic}), satisfies Condition (PWP_{sic}). Thus B_S, and so A_S satisfy Condition (PWP_{sic}). □

Proposition 3.10. *The following statements are equivalent:*

(1) *all faithful right S-acts satisfy Condition (PWP_{sic});*
(2) *all faithful finitely generated right S-acts satisfy Condition (PWP_{sic});*
(3) *all faithful right S-acts generated by at most two elements satisfy Condition (PWP_{sic});*
(4) *all faithful right S-acts generated by exactly two elements satisfy Condition (PWP_{sic});*
(5) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Let $s \in S$. If $sS = S$ then there exists $x \in S$ such that $sx = 1$, and so $sxs = s$. Thus s is regular. Now assume that $sS \neq S$. Set $A_S = S \overset{sS}{\amalg} S$. Since A_S is faithful, by assumption, A_S satisfies Condition (PWP_{sic}). By the proof of Proposition 3.2(2 $\Rightarrow$ 6), s is regular, and so S is regular.

(5) $\Rightarrow$ (1). By Proposition 3.1, all right S-acts satisfy Condition (PWP_{sic}). Thus all faithful right S-acts satisfy Condition (PWP_{sic}). □

Lemma 3.11. [9, Lemma 3.7] *The following statements are equivalent:*

(1) *there exists at least one strongly faithful right S-act;*
(2) *there exists at least one cyclic strongly faithful right S-act;*
(3) *there exists at least one monocyclic strongly faithful right S-act;*
(4) *there exists at least one finitely generated strongly faithful right S-act;*
(5) *for all $s \in S$, sS is a strongly faithful right S-act;*
(6) *there exists $s \in S$, such that sS is a strongly faithful right S-act;*
(7) *S_S is a strongly faithful right S-act;*
(8) *for all $s \in S$, the elements of sS are left cancellable;*
(9) *there exists $s \in S$ such that all elements of sS are left cancellable;*
(10) *S is left cancellable.*

Proposition 3.12. *The following statements are equivalent:*

(1) *all strongly faithful right S-acts satisfy Condition (PWP_{sic});*
(2) *all strongly faithful finitely generated right S-acts satisfy Condition (PWP_{sic});*

(3) *all strongly faithful right S-acts generated by at most two elements satisfy Condition* (PWP_{sic});

(4) *all strongly faithful right S-acts generated by exactly two elements satisfy Condition* (PWP_{sic});

(5) *S is not left cancellable or S is a group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Suppose that S be a left cancellable monoid. We claim that S is a group. It is sufficient to show that S is regular. Let $s \in S$. If $sS = S$ then s is regular. Otherwise set

$$A_S = S \overset{sS}{\amalg} S = \{(l,x)|\ l \in S \setminus sS\} \dot{\cup}\ sS \ \dot{\cup}\ \{(t,y)|\ t \in S \setminus sS\}.$$

Now we have

$$B_S = \{(l,x)|\ l \in S \setminus sS\} \dot{\cup}\ sS \cong S_S \cong \{(t,y)|\ t \in S \setminus sS\} \dot{\cup}\ \ sS = C_S.$$

Obviously $A_S =< (1,x),(1,y) >= B_S \cup C_S$. Since S is left cancellable, by Lemma 3.11, S_S is strongly faithful. By using the above isomorphisms, B_S and C_S as subacts of A_S are strongly faithful. Hence A_S is strongly faithful. Since A_S is generated by exactly two elements, by assumption A_S satisfies Condition (PWP_{sic}). It follows, from the proof of Proposition 3.2(2 $\Rightarrow$ 6), that s is regular. Thus S is regular. For every $s \in S$, from regularity S, there exists $x \in S$ such that $sxs = s$, and so $xs = 1$, since S is left cancellable. This means that every element of S is left invertible. Thus S is a group.

(5) $\Rightarrow$ (1). If S is not left cancellable, then, by Lemma 3.11, there exist no strongly faithful right S-act. Otherwise if S is a left cancellable monoid then, by assumption, S is a group. Thus S is regular, and so by Proposition 3.1, the statement (1) is obtained. □

Lemma 3.13. [9, Lemma 3.9] *Suppose that ρ be a right congruence on a monoid S. Then S/ρ is a strongly faithful right S-act if and only if $\rho = \Delta_S$ and S is left cancellable.*

Proposition 3.14. *The following statements are equivalent:*

(1) *all strongly faithful cyclic right S-acts satisfy Condition* (PWP_{sic});

(2) *S is not left cancellable or S_S satisfies Condition* (PWP_{sic});

Proof. (1) $\Rightarrow$ (2). Suppose S be a left cancellable monoid. Then, by Lemma 3.11, S_S as a cyclic right S-act is strongly faithful, and so by assumption, S_S satisfies Condition (PWP_{sic}).

