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Research Paper

ON SEMI-NI RINGS

FAHIME GHADERNAZARI AND REZA JAHANI-NEZHAD*

Abstract. In this Paper, we first introduce the concept of semi-NI rings which is a generalization of NI rings, and then we examine the characteristics of this class of rings. We investigate relationships between semi-NI rings and some other known classes of rings. We show that the class of semi-NI rings lies strictly between the class of NI rings and the class of directly finite rings. Also, we prove that this class of rings lies strictly between the class of NI rings and the class of NCI rings. In the following, we show that semi-NI rings are characterized by many equivalent conditions.

1. Introduction

Throughout this paper every ring is an associative with identity unless otherwise stated. For a ring R, we use $N_*(R)$, $N^*(R)$ and $N(R)$ to denote the lower nilradical or prime radical (i.e., the intersection of all prime ideals), the upper nilradical (i.e., the sum of all nil ideals) and the set of all nilpotent elements in R, respectively. Also, $J(R)$ is the Jacobson radical of

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∗Corresponding author

R. It is clear that $N^*(R)$ is the unique maximal nil ideal of R. Furthermore, it follows that the inclusions $N_*(R) \subseteq N^*(R) \subseteq N(R) \cap J(R)$ hold. Denote the $n \times n$ full (resp., upper triangular) matrix ring over R by $M_n(R)$ (resp., $T_n(R)$). We also use $R[x]$ for the polynomial ring over R, and we select notation $E(R)$ and $C(R)$ for the set of all idempotent elements and the set of central elements of R, respectively. Let $C_{f(x)}$ denote the set of all coefficients of given a polynomial $f(x) \in R[x]$. A ring is called *reduced* if it has no nonzero nilpotent elements. A weaker condition than "reduced" is defined by Cohn in [4]. A ring R is *reversible* if for any $a, b \in R, ab = 0$ implies $ba = 0$. Recall that a ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. One can prove through simple computation that reduced rings are reversible and reversible rings are semicommutative. A ring R is called *abelian* if $E(R) \subseteq C(R)$. Every semicommutative ring is abelian.

Duo to Marks [14], a ring R is called *NI* if $N^*(R) = N(R)$. It is obvious that a ring R is NI if and only if $N(R)$ forms an ideal of R if and only if $R/N^*(R)$ is reduced if and only if for any $a \in R,\ a^2 \in N^*(R)$ implies $a \in N^*(R)$. Semicommutative rings are NI, but the converse need not hold as can be seen by $T_2(R)$ over an NI ring R, nothing that $T_2(R)$ is NI but non-abelian. Hence $T_2(R)$ is not semicommutative.

In this paper we have defined semi-NI rings as a generalization of NI rings. We have also investigated relationships with other known rings.

2. Semi-NI rings and elementary properties

In this section we introduce a class of rings, so-called semi-NI rings, which is a generalization of NI rings. We supply an example (see Example 2.2) to show that all semi-NI rings need not be NI. Thus, the class of semi-NI rings stand as a nontrivial generalization of NI rings. Then we prove that every semi-NI is directly finite and we give an example to illustrate there are directly finite rings which are not semi-NI. Therefore, the class of semi-NI rings lies strictly between classes of NI rings and directly finite rings.

We now give our main definition.

Definition 2.1. A ring R is called *semi-NI* if for every $a \in R$, $a^2 = 0$ implies $a \in N^*(R)$.

Clearly, every NI ring is semi-NI, but the converse need not hold by the Example 2.2. Furthermore, It is obvious that reduced rings are semi-NI. For a given ring R and $k \geq 1$, we use $N_k(R) = \{a \in R \mid a^k = 0\}$.

Example 2.2. There exists a semi-NI ring which is not NI. We use the ring and argument in [5, Example 1]. Let F be a field and $A = F\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x and y over F and I denote the ideal $(x^2)^2$ of A, where (x^2) is the ideal of $A = F\langle x, y\rangle$ generated by x^2. Consider the ring $R = A/I$. We identify x and y with their images in R. Note that $N_*(R) = Rx^2R = N_2(R)$ and $N(R) = xRx + Rx^2R + Fx$. Moreover, $N_*(R) = N^*(R)$ by the computation in [9, Example 1.2(2)]. This yields $N^*(R) \subsetneq N(R)$, entailing R is not NI. Now suppose that $a^2 = 0$ for $a \in R$. Then $a \in N_2(R) = N_*(R) = N^*(R)$. Thus, R is semi-NI.