(2) $\Rightarrow$ (1). If S is not left cancellable, by Lemma 3.11, there exist no a strongly faithful right S-act. Thus (1) is obtained. If S is left cancellable, by Lemma 3.11, there exists at least one strongly faithful cyclic right S-act. Thus we can assume S/ρ is a strongly faithful cyclic right

S-act. By Lemma 3.13, $\rho = \Delta_S$, and so $S/\rho \cong S_S$, which by assumption, satisfies Condition (PWP_{sic}). □

Proposition 3.15. *The following statements are equivalent:*

(1) *there exists at least one strongly faithful cyclic right S-act which satisfies Condition (PWP_{sic});*

(2) *S is left cancellable and every strongly faithful cyclic right S-act satisfies Condition (PWP_{sic});*

(3) *S is left cancellable and S_S satisfies Condition (PWP_{sic}).*

Proof. (1) ⇒ (2). It follows, from assumption and Lemma 3.11, that S is left cancellable. Suppose that S/ρ be a strongly faithful cyclic right S-act. By Lemma 3.13, $\rho = \Delta_S$, and so $S/\rho \cong S_S$ satisfies Condition (PWP_{sic}).

(2) ⇒ (3). This is true, by Lemma 3.14.

(3) ⇒ (1). Since S is a left cancellable monoid, by Lemma 3.11, there exists at least one strongly faithful cyclic right S-act as S/ρ. Then $\rho = \Delta_S$, by Lemma 3.13, and so $S/\rho \cong S_S$, which by assumption, satisfies Condition (PWP_{sic}). □

Proposition 3.16. *The following statements are equivalent:*

(1) *all right S-acts satisfying Condition (PWP_{sic}) are (strongly) faithful;*

(2) *all finitely generated right S-acts satisfying Condition (PWP_{sic}) are (strongly) faithful;*

(3) *all cyclic right S-acts satisfying Condition (PWP_{sic}) are (strongly) faithful;*

(4) *all right Rees factor acts of S satisfying Condition (PWP_{sic}) are (strongly) faithful;*

(5) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). It follows, by Proposition 2.5, part (1), that $S/S_S \cong \Theta_S$ satisfies Condition (PWP_{sic}). Thus, by assumption, Θ_S is (strongly) faithful. Let $s, t \in S$. Then $\varpi s = \varpi t$ implies $s = t$, and so $S = \{1\}$.

(5) ⇒ (1). If $S = \{1\}$ then all right S-acts are strongly faithful, and so (1) is obtained. □

Proposition 3.17. *The following statements are equivalent:*

(1) *all right S-acts satisfying Condition (PWP_{sic}) are (projective) generator;*

(2) *all finitely generated right S-acts satisfying Condition (PWP_{sic}) are (projective) generator;*

(3) *all cyclic right S-acts satisfying Condition (PWP_{sic}) are (projective) generator;*

(4) *all right Rees factor acts of S satisfying Condition (PWP_{sic}) are (projective) generator;*

(5) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). It follows, by Proposition 2.5, part (1), that $S/S_S \cong \Theta_S$ satisfies Condition (PWP_{sic}). By assumption, Θ_S is (projective) generator. By [11, Theorem 2.3.16], there exists an epimorphism $\pi : \Theta_S \to S_S$. Thus $S = \{1\}$.

(5) ⇒ (1). If $S = \{1\}$ then all right S-acts are (projective) generator, and so (1) is obtained.

□

Proposition 3.18. *The following statements are equivalent:*

(1) *all right S-acts satisfying Condition (PWP_{sic}) are free;*

(2) *all finitely generated right S-acts satisfying Condition (PWP_{sic}) are free;*

(3) *all cyclic right S-acts satisfying Condition (PWP_{sic}) are free;*

(4) *all right Rees factor acts of S satisfying Condition (PWP_{sic}) are free;*

(5) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). By assumption, all right Rees factor acts of S satisfying Condition (PWP_{sic}) are generator. It follows, from Proposition 3.17, that $S = \{1\}$.

(5) ⇒ (1). If $S = \{1\}$ then all right S-acts are free, and so (1) is obtained. □

4. Classification by condition (PWP_{sic}) of right Rees factor acts

In this section we give a classification of monoids by Condition (PWP_{sic}) of right Rees factor acts.

Proposition 4.1. *Let K_S be a right ideal of S. The right Rees factor S-act S/K_S satisfies Condition (PWP_{sic}) if and only if K_S fulfills in the following conditions*

(I) *for all $x, y, s \in S$,*

$$[(xs = ys \in S \setminus K_S) \Rightarrow (\exists e \in E(S))(es = s \wedge xe = ye)].$$

(II) *for all $x, y, s \in S$,*

$$[(xs, ys \in K_S) \Rightarrow (\exists e \in E(S))(es = s \wedge (xe = ye \vee xe, ye \in K_S))].$$

Proof. **Necessity**. Let $xs = ys \in S \setminus K_S$, for $x, y, s \in S$. Then $(xs)\rho_K(ys)$, which implies the existence of $e \in E(S)$ such that $es = s$ and $(xe)\rho_K(ye)$, by Proposition 2.6. Thus $xe = ye$ or $xe, ye \in K_S$. If $xe, ye \in K_S$ then $ys = xs = xes \in K_S$ which is a contradiction. Hence $xe = ye$

and $es = s$, and so (I) is obtained. Now let $xs, ys \in K_S$, for $x, y, s \in S$. Thus $(xs)\rho_K(ys)$, and so Condition (PWP_{sic}) implies the existence $e \in E(S)$ such that $es = s$ and $(xe)\rho_K(ye)$. Hence $xe = ye$ or $xe, ye \in K_S$, as required.

Sufficiency. Note that if $K_S = S$ then $S/K_S \cong \Theta_S$ which satisfies Condition (PWP_{sic}), by Proposition 2.5. Assume that K_S is a proper right ideal of S and $(xs)\rho_K(ys)$, for $x, y, s \in S$. Then $xs, ys \in K_S$ or $xs = ys$. If $xs, ys \in K_S$, then by condition (II), there exists $e \in E(S)$ such that $es = s$ and $(xe)\rho_K(ye)$, as required. If $xs = ys$ then there are two cases as follows:

Case 1: $xs = ys \in K_S$. This is similar to the situation where $xs, ys \in K_S$.

Case 2: $xs = ys \in S \setminus K_S$. By condition (I), there exists $e \in E(S)$ such that $es = s$ and $xe = ye$. Hence $es = s$ and $(xe)\rho_K(ye)$. Therefore, by Proposition 2.6, S/K_S satisfies Condition (PWP_{sic}), as required. □

Remark. Note that it is easy to show that if the right ideal K_S satisfies condition (II) of Proposition 4.1, then K_S is left stabilizing.

Proposition 4.2. *Let K_S be a right ideal of S. All right Rees factor acts of S satisfying Condition (PWP_{sic}) are (weakly) flat if and only if S is right reversible.*

Proof. **Necessity**. Since $S/S_S \cong \Theta_S$ satisfies Condition (PWP_{sic}), by assumption Θ_S is (weakly) flat, and so by [11, Exercise 3.12.2], S is right reversible.

Sufficiency. Suppose that S be a right reversible monoid. Let K_S be a right ideal of S such that S/K_S satisfies Condition (PWP_{sic}). If $K_S = S$, it follows from [11, Theorem 3.12.17], $S/K_S = S/S_S \cong \Theta_S$ is (weakly) flat. Otherwise, by Proposition 4.1, K_S satisfies conditions (I) and (II). It follows from above remark that K_S is left stabilizing. Now, by [11, Theorem 3.12.17], S/K_S is (weakly) flat. □

Proposition 4.3. *The following statements are equivalent:*

(1) *all right Rees factor acts of S satisfying Condition (P) satisfy Condition (PWP_{sic});*
(2) *all WPF right Rees factor acts of S satisfy Condition (PWP_{sic});*
(3) *all strongly flat right Rees factor acts of S satisfy Condition (PWP_{sic});*
(4) *all projective right Rees factor acts of S satisfy Condition (PWP_{sic});*
(5) *all projective generator right Rees factor acts of S satisfy Condition (PWP_{sic});*
(6) *all free right Rees factor acts of S satisfy Condition (PWP_{sic});*
(7) *S_S satisfies Condition (PWP_{sic}) or S does not contain a left zero.*

Proof. Since free $\Rightarrow$ projective generator $\Rightarrow$ projective $\Rightarrow$ SF $\Rightarrow$ WPF $\Rightarrow$ (P), implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$ are obtained immediately.

(6) $\Rightarrow$ (7). Suppose that S contains a left zero as z. Then $K_S = zS = \{z\}$ is a right ideal of S. Since $|K_S| = 1$, by [11, Proposition 1.5.22], $S/K_S \cong S_S$ is free, and so by assumption, $S/K_S \cong S_S$ satisfies Condition (PWP_{sic}).

(7) $\Rightarrow$ (1). Let K_S be a right ideal of S such that S/K_S satisfies Condition (P). If $K_S = S$ then $S/K_S \cong \Theta_S$ satisfies Condition (PWP_{sic}), by Proposition 2.5. If $K_S \neq S$ then, by [11, Proposition 3.13.9], $|K_S| = 1$ and so, $K_S = zS = \{z\}$. Thus z is a left zero of S, and so by assumption, $S_S \cong S/K_S$ satisfies Condition (PWP_{sic}), as desired. □

We recall from [15, 16] that A_S is called *GP-flat*, if for every $s \in S$, and $a, a' \in A_S$, $a \otimes s = a' \otimes s$ in $A_S \otimes {}_S S$ implies the existence of a natural number n such that $a \otimes s^n = a \otimes s^n$ in $A_S \otimes_S Ss^n$. The S-act A_S is called *GPW-flat*, if for every $s \in S$, there exists $n = n_{(s,A_S)} \in \mathbb{N}$, such that the functor $A_S \otimes {}_S-$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_S S$. It is obvious that every principally weakly flat act is GPW-flat and every GPW-flat is GP-flat. Also every GP-flat act is torsion free, but the converse of these relations is not true.

Now we pose the following open questions for consideration:

1. What is the specification of S, for which every right S-act satisfying Condition (PWP_{sic}) has a certain flatness properties between free and principally weakly flat?
2. When all (GPW-flat) GP-flat right S-acts satisfy Condition (PWP_{sic})?

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