Also, the above example shows that there are semi-NI rings which are not semicommutative. Note that we have for a ring R,

$$N^*(R) = \{a \in R \mid RaR \text{ is a nil ideal of } R\}.$$

Lemma 2.3. *Let R be a ring and S be a subring of R. Then $S \cap N^*(R) \subseteq N^*(S)$.*

Proof. Let $a \in N^*(R) \cap S$. We have $SaS \subseteq RaR$. Since RaR is a nil ideal, SaS is a nil ideal of S, and so $a \in N^*(S)$. Therefore, $N^*(R) \cap S \subseteq N^*(S)$. □

We can get the following result from Lemma 2.3.

Corollary 2.4. *The class of semi-NI rings is closed under subrings.*

A ring R is said to be of *bounded index of nilpotency* if there exists a positive integer n such that $a^n = 0$ for all nilpotent elements $a \in R$. We say that R is a ring of bounded index n if n is the smallest positive integer such that $x^n = 0$ for all $x \in N(R)$. Clearly, every semi-NI ring of bounded index 2 is an NI ring.

Proposition 2.5. *Let R be a ring which each left ideal (resp. right ideal) not contained in $N^*(R)$ contains a nonzero idempotent. If R is an abelian ring, then R is semi-NI.*

Proof. Let $a^2 = 0$ for some $a \in R$. If $aR \not\subseteq N^*(R)$, then there is a nonzero idempotent $e \in aR$. Write $e = at$ for some $t \in R$. Hence,

$$e = e^2 = eat = aet = a(at)t = a^2t^2 = 0,$$

because e is central by assumption. This is a contradiction, and so $a \in N^*(R)$. Therefore, R is semi-NI. □

Proposition 2.6. *Let R be a ring in which $bac \in N^*(R)$, for any $a, b, c \in R$ such that $abc = 0$. Then R is semi-NI.*

Proof. Suppose $a^2 = 0$ for some $a \in R$. Then $raa = 0$ for all $r \in R$. By assumption, we obtain $ara \in N^*(R)$ for all $r \in R$. Since $N^*(R)$ is a semiprime ideal of R, we have $a \in N^*(R)$. □

For any subset S of a ring R, we use

$$r(S)=\{r\in R\mid Sr=0\} \text{ and } l(S)=\{r\in R\mid rS=0\},$$

to denote the right annihilator of S and left annihilator of S, respectively. If $S=\{a\}$, then we write $r(a)$ $(l(a))$ instead of $r(\{a\})$ $(l(\{a\}))$.

Let S and T be rings and M be an (S,T)-bimodule. If R is the formal triangular matrix ring $\begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$, then it is well-known that

$$N^*(R)=\begin{pmatrix} N^*(S) & M \\ 0 & N^*(T) \end{pmatrix}.$$

Proposition 2.7. *Let* $R=\begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$. *If S and T are semi-NI, then R is semi-NI.*

Proof. Assume that $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^2=0$, for some $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\in R$. Thus $a^2=0$ and $d^2=0$. Since S and T are semi-NI, $a\in N^*(S)$ and $d\in N^*(T)$. Therefore, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\in N^*(R)$, and so R is semi-NI. □

For a given ring R we usually write

$$D_n(R)=\{(a_{ij})\in T_n(R)\mid a_{11}=a_{22}=\cdots=a_{nn}\}.$$

If we set in the above proposition $S=T_{n-1}(R)$ and $T=R$, then by using induction we have the following result.

Corollary 2.8. *If R is a semi-NI ring, then $T_n(R)$, and so $D_n(R)$ are semi-NI, for any positive integer n.*

By Mohammadi et al. [15], a ring R is called a *nil-semicommutative* ring if whenever $ab=0$ for some $a,b\in N(R)$, then $aRb=0$. Clearly, semicommutative rings are nil-semicommutative. Also, nil-semicommutative rings are NI by [15, Theorem 2.5], and so these are semi-NI rings. But by Example 2.2, it is clear that semi-NI rings need not be nil-semicommutative. The Example 2.9 shows that a semi-NI ring which is not nil-semicommutative.

Example 2.9. Let R be a reduced ring. Thus R is a semi-NI ring, and so $T_4(R)$ is a semi-NI ring by Corollary 2.8. However, we see that $e_{12},e_{34}\in N(T_4(R))$, and $e_{12}e_{34}=0$, but $e_{12}e_{23}e_{34}\neq 0$. Therefore, $T_4(R)$ is not nil-semicommutative, and so $T_4(R)$ is not semicommutative.

There exists a semi-NI ring which is not abelian as shown in Example 2.10.

Example 2.10. Let F be a field. The ring $R = T_2(F)$ is a semi-NI ring by Corollary 2.8. But $ea \neq ae$, for idempotent $e = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} \in R$ and $a = e_{12} \in R$. Thus, R is not an abelian ring.

Duo to Jacobson [10, p. 218], a ring R is said to be *K-ring* if for every element $a \in R$ there exists a positive integer n such that $a^n \in C(R)$. Clearly, every K-ring is abelian. By [7, Proposition 1.3], K-rings are NI, and so these are semi-NI. But the converse of this statement does not hold in general by Example 2.10.

Theorem 2.11. *For a ring R the following conditions are equivalent*

(1) *R is semi-NI,*
(2) *$a^2 = 0$ for $a \in R$ implies $aR \subseteq N^*(R)$,*
(3) *$a^2 = 0$ for $a \in R$ implies $Ra \subseteq N^*(R)$,*
(4) *$ab = 0$ for $a, b \in R$ implies $ba \in N^*(R)$,*
(5) *$ab = 0$ for $a, b \in R$ implies $bRa \subseteq N^*(R)$,*
(6) *$ab = 0$ for $a, b \in R$ implies $aRb \subseteq N^*(R)$,*
(7) *$a^2 = 0$ for $a \in R$ implies $ab - ba \in N^*(R)$ for all $b \in R$,*
(8) *$al(a) \subseteq N^*(R)$ for all $a \in R$,*
(9) *$r(a)a \subseteq N^*(R)$ for all $a \in R$.*

Proof. Since $N^*(R)$ is an ideal of R, it follows that $(1) \Leftrightarrow (2) \Leftrightarrow (3)$.
1 ⇒ 4. Let $ab = 0$ for $a, b \in R$. Then $(ba)^2 = 0$, and so $ba \in N^*(R)$.
4 ⇒ 5. Let $ab = 0$ for $a, b \in R$. Then $a(br) = (ab)r = 0$ for every $r \in R$, and so $(br)a \in N^*(R)$ for all $r \in R$ by assumption. It implies that $bRa \subseteq N^*(R)$.
5 ⇒ 6. Let $ab = 0$ for $a, b \in R$. Then $bRa \subseteq N^*(R)$ by assumption. Hence,

$$(RarbR)R(RarbR) = Rar(bRa)rbR \subseteq N^*(R),$$

for every $r \in R$. Since $N^*(R)$ is a semiprime ideal of R, we have $RarbR \subseteq N^*(R)$ for all $r \in R$, and so $aRb \subseteq N^*(R)$.
6 ⇒ 1. Let $a^2 = 0$ for $a \in R$. Thus $aRa \subseteq N^*(R)$ by assumption. Since $N^*(R)$ is a semiprime ideal of R, $a \in N^*(R)$.
1 ⇒ 7. Let $a^2 = 0$ for $a \in R$. Then $a \in N^*(R)$ because R is semi-NI, and so $ab - ba \in N^*(R)$ for all $b \in R$.
7 ⇒ 1. Let $a^2 = 0$ for $a \in R$. Then $aba = a(ab - ba) \in N^*(R)$ for all $b \in R$ by assumption. Since $N^*(R)$ is semiprime, we have $a \in N^*(R)$.
5 ⇒ 8. Let $a \in R$. Then $ba = 0$ for any $b \in l(a)$, and so $aRb \subseteq N^*(R)$ for any $b \in l(a)$ by assumption. Therefore, $al(a) = aRl(a) \subseteq N^*(R)$.

8 ⇒ 9. Let $a \in R$. For every $b \in r(a)$ we have $a \in l(b)$, and so $ba \in bRa \subseteq bl(b) \subseteq N^*(R)$. This implies that $r(a)a \subseteq N^*(R)$.
9 ⇒ 1. Assume that $a^2 = 0$ for some $a \in R$. Then $a \in r(a)$, and so

$$aRa \subseteq r(a)Ra = r(a)a \subseteq N^*(R),$$

by assumption. Since $N^*(R)$ is a semiprime ideal of R, we have $a \in N^*(R)$, which shows that R is a semi-NI ring. □

Let I be an ideal of a ring R and $\bar{R} = R/I$. Suppose that $\bar{x}$ is an idempotent element of $\bar{R}$, we say that $\bar{x}$ can be *lifted modulo* I if there is an idempotent element $e \in R$ such that $\bar{e} = \bar{x}$. By [13, Theorem 21.28], if I is a nil ideal (so $I \subseteq N^*(R)$), then every idempotent element of $\bar{R}$ can be lifted modulo I. In this direction we have the following result.

Proposition 2.12. *Let R be a semi-NI ring. Then $\bar{R} = R/N^*(R)$ is an abelian ring.*

Proof. Let $\bar{x}$ be an idempotent element of $\bar{R}$. By [13, Theorem 21.28], $\bar{x}$ can be lifted modulo $N^*(R)$, hence there is an idempotent element $e \in R$ such that $\bar{e} = \bar{x}$. Thus, $(er(1-e))^2 = 0$ and $((e-1)re)^2 = 0$ for any $r \in R$. Since R is semi-NI, we have $er(1-e)$, $(e-1)re \in N^*(R)$ for all $r \in R$, and so $er - re = er(1-e) + (e-1)re \in N^*(R)$ for any $r \in R$. Hence, $\bar{x}\bar{r} = \bar{r}\bar{x}$ for all $\bar{r} \in \bar{R}$. Therefore, $\bar{R} = R/N^*(R)$ is an abelian ring. □

A ring R is called *locally finite* if every finite subset of R generates a finite semigroup multiplicatively.

Proposition 2.13. *Let R be a locally finite ring of bounded index of nilpotency. If R is a semi-NI ring, then R is NI with $N^*(R) = N(R) = J(R)$.*

Proof. Let R be a locally finite ring. Thus, $R/N^*(R)$ is locally finite by [6, Proposition 2.1(1)]. If R is semi-NI, then $R/N^*(R)$ is abelian by Proposition 2.12. Since R is of bounded index of nilpotency and $N^*(R)$ is nil ideal, $R/N^*(R)$ is of bounded index of nilpotency. Hence, $R/N^*(R)$ is a locally finite abelian ring of bounded index of nilpotency. Therefore,

$$N_*(R/N^*(R)) = N^*(R/N^*(R)) = J(R/N^*(R)),$$

by [6, Proposition 2.11]. Hence, $N^*(R) = N(R) = J(R)$, and so R is an NI ring. □

Proposition 2.14. *For a ring R the following statements are equivalent*

(1) *R is semi-NI,*
(2) *$ab \in E(R)$ for $a, b \in R$ implies $\overline{ba} \in E(R/N^*(R))$,*

(3) *$ab \in E(R)$ for $a, b \in R$ implies $\overline{ab} = \overline{ba}$ in $R/N^*(R)$,*

(4) *$ab \in E(R)$ for $a, b \in R$ implies $\overline{arb} = \overline{arbab}$ in $R/N^*(R)$ for all $r \in R$.*

Proof. **1** $\Rightarrow$ **2**. If $ab \in E(R)$ for $a, b \in R$, then $ab(1-ab) = 0$. Since R is semi-NI, we have $b(1-ab)a \in N^*(R)$ by Theorem 2.11(4), and so $\overline{ba} = (\overline{ba})^2$. Thus $\overline{ba} \in E(R/N^*(R))$.

2 $\Rightarrow$ **3**. If $ab \in E(R)$ for $a, b \in R$, then $\overline{ba} \in E(R/N^*(R))$ by assumption. Since $R/N^*(R)$ is abelian by Proposition 2.12, we have

$$\overline{ba} = (\overline{ba})^2 = (\overline{ba})(\overline{ba}) = \bar{b}(\overline{ab})\bar{a} = (\overline{ab})(\overline{ba}) = \bar{a}(\overline{ba})\bar{b} = (\overline{ab})(\overline{ab}) = \overline{(ab)^2} = \overline{ab}.$$

3 $\Rightarrow$ **1**. Let $ab = 0$ for $a, b \in R$. Thus, $\bar{0} = \overline{ab} = \overline{ba}$ in $R/N^*(R)$ by assumption. Hence $ba \in N^*(R)$, and so R is semi-NI by Theorem 2.11(4).

1 $\Rightarrow$ **4**. If $ab \in E(R)$ for $a, b \in R$, then $ab(1-ab) = 0$. It follows from Theorem 2.11(6) that $arb(1-ab) \in N^*(R)$ for all $r \in R$. Therefore, $\overline{arb} = \overline{arbab}$ for all $r \in R$.

4 $\Rightarrow$ **1**. Let $ab = 0$ for $a, b \in R$. Then for all $r \in R$, $\overline{arb} = \overline{arbab} = \bar{0}$ by assumption. Hence $aRb \subseteq N^*(R)$. It follows from Theorem 2.11(6) that R is a semi-NI ring. □

Proposition 2.15. *Let R be a ring and $\bar{R} = R/N^*(R)$. Assume that R satisfies the following condition:*

$$\bar{a}\bar{R}\bar{b} \subseteq E(\bar{R}) \text{ for any } a, b \in R \text{ which } ab \in E(R).$$

Then R is semi-NI, whenever $\bar{R}$ is abelian.

Proof. Let $ab = 0$ for $a, b \in R$. Then by supposition, $\overline{arb} \in E(\bar{R})$ for all $r \in R$. Hence,

$$(\overline{barbar})^2 = \bar{b}\ (\overline{arb}\ \overline{arb}\ \overline{arb})\ \bar{a}\bar{r} = \bar{b}\ (\overline{arb})\ \bar{a}\ \bar{r} = \overline{barbar},$$

and so $\overline{barbar} \in E(\bar{R})$. Thus,

$$\overline{arb} = (\overline{arb})^3 = \overline{ar}\ (\overline{barbar})\bar{b} = \bar{a}\ (\overline{barbar})\ \bar{r}\ \bar{b} = \overline{ab}\ \overline{arbarrb} = \bar{0},$$

because $\bar{R}$ is abelian and $ab = 0$. Therefore, $arb \in N^*(R)$ for all $r \in R$, and so R is semi-NI by Theorem 2.11(6). □

Let R be a ring. An idempotent element e of R is called *right* (*left*) *semicentral*, if $ea = eae$ ($ae = eae$) for all $a \in R$. Write

$$ME_r(R) = \{e \in E(R) \mid eR \text{ is a minimal right ideal of } R\}.$$

Similarly, we can define $ME_l(R)$. A ring R is called *right* (*left*) *min-abelian* if every element $ME_r(R)$ ($ME_l(R)$) is right (left) semicentral in R.

Proposition 2.16. *Semi-NI rings are right min-abelian.*

Proof. Let $e \in ME_r(R)$ and $a \in R$. Set $h = ea - eae$. Thus $eh = h$, $he = 0$, and so $hR \subseteq eR$. If $h \neq 0$, then $hR = eR$ because eR is a minimal right ideal of R. Since R is semi-NI, it follows from Theorem 2.11(9) that $h = eh \in r(h)h \subseteq N^*(R)$. Hence $eR = hR \subseteq N^*(R)$, and so $e \in N(R) \cap E(R)$, which implies that $e = 0$. This is a contradiction. Therefore, $h = 0$, and consequently e is right semicentral. Thus, R is right min-abelian. □

By the same argument, we can prove that semi-NI rings are left min-abelian. A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. The fact that abelian rings are directly finite is well-known. NI rings are directly finite by [8, Proposition 2.7]. We extend this result to semi-NI rings in the following.

Proposition 2.17. *Semi-NI rings are directly finite.*

Proof. Let R be a semi-NI ring and $ab = 1$ for $a, b \in R$. Then $(ba)^2 = ba$, and so $e = ba \in E(R)$. Since R is semi-NI and $(1-e)b = 0$, we have $1-e = (1-e)ab \in (1-e)Rb \subseteq N^*(R)$ by Theorem 2.11(6). Hence $1 - e \in E(R) \cap N(R)$, which implies that $1 - e = 0$, and so $ba = 1$. Therefore, R is directly finite. □

In the following we see that $M_n(R)$ cannot be semi-NI for every ring R and $n \geq 2$.

Example 2.18. Let R be a ring and $n \geq 2$ be a natural element. It is clear that for $e_{11}, e_{12} \in M_n(R)$, we have $e_{12}(e_{11} + e_{12}) = 0$, but

$$e_{12}(e_{21} + e_{22})(e_{11} + e_{12}) = e_{11} + e_{12} \notin N(M_n(R)).$$

Hence, $M_n(R)$ is not semi-NI for $n \geq 2$ by Theorem 2.11(6).

It is well-known that for any positive integer n, $M_n(\mathbb{R})$, the full matrix ring over real number field $\mathbb{R}$ is directly finite. But by Example 2.18, we know that $M_n(\mathbb{R})$ is not semi-NI for $n \geq 2$. Hence, the converse statement of Proposition 2.17 is not true in general.

3. Semi-NI rings and related topics

In this section we verify relationships between semi-NI rings and some other rings. In previous section we see that all NI rings, reduced rings, reversible rings and semicommutative rings are semi-NI. Now, we prove that semi-NI rings are NCI rings, but NCI rings need not be semi-NI in general. Hence, the class of semi-NI rings lies strictly between classes of NI rings and NCI rings.

A ring R is called a *right p.p. ring* if each principal right ideal of R is projective, or equivalently, if the right annihilator of each element of R is generated by an idempotent. A ring is called a *p.p. ring* if it both right and left *p.p. ring*. The next result provides some conditions under which a semi-NI ring becomes reduced.

Proposition 3.1. *Let R be a right p.p. ring. The following conditions are equivalent*

(1) *R is a semi-NI ring,*
(2) *For every $a \in R$ and $e \in E(R)$, if $ae = 0$, then $Rera \subseteq N^*(R)$ for all $r \in R$,*
(3) *$eR(1-e) \subseteq N^*(R)$ for any $e \in E(R)$,*
(4) *For every $a \in R$ and $e \in E(R)$, if $ae = 0$, then $Rera^n \subseteq N^*(R)$ for all $r \in R$ and $n \in \mathbb{N}$,*
(5) *For every $a \in R$ and $e \in E(R)$, if $ae = 0$, then $Rera \subseteq N^*(R)$ for all $r \in R$,*
(6) *$eN(R)(1-e) \subseteq N^*(R)$ for any $e \in E(R)$.*

Proof. **1 ⇒ 2**. If $ae = 0$ for $a \in R$ and $e \in E(R)$, then $aer = 0$ for all $r \in R$. Since R is semi-NI, it follows from Theorem 2.11(4) that $era \in N^*(R)$. Hence, $Rera \subseteq N^*(R)$ for any $r \in R$.
2 ⇒ 3. Let $e \in E(R)$. Since $(1-e)e = 0$, we conclude from assumption that

$$er(1-e) \in Rer(1-e) \subseteq N^*(R)$$

for all $r \in R$. Hence $eR(1-e) \subseteq N^*(R)$.
3 ⇒ 4. Let $ae = 0$ for $a \in R$ and $e \in E(R)$. Then $a^n = a^n(1-e)$ for all $n \in \mathbb{N}$. Hence, $eRa^n = eRa^n(1-e) \subseteq eR(1-e) \subseteq N^*(R)$ for all $n \in \mathbb{N}$ by assumption. Thus, $era^n \in N^*(R)$ for any $r \in R$, and so $Rera^n \subseteq N^*(R)$ for all $n \in \mathbb{N}$.
4 ⇒ 5. It is obvious.
5 ⇒ 1. Let $a^2 = 0$ for $a \in R$. Since R is a right p.p. ring, there exists $e \in E(R)$ such that $r(a) = eR$, and so $ae = 0$ and $a = ea$. Hence, $Rera \subseteq N^*(R)$ for all $r \in R$ by assumption. Especially $Rara = Re(ar)a \subseteq N^*(R)$ for any $r \in R$. Hence $aRa \subseteq N^*(R)$. Since $N^*(R)$ is a semiprime ideal of R, we have $a \in N^*(R)$. Therefore, R is a semi-NI ring.
5 ⇒ 6. Let $e \in E(R)$ and $b \in N(R)$. Set $a = eb(1-e)$. Since $ae = 0$, we obtain

$$Rereb(1-e) = Rera \subseteq N^*(R),$$

for all $r \in R$ by assumption. Especially $Reb(1-e) \subseteq N^*(R)$. Hence $eN(R)(1-e) \subseteq N^*(R)$.
6 ⇒ 3. It is clear that $eR(1-e) \subseteq N(R)$ for any $e \in E(R)$. Then by assumption

$$eR(1-e) = e(eR(1-e))(1-e) \subseteq eN(R)(1-e) \subseteq N^*(R).$$

□

We can state and prove a result similar to the above proposition for a left p.p. ring. We know that every reduced ring is semi-NI not conversely, unless the following conditions are hold.

Proposition 3.2. *Let R be a semi-NI such that $N^*(R) \subseteq C(R)$. Then R is reduced ring if R satisfies any of the following conditions.*

(1) *R is a semiprime ring.*
(2) *R is a right (left) p.p. ring.*

Proof. We assume that R is semi-NI and $N^*(R) \subseteq C(R)$. If $a \in R$ with $a^2 = 0$, then $a \in N^*(R)$. Now consider the following cases;
1. Let R be a semiprime ring. Since $N^*(R) \subseteq C(R)$, $aRa = Ra^2 = 0$, and so $a = 0$. Thus, R is reduced.
2. Let R be a right p.p. ring. Then there exists an idempotent $e \in R$ such that $r(a) = eR$. Hence $ae = 0$. Since $a^2 = 0$ and $N^*(R) \subseteq C(R)$, we have $ea = a$, and so $a = ea = ae = 0$. Therefore, R is reduced. A similar proof can be given for left p.p. rings. □

A ring R is called (in [19]) a *semi-IFP* ring if for any $a \in R$, $a^2 = 0$ implies $aRa = 0$. Clearly, semicommutative rings are semi-IFP.

Proposition 3.3. *Let R be a ring. The following statements are hold.*

(1) *If $l(a)$ is an ideal of R for any $a \in N(R)$, then R is a semi-IFP ring.*
(2) *If R is a semi-IFP ring, then R is a semi-NI ring.*

Proof. **1.** Let $a^2 = 0$ for $a \in R$. Then $a \in l(a)$, and so $aR \subseteq l(a)$ because $l(a)$ is an ideal of R. Therefore $aRa = 0$, as desired.
2. Let $a^2 = 0$ for $a \in R$. Then $aRa = 0 \subseteq N^*(R)$. Since $N^*(R)$ is a semiprime ideal of R, we have $a \in N^*(R)$. This implies that R is a semi-NI ring. □

Example 3.4. Let R be a ring. Then $T_n(R)$ cannot be semi-IFP for $n \geq 4$ by [19, Example 1.5], but if R is a semi-NI ring, then $T_n(R)$ is semi-NI for any positive integer n by Corollary 2.8. Therefore, semi-NI rings are not semi-IFP in general.

In [17] R is said to be *central semicommutative* if $ab = 0$ implies $aRb \subseteq C(R)$ for $a, b \in R$. Every central semicommutative ring is NI by [17, Theorem 2.4], and so every central semicommutative ring is semi-NI. But the converse is not hold in general by Example 2.2. If R is a semi-NI ring, then for any $a, b \in R$ with $ab = 0$ we have $aRb \subseteq N^*(R)$ by Theorem 2.11(6). Thus we obtain

Proposition 3.5. *Let R be a semi-NI ring such that $N^*(R) \subseteq C(R)$. Then R is a central semicommutative ring.*

A ring R is called *central reversible* if for any $a, b \in R$, $ab = 0$ implies $ba \in C(R)$. By [12, Theorem 2.19], central reversible rings are NI, and so these are semi-NI. But the converse is not hold in general by Example 2.2. Also, we Know from [12, Lemma 2.13] that every central reversible ring is abelian. Now, if R is a semi-NI ring, then for any $a, b \in R$ with $ab = 0$, $aRb \subseteq N^*(R)$ by Theorem 2.11(4). Therefore, we have the following result.

Proposition 3.6. *Let R be a semi-NI ring. If $N^*(R) \subseteq C(R)$, then R is a central reversible ring and so R is an abelian ring.*

Therefore we obtain.

Corollary 3.7. *If R is a ring such that $N^*(R) \subseteq C(R)$, then the following statements are equivalent:*

(1) *R is a semi-NI ring.*
(2) *R is a central reversible ring.*
(3) *R is a central semicommutative ring.*

According to [9], a ring R is called *NCI* if $N(R) = 0$ or there exists a nonzero ideal of R contained in $N(R)$. Clearly, NI rings are NCI rings.

Proposition 3.8. *A semi-NI ring is an NCI ring.*

Proof. Let R be a semi-NI ring where $N(R) \neq 0$ and $0 \neq a \in N(R)$. Then there exists an integer number $n \geq 2$ such that a is nilpotent of index n (i.e., $a^n = 0$, but $a^{n-1} \neq 0$). Hence $a^{n-1} \in l(a)$. Since R is semi-NI, according to Theorem 2.11(8), $al(a) \subseteq N^*(R)$. If $al(a) \neq 0$, then $N(R)$ contains nonzero ideal $Ral(a)R$. If $al(a) = 0$, then $aRa^{n-1} \subseteq al(a) = 0$. For every element $r, s \in R$, $(ra^{n-1}s)^2 = ra^{n-2}(asra^{n-1})s \in RaRa^{n-1}R = 0$. Since R is semi-NI, $ra^{n-1}s \in N^*(R)$ for all $r, s \in R$. Hence $Ra^{n-1}R \subseteq N^*(R) \subseteq N(R)$. Therefore, $N(R)$ contains nonzero ideal $Ra^{n-1}R$. This implies that R is an NCI ring. □

According to [9, Example 2.5], we know that there exists an NCI ring which is not directly finite. Hence, by Proposition 2.17, the converse statement of Proposition 3.8 is not true in general, and so semi-NI rings lies strictly between NI rings and NCI rings.

Proposition 3.9. *Let R be a ring of bounded index of nilpotency. Then the following statements are equivalent*

(1) *R is reduced,*

(2) *R is NI and semiprime,*

(3) *R is semi-NI and semiprime,*

(4) *R is NCI and semiprime.*

Proof. (1), (2) and (4) are equivalent by [9, Proposition 1.3].

2 $\Rightarrow$ **3**. It is obvious.

3 $\Rightarrow$ **4**. It follows from Proposition 3.8.

Hence, the proof is complete. □

In [18], a ring R is said to be *Armendariz* if whenever two polynomials $f(x)$, $g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. Every reduced ring is Armendariz and Armendariz rings are abelian by the proof of [1, Theorem 6].

The concepts of Armendariz ring and semi-NI ring are independent of each other by the following example.

Example 3.10. 1. Let R be a semi-NI ring. Then $T_2(R)$ is semi-NI by Corollary 2.8. But this is non-abelian, and so $T_2(R)$ is not Armendariz.

2. We use construction and argument in [3, Theorem 4.7]. Let F be a field and $A = F\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x and y over F. Consider $R = A/(x^2)$. Thus, R is Armendariz by [3, Theorem 4.7]. If we identify x and y with their images in R, then we have $yxx = 0$, but $yxyx$ is not nilpotent because $N(R) = xRx + Fx$ by [5, Example 1]. Hence $yxyx \notin N^*(R)$, and so R is not semi-NI by Theorem 2.11(6).

In [11], a ring R is said to be *power-serieswise Armendariz* if whenever power series $f(x), g(x) \in R[[x]]$ satisfy $f(x)g(x) = 0$, then $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. Power-serieswise Armendariz rings are clearly Armendariz, but the converse is false by [2, Example 2]. We know that Power-serieswise Armendariz rings are semicommutative by [11, Lemma 2.3(2)]. Hence these are semi-NI rings. The converse of the above statement is not true in general by the following example.

Example 3.11. Let R be a ring. We know that $D_4(R)$ is not power-serieswise Armendariz by [16, Example 2.2]. However, if R is a semi-NI ring, then $D_4(R)$ is semi-NI by Corollary 2.8.

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Reza Jahani-Nezhad

Faculty of Mathematical Sciences, Department of Pure Mathematics,

University of Kashan, Kashan,

P. O. Box 87317-53153, I. R. Iran.

jahanian@kashanu.ac.ir

Fahime Ghadernazari

Faculty of Mathematical Sciences, Department of Pure Mathematics,

University of Kashan, Kashan,

P. O. Box 87317-53153, I. R. Iran.

f.ghadernazari@gmail.com